SOME NEW GEOMETRICAL SIGNIFICANCES OF THE PROJECTIVE CURVATURES AND THE CURVATURE FORM OF A SPACE CURVE

By Te-Chih Fon[1] (方德植)

Su has recently established[2] the projective theory of space curves by a purely geometrical method and has shown among other things that the projective invariants of a curve can simply be expressed by certain double ratios. In my former paper[3] I have interpreted the curvature form by the Von Staudt's double ratios of the tangent of the space curve C at a point infinitely near an ordinary point P with respect to fundamental tetrahedron of Sannia at P.

The object of this paper is to give some simpler geometrical significances of the projective curvatures and the curvature form of a space curve C.

Let the coordinates of the vertices P, P_1, P_2, P_3 of the normal tetrahedron of Su be (x), (x_1), (x_2), (x_3) respectively, then the projective Frenet-Serret Formulae take on the form[4]

$$(1)\qquad \begin{cases} x' = x_1, \\ x_1' = -3Ix + x_2, \\ x_2' = -\dfrac{16}{5}x - 4Ix_1 + x_3, \\ x_3' = -Jx - \dfrac{4}{5}x_1 - 3Ix_2, \end{cases}$$

(1) Research Fellow of the China Foundation for the Promotion of Education and Culture.

(2) B. Su, Note on the projective differential geometry of space curves, *Journal Chinese Math. Soc.*, 2 (1937), 98-137.

(3) Te-Chih Fon, Note on the projective differential geometry of space curves, *Annali di Mat.* (1939, 97-106.

(4) Cfr. Su, loc. lit. 130.

where the dash represents the derivative with respect to the projective arc σ; I and J denote respectively the first and second projective curvatures of Sannia.

It is well known that any seven-point quadric of a space curve C at a point P always passes through the eighth fixed point S, namely, the point of Sannia.

If we express the projective homogeneous coordinates of any point $M(Y)$ in space by the form

$$Y = y_1 x + y_2 x_1 + y_3 x_2 + y_4 x_3 , \tag{2}$$

where y_1, y_2, y_3, y_4 denote the local coordinates of M with respect to the normal tetrahedron $\{PP_1P_2P_3\}$ of C at P with the unit point $(x+x_1+x_2+x_3)$, then the coordinates of S are

$$\begin{cases} y_1 = 1 - 9\, x_0^3 , \\ y_2 = 3\, x_0 , \\ y_3 = \dfrac{9}{2}\, x_0^2 , \\ y_4 = \dfrac{9}{2}\, x_0^2 , \end{cases} \tag{3}$$

x_0 being defined by

$$x_0 = \frac{4}{5\,J} . \tag{4}$$

Whence the coordinates of the point of intersection $\overline{S}$ of the plane $[P_2P_3, S]$ and the tangent PP_1 are

$$y_1 = 1 - 9\, x_0^3 , \quad y_2 = 3\, x_0 , \quad y_3 = 0 , \quad y_4 = 0 ; \tag{5}$$

and the principal point of Sannia $\bar{R}$ is

$$y_1 = 5\,J , \quad y_2 = -4 , \quad y_3 = 0 , \quad y_4 = 0 . \tag{6}$$

From (5) and (6) it follows that the double ratio of the four points P, P_1, $\overline{R}$, $\overline{S}$ is equal to

$$D \equiv (P P_1, \bar{R} \bar{S}) = -\frac{1}{3} + \frac{3 \cdot 2^6}{5^3 J^3}, \tag{7}$$

or

$$J^3 = \frac{3^2 \cdot 2^6}{5^3 (1 + 3D)}. \tag{I}$$

Thus we arrive at the following

Theorem I. *Let $\bar{S}$ be the point where the tangent PP_1 intersects the plane determined by the edge P_2P_3 of the normal tetrahedron and the Sannia point S. If $\bar{R}$ be the principal point of Sannia and D the double ratio of the four points P, P_1, $\bar{R}$, $\bar{S}$, then the second projective curvature J is given by the equation (I).*

In the next place we shall express the first projective curvature I by another simple double ratio of four elements of a primitive geometric form.

The equation of the osculating conic C_2 of C at P is given by

$$y_2^2 - \frac{8}{3} y_1 y_3 = 0, \tag{8}$$

and from (1) we obtain that the equation of the tangent t_1 to the curve (P_1) at P_1 is

$$y_1 + 3 I y_3 = 0. \tag{9}$$

Let Q_ε denote the points of intersection of t_1 and C_2, then their coordinates are

$$y_1 = -3I, \quad y_2 = \varepsilon \sqrt{-8I}, \quad y_3 = 1, \quad y_4 = 0, \tag{10}$$

where $\varepsilon = \pm 1$. On the other hand the tangent of the curve (P_3) at P_3 meets the osculating plane of the curve at the point P_3^* with the coordinates

$$y_1 = J, \quad y_2 = \frac{4}{5}, \quad y_3 = 3I, \quad y_4 = 0. \tag{11}$$

Therefore the double ratio of the four lines PP_1, PP_2, PP_3^*, PQ_ε is equal to

$$P(P_1 P_2, P_3^* Q_\varepsilon) = \frac{15}{4} \varepsilon I \sqrt{-8I},$$

or

(II) $$I^3 = -\frac{2}{3^2 5^2}\Big[P(P_1P_2, P_3^* Q_\varepsilon)\Big]^2$$

Thus we are led to

Theorem II. *Let Q_ε ($\varepsilon=1$ or -1) denote one of the points where the tangent t_1 of the curve (P_1) at P_1 intersects the osculating conic C_2 of C at P, and let P_3^* be the point where the tangent of the curve (P_3) at P_3 meets the osculating plane; then the cube of the first projective curvature I of C is, except a numerical factor $-\frac{2}{3^2 5^2}$, equal to the square of the double ratio of the four lines PP_1, PP_2, PP_3,* PQ_ε.*

Finally, we shall give a simple geometric significance of the curvature form

The consecutive point P' $(x(\sigma+d\sigma))$ of P of the curve C may be regarded as a point on the osculating plane provided that the infinitesimals of order $\geqq 3$ be neglected. Thus

$$x(\sigma+d\sigma) = (1-\frac{3}{2}I\,d\sigma^2)\,x + d\sigma\, x_1 + \frac{d\sigma^2}{2}\,x_2\,,$$

and consequently the local coordinates of P' are given by

$$y_1 = 1-\frac{3}{2}I\,d\sigma^2,\quad y_2 = d\sigma,\quad y_3 = \frac{d\sigma^2}{2}\,,\quad y_4 = 0\,.$$

Any point on the line PP' is of the coordinates

(12) $$y_1 = 1-\frac{3}{2}I\,d\sigma^2-\varrho,\quad y_2 = d\sigma,\quad y_3 = \frac{d\sigma^2}{2}\,,\quad y_4 = 0\,,$$

ϱ being a parameter. If Q, Q' denote the points of intersection of PP' with P_1P_2 and the tagent t_1 of (P_1) respectively, then the corresponding parameters of Q, Q' are $1-\frac{2}{3}I\,d\sigma^2$ and 1 respectively. Hence the double ratio of P, P', Q', Q is

$$\Delta \equiv (P\,P',\,Q'\,Q) = 1-\frac{3}{2}I\,d\sigma^2.$$

Therefore we have the following theorem:

Theorem III. *Let P' be a point on C so near the point P that on the osculating plane but not on the tangent of C at P. If the line PP' intersects the line P_1P_2 and the tangent t_1 of (P_1) at Q and Q' respectively, and if Δ denote the double ratio of the four points P, P', Q', Q, then*

$$I\,d\sigma^2 = \frac{2}{3}(1 - \Delta). \tag{III}$$

National University of Chekiang.

(Received 8 March, 1939).