ON THE MEYER'S THEOREM AND THE DECOMPOSITION OF QUADRATIC FORMS

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Meyer(1) proved in 1883 the

Theorem M. *Every indefinite quadratic form in more than four variables, with determinant not zero and integer coefficients, always represents zero with the variables not all zero.*

The proof is based upon the deeper arithmetical theory of indefinite ternary quadratic forms. Up to the present, the corresponding problem (2) for cubic forms is not yet solved. It is desirable to find some new proofs for Meyer's Theorem which may enable to generalize to cubic forms. On using a method of one of my papers, (3) I succeed to find a new proof of the Theorem which I shall give in §1, but unfortunately, it seems also very difficult to extend to cubic forms. Although the proof requires some known results, none of them assume Meyer's Theorem.

By using Meyer's Theorem and my result about the even definite quadratic forms, (4) Dr. Zilinskas (5) recently proved the

(1) Cf. Bachmann, *Die Arithmetik der quadratischen Formen*, part I. (1898) pp. 266-267. On pp. 551-553, referesce is made to Minkowski's proof depending on the theory of the general quadratic form. An account is given in Dickson's Studies, pp. 68-70. Also see L. J. Mordell, 'On the condition for integer solutions of the equation $ax^2+by^2+cz^2+dt^2=0$', *J. R. Angew. Math.*, 164, (1931) 40-49.

(2) L. J. Mordell, A remark on indeterminate equations in several variables, *J. London Math. Soc.*, 12, (1937) 127-129.

(3) C. Ko, On the representation of a quadratic form as a sum of squares of linear forms, *Quart. J. of Math.* (Oxford), 8, (1937) 81-98.

(4) C. Ko, On the positive definite quadratic forms with determinant unity, *Acta Arithmetica*, 3, (1938) 79-85.

(5) G. Zilinskas, On the class number of indefinite quadratic forms in n variables with determinant ±1, *J. London Math. Soc.*, 13, (1938) 225-240.

Theorem Z. *Every properly primitive indefinite quadratic form in n variables with determinant ± 1 is equivalent to one of the set*

$$F_t(x_1, \cdots, x_n) = \sum_{1}^{n-t} x_i^2 - \sum_{n-t+1}^{n} x_i^2,$$

where t is the index of the form.

Similar to the proof of Meyer's Theorem in §1, by using Theorem Z, I shall prove in §2, the following

Theorem 1.(6) *Every quadratic form*

$$f_n = \sum_{i,j=1}^{n} a_{ij} x_i x_j \qquad (a_{ij} = a_{ji}),$$

the a's being integers, can be expressed as an algebraic sum of $n+3$ squares of linear forms with integer coefficients.

§1. The proof of Theorem M requires the following lemmas.

Lemma 1. *If f_n $(n<8)$ is a quadratic form with determinant ± 1, then a unitary transformation with integer coefficients carries f_n into a form $f'_n = \sum_{i,j=1}^{n} a'_{ij} x_i x_j$ with $|a'_{11}| < 2$.*

We need only illustrate the proof by proving the lemma for the case f_n having the canonical form

$$f_n = \sum_{i=1}^{3} V_i^2 - \sum_{i=4}^{7} V_i^2,$$

where $V_i = \sum_{j=1}^{n} \pi_{ij} x_j$ and π_{ij} being real.

Let

$$g_n = \sum_{i=1}^{7} V_i^2$$

(6) For definite quadratic forms, see C. Ko, (3), L. J. Mordell, The representation of a definite quadratic form as a sum of two others, *Annals of Math.*, 38, (1937) 751-757, C. Ko, On the decomposition of quadratic forms in six variables, *Acta Arithmetica*, 3, (1938) 64-78, and for indefinite quadratic forms in two variables, see C. Ko, On a Waring's problem with squares of linear forms, *Proc. London Math. Soc., Series* 2, 42, (1937), 171-185.

be the definite form associated to f_n, then obviously it has the determinant unity. Since it is known that the minimum of a positive definite quadratic form with determinant unity in less than eight variables is less thun or equal to $\sqrt[7]{64}$, (7) there exists a unitary transformation with integer coefficients which carries g_n into

$$g_n' = \sum_{i=1}^{7} V_i'^2,$$

where $V_i' = \sum_{j=1}^{7} \pi_{ij}' x_j$ and $\sum_{i=1}^{7} \pi_{i1}'^2 \leq \sqrt[7]{64}$. The same transformation carries f_n into $f_n' = \sum_{i=1}^{3} V_i'^2 - \sum_{i=4}^{7} V_i'^2 = \sum_{i,j=1}^{n} a_{ij}' x_i x_j$ with

$$a_{11}' = \sum_{i=1}^{3} \pi_{i1}'^2 - \sum_{i=4}^{7} \pi_{i1}'^2 \leq \sum_{i=1}^{7} \pi_{i1}'^2 \leq \sqrt[7]{64} < 2.$$

Lemma 2. *The indefinite quadratic forms in 3, 5, 7 variables with determinants* ± 1 *and canonical forms, respectively,*

$$f_3 = V_1^2 - V_2^2 - V_3^2, \quad f_5 = V_1^2 + V_2^2 - V_3^2 - V_4^2 - V_5^2, \quad f_7 = \sum_{i=1}^{3} V_i^2 - \sum_{i=4}^{7} V_i^2,$$

are null-forms, i.e. each of them represents zero with the variables not all zero.

Let us prove the lemma for f_3 for an illustration.

Suppose f_3 does not represent zero except all the variables being zero. By lemma 1, one of the forms $\pm f_3$ represents 1 and so

$$\pm f_3 \sim x_1^2 + f_2(x_2, x_3),$$

where f_2 has the canonical form $V_1^2 - V_2^2$ or $-V_1^2 - V_2^2$. Again, by lemma 1, one of the forms $\pm f_2$ represents 1, we have, respectively,

$$\pm f_2 \sim x_2^2 - x_3^2 \text{ or } x_2^2 + x_3^2.$$

(7) Blichfeldt, The minimum values of positive quadratic forms in six, seven and eight variables, *Math. Z.*, **39**, (1934), 1-15.

Hence

$$f_3 \sim x_1^2 - x_2^2 - x_3^2,$$

which represents zero for $x_1 = x_2$, $x_3 = 0$.

Lemma 3. *An indefinite quadratic form f_7 in seven variables with determinant* 1 *and canonical form* $\sum_{i=1}^{3} V_i^2 - \sum_{i=4}^{7} V_i^2$ *is equivalent to one of the following forms:*

$$2x_1 x_2 + \rho_1 x_2^2 + 2x_3 x_4 + \rho_2 x_4^2 + 2x_5 x_6 + \rho_3 x_6^2 - x_7^2,$$

where $\rho_i = 0$ *or* 1 $(i = 1, 2, 3)$.

By lemma 2, f_7 is a null-form,

$$f_7 \sim g_7 = 2x_1 x_2 + \rho_1 x_2^2 + f_5(x_3, \cdots, x_7),$$

where f_5 has the canonical form $V_1^2 + V_2^2 - V_3^2 - V_4^2 - V_5^2$ Again, by lemma 2, f_5 is a null-form,

$$f_5 \sim g_5 = 2x_3 x_4 + \rho_2 x_4^2 + f_3(x_5, x_6, x_7),$$

where f_3 has the canonical form $V_1^2 - V_2^2 - V_3^2$ which, by lemma 2, is also a null-form. Hence

$$f_3 \sim g_3 = 2x_5 x_6 + \rho_3 x_6^2 - x_7^2$$

and the lemma is established.

Lemma 4. *Let f_5 be a quadratic form in five variables with determinant D_5 and let an odd prime p be a divisor of d_4, the g. c. d. of all the 4-rowed minors in D_5. Then the transformations of the types*

(1) $$x_i = y_i \quad (i = 1, 2, 3, 4), \qquad p x_5 = y_5;$$

(1') $$x_i = y_i \quad (i = 1, 2, 3), \qquad p x_i = y_i \quad (i = 4, 5)$$

carry a form $f'_5 \sim p^{\varrho} f_5$, where $\varrho = 0$ or 1, into a form with integral coefficients and determinant D_5/p^2 or D_5/p.

With $t > 2$, we have $f''_5 \sim f_5$ and[8]

$$f''_5 \equiv p^{\varrho_1} \alpha_1 \xi_1^2 + p^{\varrho_2} \alpha_2 \xi_2^2 + \cdots + p^{\varrho_5} \alpha_5 \xi_5^2 \qquad (\text{mod } p^t)$$

$$(0 \leq \varrho_1 \leq \varrho_2 \leq \cdots \leq \varrho_5, \quad (\alpha_i, p) = 1 \quad \text{for} \quad i = 1, \cdots, 5).$$

The lemma is evident if one of the ϱ's is > 1. Suppose then all the ϱ's are not greater than one.

If three of the ϱ's are 1, say $\varrho_3 = \varrho_4 = \varrho_5 = 1$. Then the unitary transformation

$$\xi_1 = \eta_1, \quad \xi_2 = \eta_2, \quad \xi_3 = \eta_3 + z_3 \eta_5, \quad \xi_4 = \eta_4 + z_4 \eta_5, \quad \xi_5 = \eta_5 \tag{2}$$

carries f''_5 into f'_5 of the type

$$f'_5 \equiv p^{\varrho_1} \alpha_1 \eta_1^2 + p^{\varrho_2} \alpha_2 \eta_2^2 + p\alpha_3 \eta_3^2 + p\alpha_4 \eta_4^2 + 2p(\alpha_3 z_3 \eta_3 \eta_5 + \alpha_4 z_4 \eta_4 \eta_5)$$

$$+ p(\alpha_3 z_3^2 + \alpha_4 z_4^2 + \alpha_5) \eta_5^2 \qquad (\text{mod } p^t).$$

Since $(\alpha_3, p) = (\alpha_4, p) = 1$,

$$\alpha_3 z_3^2 + \alpha_4 z_4^2 + \alpha_5 \equiv 0 \qquad (\text{mod } p)$$

is solvable in z_3, z_4,[9] and so the lemma is proved, since the coefficients of $\eta_i \eta_5$ $(i=1, 2, 3, 4)$ are $\equiv 0 \pmod{p}$ and that of $\eta_5^2 \equiv 0 \pmod{p^2}$.

Since $p \mid d_4$, the only remaining case is that only two of the ϱ's are 1, say

$$\varrho_1 = \varrho_2 = \varrho_3 = 0, \qquad \varrho_4 = \varrho_5 = 1.$$

Then

$$pf''_5 \equiv p\alpha_1 \xi_1^2 + p\alpha_2 \xi_2^2 + p\alpha_3 \xi_3^2 + p^2\alpha_4 \xi_4^2 + p^2\alpha_5 \xi_5^2 \; (\text{mod } p^{1+t}). \tag{3}$$

(8) Bachmann (1), p. 434.

(9) Cf. Landau, *Vorlesungen über Zahlentheorie*, Satz 155.

After the transformation

$$\xi_i = \eta_i \quad (i = 1, 2, 3) \quad p\,\xi_4 = \eta_4, \quad p\,\xi_5 = \eta_5.$$

(3) becomes

$$f_5''' \equiv p\alpha_1 \eta_1^2 + p\alpha_2 \eta_2^2 + p\alpha_3 \eta_3^2 + \alpha_4 \eta_4^2 \quad \alpha_5 \eta_5^2 \qquad (\text{mod } p^{t-1}).$$

with determinant $p^5 D_5/p^4 = pD_5$.

As above, the transformations

$$\eta_1 = \zeta_1, \quad \eta_2 = \zeta_2 + z_2\,\zeta_1, \quad \eta_3 = \zeta_3 + z_3\,\zeta_1, \quad \eta_4 = \zeta_4, \quad \eta_5 = \zeta_5$$

and

$$\zeta_i = y_i \quad (i = 2, 3, 4, 5), \qquad p\,\zeta_1 = y_1,$$

carry f_5'' into f_5' with determinant $pD_5/p^2 = D_5/p$.

Lemma 5. *If* $2 \mid D_5$, *then the transformations of the types*

(4) $$x_i = y_i \quad (i = 1, 2, 3, 4), \qquad 2x_5 = y_5;$$

(4') $$x_i = y_i \quad (i = 1, 2, 3), \qquad 2x_i = y_i \quad (i = 4, 5)$$

carry a form $f_5' \sim 2^\varrho f_5$, *where* $\varrho = 0$ *or* 1, *into a form with integer coefficients and determinant* $D_5/4$ *or* $\frac{1}{2}D_5$.

With $t > 2$, we have $f_5'' \sim f_5$, f_5'' being one of the following three types: (10)

(5.1) $$f_5'' \equiv 2^{\varrho_1} \alpha_1 \xi_1^2 + 2^{\varrho_2} \alpha_2 \xi_2^2 + \cdots + 2^{\varrho_5} \alpha_5 \xi_5^2,$$

(5.2) $$f_5'' \equiv 2^{\varrho_1} \alpha_1 \xi_1^2 + 2^{\varrho_2} \alpha_2 \xi_2^2 + 2^{\varrho_3} \alpha_3 \xi_3^2 + 2^{\varrho_4} (2\alpha_4 \xi_4^2 + 2\alpha_5 \xi_4 \xi_5 + 2\beta \xi_5^2),$$

(5.3) $$f_5'' \equiv 2^{\varrho_1} \alpha_1 \xi_1^2 + 2^{\varrho_2} (2\alpha_2 \xi_2^2 + 2\alpha_3 \xi_2 \xi_3 + 2\beta_1 \xi_3^2)$$
$$+ 2^{\varrho_3} (2\alpha_4 \xi_4^2 + 2\alpha_5 \xi_4 \xi_5 + 2\beta_2 \xi_5^2) \qquad (\text{mod } 2^t),$$

where the α's are odd.

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(10) Bachmann (1), p. 444.

Suppose first f_5'' is of the type (5.1.). The lemma is evident when one of the ρ's is greater than 1. Suppose then $\rho_i \leqq 1$ $(i=1,\cdots, 5)$. If two of the ρ's are 1, say $\rho_4 = \rho_5 = 1$. Then the unitary transformation

$$\xi_1 = \eta_1, \quad \xi_2 = \eta_2, \quad \xi_3 = \eta_3, \quad \xi_4 = \eta_4 + \eta_5, \quad \xi_5 = \eta_5$$

carries f_5'' into

$$f_5' \equiv 2^{\rho_1} \alpha_1 \eta_1^2 + 2^{\rho_2} \alpha_2 \eta_2^2 + 2^{\rho_3} \alpha_3 \eta_3^2 + 2\alpha_4 \eta_4^2 + 2.2\alpha_4 \eta_4 \eta_5 + 2(\alpha_4 + \alpha_5) \eta_5^2$$

$$(\text{mod } 2^t),$$

and so the lemma is proved since $\alpha_4+\alpha_5 \equiv 0 \pmod 2$.

Since $2 | D_5$, the remaining case is that only one of the ρ's is 1, say $\rho_5=1$. Then

$$(6) \qquad 2f_5'' \equiv 2\alpha_1 \xi_1^2 + 2\alpha_2 \xi_2^2 + 2\alpha_3 \xi_3^2 + 2\alpha_4 \xi_4^2 + 2^2\alpha_5 \xi_5^2 \qquad (\text{mod } 2^{t+1}).$$

After the transformation (4), (6) becomes

$$f_5''' \equiv 2\alpha_1 \xi_1^2 + 2\alpha_2 \xi_2^2 + 2\alpha_3 \xi_3^2 + 2\alpha_4 \xi_4^2 + 2\alpha_5 \xi_5^2 \qquad (\text{mod } 2^t)$$

with the determinant $2^5D_5/2^2 = 2^3D_5$. Then the transformations

$$\xi_1 = \eta_1, \quad \xi_2 = \eta_1 + \eta_2, \quad \xi_3 = \eta_3, \quad \xi_4 = \eta_3 + \eta_4, \quad \xi_5 = \eta_5$$

and

$$2\eta_1 = y_1, \ \eta_2 = y_2, \ 2\eta_3 = y_3, \ \eta_4 = y_4, \ \eta_5 = y_5$$

carry f_5''' into f_5' with determinant $2^3D_5 \ 2^4 = D_5/2$.

Suppose next f_5'' is of the type (5.2). The lemma is evident if either one of the ρ_i $(i=1, 2, 3)$ is greater than 1, or $\rho_4>0$. Suppose then $\rho_i \leqq 1$ $(i=1, 2, 3)$ and $\rho_4=0$. If two of the ρ_i $(i=1, 2, 3)$ are 1, say $\rho_1 = \rho_2=1$. As above, by a unitary transformation

$$\xi_1 = \eta_5 \quad \xi_2 = \eta_6 + \eta_2, \quad \xi_3 = \eta_3, \quad \xi_4 = \eta_4, \quad \xi_5 = \eta_1,$$

we can make all the coefficients of the terms involving $\eta_4 \equiv 0 \pmod 4$, and so the lemma is proved.

Since $2|D_5$, the remaining case is that only one of the ρ_i $(i=1, 2, 3)$ is 1, say $\rho_1=1$. Then

(7) $$2f_5'' \equiv 2^2\alpha_1\xi_1^2 + 2\alpha_2\xi_2^2 + 2\alpha_3\xi_3^2 + 2^2\alpha_4\xi_4^2 + 2^2\alpha_5\xi_4\xi_5 + 2^2\alpha_6\xi_5^2 \pmod{2^{t+1}}.$$

After the transformation

$$2\xi_1 = \eta_1,\quad \xi_2 = \eta_2,\quad \xi_3 = \eta_3,\quad \xi_4 = \eta_4,\quad 2\xi_5 = \eta_5,$$

(7) becomes

$$f_5''' \equiv \alpha_1\eta_1^2 + 2\alpha_2\eta_2^2 + 2\alpha_3\eta_3^2 + 2^2\alpha_4\eta_4^2 + 2\alpha_5\eta_4\eta_5 + \alpha_6\eta_5^2 \pmod{2^{t-1}}.$$

with determinant $2^5D_5/2^4=2D_5$. Then as above, the transformations

$$\eta_1 = \zeta_1,\quad \eta_2 = \zeta_2+\zeta_3,\quad \eta_3 = \zeta_3,\quad \eta_4 = \zeta_4,\quad \eta_5 = \zeta_5,$$

and

$$\zeta_1 = y_1,\quad \zeta_2 = y_2,\quad 2\zeta_3 = y_3,\quad \zeta_4 = y_4,\quad \zeta_5 = y_5,$$

carry f_5''' into f_5' with determinant $2D_5/2^2=D_5/2$.

Similarly, we can prove the third case without any difficulty.

Lemma 6. *If $d_4 = 1$, the transformations of the type (1) carries $f_5' \sim pf_5$ with a suitable odd prime p into a form with determinant pD_5 and $d_4'= 1$, where d_4' is the g.c.d. of all the 4-rowed minors of pD_5.*

Since $d_4=1$, the g.c.d. of the 3-rowed minors of the determinant of pf_5 is $d_3'=p^3$. Write

(8) $$pf_5 \equiv p^{\rho_1}\alpha_1\xi_1^2 + p^{\rho_2}\alpha_2\xi_2^2 + \cdots + p^{\rho_5}\alpha_5\xi_5^2 \pmod{p^t}.$$

Since $\rho_5 \geq \rho_4 \geq \rho_3 \geq 1$, as in the proof of the first part of lemma 4, the transformations (1) and (1′) (if necessary, combined with (2)) carry (8) into f_5'' with $d_3''=p$ and determinant $p^5D_5/p^2=p^3D_5$.

With $t>1$, we have $f_5''' \sim f_5''$ and

$$f_5''' \equiv p^{\rho_1'}\alpha_1'\xi_1^2 + \cdots + p^{\rho_5'}\alpha_5'\xi_5^2 \pmod{p^t}.$$

Since $d_3''=p$, at least 3 of the ϱ's must be ≥ 1, as before, the transformations (1) and (1′) (if necessary, combined with (2)) carry (8) into f_5' with determinant $p^3D_5/p^2=pD_5$.

Now, if $d_4'=p$, we have $pD_5\equiv 0 \pmod{p^2}$. Hence the lemma is proved if we choose the prime p such that $D_5\not\equiv 0 \pmod{p}$.

Denote the minor determinant of the matrix (a_{ij}) $(i, j=1,\cdots, n)$ of the quadratic form $f(x_1,\cdots, x_n)$ formed by the elements at the intersections of rows $i_1, i_2, \cdots, i_k$ and columns $j_1, j_2, \cdots, j_k$ by $\mathfrak{A}^{(k)}_{i_1,\cdots i_k;\, j_1,\cdots j_k}$; the g.c.d. of all the minors of order k by d_k; the g.c.d. of all the integers

$$\mathfrak{A}^{(k)}_{i_1,\cdots, i_k;\, i_1,\cdots, i_k}\ d_k, \qquad 2\,\mathfrak{A}^{(k)}_{i_1,\cdots, i_k;\, j_1,\cdots, j_k}\ d_k$$

by $\sigma_k = 1$ or 2 $(k = 1, 2, \cdots, n)$. Let $d_{-1} = 0$, $d_0 = 1$, $d_{n+1} = 0$,

and define the numbers

$$o_k = d_{k+1}\, d_{k-1}/d_k^2 \qquad (k = 0, \cdots, n).$$

Then we have the known theorem:(11)

Lemma 7. *Let*

$$\mathfrak{A}^{(k)}_{1,2,\cdots,k;\,1,2,\cdots,k} = \sigma_k\, d_k\, B_k \quad (k = 1, \cdots, n)$$

so that $B_n = 1$. *Define* $B_0 = 1$. *Then there exists a form* $f_n' \sim f_n$ *with*

$$(B_k',\ 2B_{k+1}'\, B_{k-1}'\, o_1\, o_2 \cdots o_{n-1}) = 1 \qquad (k = 1, \cdots, n-1).$$

Lemma 8. *If* $d_4 = 1$ *and* $d_5 = D_5$ *is odd, then a unitary substitution carries* f_5 *into a form*

$$f_5' = \sum_{i,j=1}^{5} a_{ij}'\, x_i\, x_j \qquad (a_{ij}' = a_{ij}'),$$

(11) Bachmann (1), p. 453.

Where $A = \begin{vmatrix} a'_{22} & a'_{23} & \cdots & a'_{25} \\ \cdot & \cdot & \cdots & \cdot \\ a'_{52} & a'_{53} & \cdots & a'_{55} \end{vmatrix}$ *is relatively prime to* D_5.

Since $d_4=1$, we have

$$o_1 = o_2 = o_3 = 1, \qquad o_4 = d_5 = D_5,$$

and so by lemma 7 with $k=4$, there exists a form $f_5'' \sim f_5$ with

$$(\mathfrak{A}^{(4)}_{1,2,3,4;1,2,3,4}, o_4, D_5) = 1, \text{ or } (A, D_5) = 1,$$

since D_5 is odd and $\mathfrak{A}^{(4)}_{1,2,3,4;1,2,3,4} = A$ if we replace x_i by x_{i+1} for $i = 1, 2, 3, 4$ and x_5 by x_1 in f_5''. Hence the lemma is established.

Lemma 9. *Let* $f_5 = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ *satisfy the conditions:*

(i) *Whose canonical form is either (a)* $V_1^2 + V_2^2 + V_3^2 - V_4^2 - V_5^2$, *or (b)* $V_1^2 - V_2^2 - V_3^2 - V_4^2 - V_5^2$,

(ii) *whose determinant* $D_5 \neq 8m - 1$ *and is odd.*

(iii) $(A, D_5) = 1$, *then there exist integers* a_{66}, a_{67}, a_{77} *such that*

$$f_7 = f_5 + 2x_1 x_7 + a_{66} x_6^2 + 2a_{67} x_6 x_7 + a_{77} x_7^2$$

is a form with determinant unity and whose canonical form is

$$V_1^2 + V_2^2 + V_3^2 - V_4^2 - V_5^2 - V_6^2 - V_7^2. \tag{9}$$

For this means

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{15} & 0 & 1 \\ a_{21} & a_{22} & \cdots & a_{25} & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ a_{51} & a_{52} & \cdots & a_{55} & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_{66} & a_{67} \\ 1 & 0 & \cdots & 0 & a_{67} & a_{77} \end{vmatrix} = 1,$$

whence

$$a_{67}^2 D_5 + 1 = a_{66}(a_{77} D_5 - A). \tag{10}$$

Since $(A, D_5)=1$, there exist for variable integers a_{77} infinitely many primes p or $2p$ of the form $|D_5 a_{77} - A|$. We show that (10) is solvable by selecting a_{77} so that $|D_5 a_{77} - A| = p$ or $2p$, where p is an odd prime >0, and $(-D_5/p)=1$, the symbol being that of quadratic reciduacity.

Suppose first $D_5 \equiv 1 \pmod 4$. Take $|a_{77}D_5 - A| = p$. Then

$$(-D_5/p) = (-1)^{\frac{1}{2}(p-1)}(p/D_5) = (-1)^{\frac{1}{2}(p-1)}(\pm A/D_5).$$

Since we can choose a_{77} such that $p\equiv 1$ or $3 \pmod 4$, we can always make $(-D_5/p)=1$.

Suppose next $D_5 \equiv 3 \pmod 8$. If $(\pm A/D_5)=1$, take $|a_{77}D_5 - A| =$ p, and we have $(-D_5/p)=(\pm A/D_5)=1$. If $(\pm A/D_5)=-1$, we take $|a_{77}D_5 - A| = 2p$, then $(2/D_5)=-1$ and so

$$(-D_5/p) = -(p/D_5)(2/D_5) = -(2p/D_5) = -(\pm A/D) = 1.$$

Hence we can always make $(-D_5/p)=1$, and (10) is solvable in a_{66}, a_{67}, a_{77}.

For the case (a), we choose $a_{77}<0$ and such that $D_5 a_{77} - A<0$, Then $a_{66}<0$, since the left-hand side of (10) is positive. Let us call

$$a_{11} = D_1, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = D_2, \quad \cdots, \quad \begin{vmatrix} a_{11} & \cdots & a_{15} & 0 \\ a_{21} & \cdots & a_{25} & 0 \\ \cdot & \cdots & \cdot & \cdot \\ a_{51} & \cdots & a_{55} & 0 \\ 0 & \cdots & 0 & a_{66} \end{vmatrix} = D_6, \quad \text{and} \quad D_7 = 1,$$

then

$$f_7 = D_1 X_1^2 + D_2 X_2^2 / D_1 + \cdots + D_5 X_5^2 / D_4 + D_6 X_6^2 / D_5 + D_7 X_7^2 / D_6, \tag{11}$$

where X's are linear functions in $x_1, \cdots, x_n$ and by lemma 7, we can assume that $D_i \neq 0$ for $i=1, 2, 3, 4$. Since $D_5>0$, $a_{66}<0$, we have $D_6<0$ and so the canonical form of f_7 is exactly (9).

For the case (b), we choose $a_{77}>0$ and such that $D_5a_{77}-A>0$. Then $a_{66}>0$, since the left-hand side of (10) is positive. Hence from (11), the canonical form of f_7 is again (9), since now D_5, D_6, D_7 are all positive.

Now the *Meyer's theorem* can be easily proved.

By lemmas 4 and 5, we need only prove those forms with D_5 odd and $d_4=1$. By lemma 6, we need only to consider those forms D_5 odd and $\neq 8m-1$ and $d_4=1$. Then by lemma 8, we can also assume that $(D_5, A)=1$. By lemma 9, we have a form f_7 which is f_5 by putting $x_6=x_7=0$ and the determinant of f_7 is 1. By lemma 3,

$$f_7 \sim 2y_1y_2+\varrho_1y_2^2+2y_3y_4+\varrho_2y_4^2+2y_5y_6+\varrho_3y_6^2-y_7^2.$$

Hence by putting $x_6=x_7=0$, we can write

$$f_7=2(\sum_{i=1}^{5}c_{1i}x_i)(\sum_{i=1}^{5}c_{2i}x_i)+\varrho_1(\sum_{i=1}^{5}c_{2i}x_i)^2+\cdots$$
$$+2(\sum_{i=1}^{5}c_{5i}x_i)(\sum_{i=1}^{5}c_{6i}x_i)+\varrho_3(\sum_{i=1}^{5}c_{6i}x_i)^2-(\sum_{i=1}^{5}c_{7i}x_i)^2,$$

were c_{ij} are integers. Since the determinant of the system of four linear equations in five variables

$$\sum_{j=1}^{5}c_{ij}x_j=0 \qquad (i=2,4,6,7)$$

is always solvable with the x's not all zero, we have a set of non-trivial integer solutions satisfying $f_5=0$.

§2. The proof of theorem 1 requires the following lemmas:

Lemma 10. *Let f_n be a quadratic form with determinant D_n and let the odd prime p be a divisor of d_{n-2}. Then the transformation*

$$x_i=y_i \ (i=1,2,\cdots,n-1),\ px_n=y_n$$

carries a form $f'_n\sim f_n$ into a form with integer coefficients and determinant D_n/p^2.

Lemma 11. *Let 2 be a divisor of d_{n-1}. Then the transformation*

$$x_i = y_i \quad (i = 1, 2, \cdots, n-1). \qquad 2x_n = y_n$$

carries a form $f'_n \sim f_n$ *into a form with integer coefficients and determinant* D_n 4.

Lemma 12. *If* $d_{n-2}=1$ *and* d_{n-1} *is odd, then a unitary substitution carries* f_n *into a form*

$$f'_n = \sum_{i,j=1}^{n} a'_{ij}\, x_i\, x_j \qquad (a'_{ij} = a'_{ji}).$$

Where
$$\begin{vmatrix} a'_{22} & \cdots & a'_{2n} \\ \cdots & \cdots & \cdots \\ a'_{n2} & \cdots & a'_{nn} \end{vmatrix} \not\equiv 0 \pmod 4$$

and is relatively prime to
$$\begin{vmatrix} a'_{33} & \cdots & a'_{3n} \\ \cdots & \cdots & \cdots \\ a'_{n3} & \cdots & a'_{nn} \end{vmatrix}.$$

Lemma 13. *Given a quadratic form* f_n *with*

$$A_{11}^{(n)} = \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \not\equiv 0 \pmod 4$$

and relatively prime to

$$\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots \\ a_{n3} & \cdots & a_{nn} \end{vmatrix},$$

then we can find two integers $a_{1,n+1}$, $a_{n+1,n+1}$ *such that the form*

$$f_{n+1} = f_n + 2a_{1,n+1}\, x_1\, x_{n+1} + 2x_2\, x_{n+1} + a_{n+1,n+1}\, x_{n+1}^2$$

(i) *has determinant* $D_{n+1} \neq 4^k(8m-1)$

and (ii) $(A_{11}^{(n+1)}, D_{n+1}) = 1$, *where*

$$A_{11}^{(n+1)} = \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} & 1 \\ a_{32} & a_{33} & \cdots & a_{3n} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} & 0 \\ 1 & 0 & \cdots & 0 & a_{n+1,\, n+1} \end{vmatrix} = a_{n+1,\, n+1}\, A_{11}^{(n)} - \begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots \\ a_{3n} & \cdots & a_{nn} \end{vmatrix}$$

$= p$, *an old prime.*

Lemma 14. *If in lemma* 13, $D_{n+1} \equiv 0 \pmod 4$, *the transformation*

(12) $$x_i = y_i \ (i = 1, \cdots, n),\ 2x_{n+1} = y_{n+1}$$

carries a form $f'_{n+1} \sim f_{n+1}$ *into a form with integer coefficients, determinant* $D_{n+1}/4$ *and* $d_n = 1$, $\sigma_n = 1$.

The proofs of the above lemmas are entirely similar to that already given by the author.(12)

Lemma 15. *Let* f_{n+1} *be a quadratic form with* $(A_{11}^{(n+1)}, D_{n+1}) = 1$, *where* $D_{n+1} \not\equiv 0 \pmod 4$, *being positive, is the determinant of* f_{n+1} *and* $A_{11}^{(n+1)}$, *the cofactor of* a_{11} *in* D_{n+1}, *is odd. If* $D_{n+1} \not\equiv 7 \pmod 8$, *there exist integers* $a_{n+2,n+2}$, $a_{n+2,n+3}$, $a_{n+3,n+3}$ *such that the form in* $n+3$ *variables*

$$f_{n+3} = f_{n+1} + 2x_1 x_{n+3} + a_{n+2,\, n+2}\, x_{n+2}^2 + 2a_{n+2,\, n+3}\, x_{n+2}\, x_{n+3} + a_{n+3,\, n+3}\, x_{n+3}^2$$

is an indefinite properly primitive form with determinant unity.

It requires to solve the Diophantine equation

(13) $$a_{n+2,\, n+3}^2\, D_{n+1} + 1 = a_{n+2,\, n+2}(a_{n+3,\, n+3}\, D_{n+1} - A_{11}^{(n+1)}).$$

Similar to the proof of lemma 9, (13) is always solvable with either $a_{n+2,\, n+2}$, $a_{n+3,\, n+3}$ both positive or both negative at our disposal, unless $D_{n+1} = 2D'_{n+1}$, $D'_{n+1} \equiv 1 \pmod 2$. In the latter case, if we set

(12) Ko (3), Lemmas 8, 9, 10, 12 and 13.

$\left| a_{n+3,\, n+3} D_{n+1} - A_{11}^{(n+1)} \right| = p$, being a prime, then

$$(-D_{n+1}/p) = (2/p)\,(-D'_{n+1}/p).$$

But

$$(2/p) = (-1)^{(p^2-1)/8},\ (-D'_{n+1}/p) = (-1)^{\frac{1}{2}(p-1).\,\frac{1}{2}(D'_{n+1}+1)}(p/D'_{n+1})$$

$$= (-1)^{\frac{1}{2}(p-1).\,\frac{1}{2}(D'_{n+1}+1)}(\pm A_{11}^{(n+1)}/D'_{n+1}),$$

and so

$$(-D_{n+1}/p) = (-1)^{(p^2-1)/8+\frac{1}{2}(p-1).\,\frac{1}{2}(D'_{n+1}+1)}(\pm A_{11}^{(n+1)}/D'_{n+1}).$$

Choose $p \equiv 1 \pmod 8$, when $(\pm A_{11}^{(n+1)}/D'_{n+1}) = 1$; and $p \equiv 5 \pmod 8$, when $(\pm A_{11}^{(n+1)}/D'_{n+1}) = -1$. Then always $(-D_{n+1}/p) = 1$ and (13) is still solvable with either $a_{n+2,n+2}$, $a_{n+3,n+3}$ both positive or both negative at our disposal. Hence we can always make f_{n+3} to be an indefinite form.

As in the proof of lemma 9, in case $D_{n+1} \equiv 3 \pmod 8$, $|a_{n+3,n+3} D_{n+1} - A_{11}^{(n+1)}| = 2p$, we have $a_{n+3,\,n+3}$ odd, since $A_{11}^{(n+1)}$ is odd. In all the other cases $|a_{n+3,\,n+3} D_{n+1} - A_{11}^{(n+1)}| = p$. If we put $a_{n+2,\,n+3} = 2a'_{n+2,\,n+3}$, the solvability of (13) is not effected, since $(4/p) = 1$ and then $a_{n+2,\,n+2}$ must be odd, sisce the left-hand side of (13) is odd. Hence one of $a_{n+2,\,n+2}$, $a_{n+3,\,n+3}$ can be chosen to be odd in this lemma in order to make f_{n+3} to be a properly primitive form.

Proof of the theorem 1.

From lemmas 10, 11 and 12, it is clear that we need only to prove the theorem for the from

$$f_n = \sum_{i,j=1}^{n} a_{ij} x_i x_j \qquad (a_{ij} = a_{ji}),$$

where $A_{11}^{(n)}$ is relatively prime to

$$\begin{vmatrix} a_{33} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n3} & \cdots & a_{nn} \end{vmatrix}.$$

Since from Zilinskas theorem, every properly primitive indefinite quadratic form in $n+3$ variables with determinant unity is equivalent to one of the set $F_t(x_1, \cdots, x_{n+3})$, then by lemmas 13 and 15, if $D_{n+1} \not\equiv 0 \pmod 4$ in lemma 13,

$$f_{n+3} = f_n + 2x_1 x_{n+3} + 2x_2 x_{n+1} + 2a_{1,\, n+1} x_1 x_{n+1} + 2a_{n+2,\, n+3} x_{n+2} x_{n+3}$$
$$+ a_{n+1,\, n+1} x_{n+1}^2 + a_{n+2,\, n+2} x_{n+2}^2 + a_{n+3,\, n+3} x_{n+3}^2$$

is an indefinite properly primitive form with determinant unity and so

$$f_{n+3} = \sum_{i=1}^{n+3} \varepsilon_i L_i^2$$

where the L's are linear forms in $x_1, \cdots, x_{n+3}$ and $\varepsilon_i = +1$ or -1. Putting $x_{n+1} = x_{n+2} = x_{n+3} = 0$, we have a representation of f_n as an algebraic sum of $n+3$ squares of linear forms with integer coefficients.

If, in lemma 13, $D_{n+1} \equiv 0 \pmod 4$, then by lemma 14, we can carry f_{n+1} in the lemma 13, by some unitary substitutions combined with those of the type (12), into a form f'_{n+1} with $D'_{n+1} \not\equiv 0 \pmod 4$ and $d'_{n+1} = \sigma'_{n+1} = 1$. Then by similar argument to that in lemma 12, we can assume that f'_{n+1} satisfies all the conditions of lemma 15. Then there exist integers $a_{n+2,\, n+2}$, $a_{n+2,\, n+3}$, $a_{n+3,\, n+3}$ such that

$$f'_{n+1} + 2x_1 x_{n+3} + a_{n+2,\, n+2} x_{n+2}^2 + 2a_{n+2,\, n+3} x_{n+2} x_{n+3} + a_{n+3,\, n+3} x_{n+3}^2$$
$$= \sum_{i=1}^{n+3} \varepsilon'_i L_i'^2 \cdot \qquad (\varepsilon'_i = +1 \text{ or } -1).$$

Putting $x_{n+2} = x_{n+3} = 0$ and transforming f'_{n+1} back to f_{n+1}, we have

$$f_{n+1} = \sum_{i=1}^{n+3} \varepsilon'_i L_i''^2 \cdot \tag{14}$$

Putting $x_{n+1} = 0$ in (14), we have a representation of f_n as an algebraic sum of $n+3$ squares of linear forms with integer coefficients. Hence theorem 1 is completely proved.

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(Received 28, March, 1939).