

# Intuitionistic quantum logic of an n-level system

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## Abstract

A decade ago, Isham and Butterfield proposed a topos theoretic approach to quantum mechanics, which meanwhile has been extended by Doering and Isham so as to provide a new mathematical foundation for all of physics. Last year, three of the present authors redeveloped and refined these ideas by combining the  $C^*$ -algebraic approach to quantum theory with the so-called internal language of topos theory (see arXiv:0709.4364). The goal of the present paper is to illustrate our abstract setup through the concrete example of the  $C^*$ -algebra  $M_n(\mathbb{C})$  of complex  $n \times n$  matrices. This leads to an explicit expression for the pointfree quantum phase space and the associated logical structure and Gelfand transform of an n-level system. We also determine the pertinent non-probabilistic state-proposition pairing (or valuation) and give a very natural topos-theoretic reformulation of the Kochen-Specker Theorem.

In our approach, the nondistributive lattice  $P(M_n(\mathbb{C}))$  of projections in  $M_n(\mathbb{C})$  (which forms the basis of the traditional quantum logic of Birkhoff and von Neumann) is replaced by a specific distributive lattice of functions from the poset of all unital commutative  $C^*$ -subalgebras of  $M_n(\mathbb{C})$  to  $P(M_n(\mathbb{C}))$ . The latter lattice is essentially the (pointfree) topology of the quantum phase space mentioned above, and as such defines a Heyting algebra. Each element of the lattice corresponds to a "Bohrified" proposition, in the sense that to each classical context it associates a yes-no question pertinent to this context, rather than being a single projection as in standard quantum logic. Distributivity is recovered at the expense of the law of the excluded middle (Tertium Non Datur), whose demise is in our opinion to be welcomed, not just in intuitionistic logic in the spirit of Brouwer, but also in quantum logic in the spirit of von Neumann.

**Keywords:** quantum logic; topos theory; intuitionistic logic

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