<u>The Problem of Transition Across Levels in the van Hiele Theory of Geometric Reasoning</u>

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The most outstanding characteristic of the van Hiele theory of geometric reasoning (Hoffer, 1983; Perdikaris, 1994; van Hiele,1986) is perhaps the relationship between the levels of reasoning, a hierarchical sequence of levels of cognitive development, and the phases of learning, a cyclical sequence of stages of learning within levels.

The levels of reasoning constitute a description of the ways of student reasoning in Euclidean geometry. Students can progress through five levels of increasing structural complexity hierarchically. A higher level contains all knowledge of any lower level and some additional knowledge that is not explicit at the lower levels. Each level appears as a metatheory of the previous one (Freudenthal, 1973).

The phases of learning constitute a prescription for organising learning that helps students to pass from the current level to the next one. The phases are: inquiry, directed orientation, expliciting, free orientation and integration. According to van Hiele (1986), this functor (Hoffer, 1983) is the transition mechanism that prompts the transition to the next level.

The transition mechanism proposed by van Hiele does not involve any epistemologically sound prescriptive procedures for managing the uncertainty available. Besides, it does not imply any measure of uncertainty that will operationalise prescriptive procedures so that they will be useful in praxis.

During the van Hiele process, students use linguistic terms and concepts that are inherently ambiguous and hence their geometric reasoning constitutes a source of conflict. It is this conflict that plays a central role in the transition. Since conflict is a type of uncertainty, it is ultimately connected with information, that is, any reduction of conflict is an equal gain of information (Klir, 1995).

Two epistemologically sound prescriptive procedures, the principle of maximum uncertainty and the principle of minimum uncertainty (Klir, 1995), are intuitively used, as useful principles of wisdom, to manage the conflict involved in the integration phase at any van Hiele level. Thus, an appropriate generalisation, that produces the transition, can be chosen. Then a measure of possibilistic uncertainty is used to operationalise the principles of uncertainty by calculating the conflict of each of three student groups, with the same mathematical background, in the work of Gutierrez, Jaime and Fortuny (1991). These values of conflict are used to compare the geometric information of the student groups and thus explain why they had acquired different van Hiele levels in a geometric task.

Managing the conflict

Student reasoning, in the integration phase at a van Hiele level, must employ all available information of the level but no additional information (Hoffer, l983; van Hiele, 1986) to call into being inferences. That is, the inferences should be based on all relevant information contained in the available evidence. These inferences imply conclusions for formulating an appropriate generalisation that produces the transition to the next level.

However, students may use ampliative reasoning that involves ampliative inferences whose content is beyond the available evidence and hence conclusions not entailed in the given premises. Any information not supported by evidence is unwarranted on epistemological and pedagogical grounds and must be avoided. The proper way of dealing with this is to use the principle of maximum uncertainty. This principle requires that conclusions resulting from any ampliative inference maximise the relevant conflict (or minimise the relevant information) subject to constraints expressed by the premises. Thus, an appropriate generalisation is chosen (or constructed) whose amount of information does not exceed the amount of information in the level of functioning and the transition to the next level is attained.

The principle of maximum uncertainty may be violated and this will invariably lead to conflict in the conclusions, that is, contradictions either between the data and the conclusions or between different possible conclusions. These conclusions may produce a generalisation whose amount of information will exceed the amount of information in the level of functioning, that is, an overgeneralisation.

The appearance of conflict in the conclusions requires that the conclusions be appropriately adjusted so that the resulting generalisation is free of conflict. It is likely that some information, contained in the conclusions, is lost by these adjustments. This is undesirable and must be avoided. Hence, the loss of information should be minimised. The principle of minimum uncertainty is used to facilitate the selection of only those adjustments for which the total loss of information (or total gain of conflict) is minimal. The principle quarantees that conflict resolution is achieved with minimum information loss. Thus, a generalisation is chosen (or constructed) that is free of conflict and the transition to the higher level is attained.

Measuring the conflict

Gutierrez et al. (1991) evaluated the acquisition of the van Hiele levels in a geometric task. Their sample consisted of 50 students in groups A, B and C with 20, 21 and 9 students respectively. The authors summarised the various degrees of acquisition of the levels and tabulated the number of students attaining degrees of acquisition of each level in Table 1.

Table 1

Number of students attaining degrees of acquisition of each van Hiele level (Adopted from Gutierrez et al. (1991))

Group	van Hiele level	Degree of acquisition No	Low	Intermediate	High	Complete
A	I	0	0	0	0	20
A	II	I	0	3	6	10
A	III	2	3	6	6	3
B	I	0	0	1	2	18
B	II	0	3	4	13	1
B	III	9	6	5	1	0
C	I	0	2	4	2	1
C	II	3	4	2	0	0
C	III	9	0	0	0	0

The attributes, acquisition of Level 1, acquisition of Level 2 and acquisition of Level 3 are observed. These attributes can be expressed in terms of the variables v_1, v_2 and v_3, respectively. Each variable has five states that may represent fuzzy sets (Klir & Folger, 1988; Perdikaris, 1996b) whose linguistic labels, no, low, intermediate, high and complete, can be represented by a, b, c, d and e respectively. It should be noted that only the first three van Hiele levels are considered since the higher levels rarely appear in secondary classrooms. Membership degrees of fuzzy sets (Klir & Folger, 1988) corresponding to these linguistic labels may be found by the following method of mathematical psychology (Klir & Folger, 1988; Perdikaris, 1996a).

The fuzzy data for the three variables, each with five states (values, labels) is shown in Table 2. Observations are distinguished by the student groups A, B and C. Each observation consists of three 5-tuples of membership degrees, one 5-tuple for each variable and one membership degree for each fuzzy set. For group A, for example, the membership degree of fuzzy set d of variable v_2 is 6/20. This means that 6 of the 20 students had high acquisition of van Hiele Level 2.

Table 2

Fuzzy data of three variables each with five states

			A	**B**	**C**
$v_1=$	$\{$	a	0	0	0
		b	0	0	2/9
		c	0	1/21	4/9
		d	0	2/21	2/9
		e	1	18/21	1/9
$v_2=$	$\{$	a	1/20	0	3/9
		b	0	3/21	4/9
		c	3/20	4/21	4/9
		d	6/20	13/21	0
		e	10/20	1/21	0
$v_3=$	$\{$	a	2/20	9/21	1
		b	3/20	6/21	0
		c	6/20	5/21	0
		d	6/20	1/21	0
		e	3/20	0	0

Research (Gutierrez et al., 1991) has shown that students used several levels of reasoning at the same time during a geometric task. Thus, an overall system is more appropriate in the treatment of fuzzy data. It is observed that the sample involves relatively few data and nonrandom and unpredicted variation in behaviour. This implies that the conflict involved in the van Hiele process is possibilistic in nature and consequently possibility distributions (Klir & Folger, 1988) can be used in this analysis. This is reinforced by Shackle (1961,1979) who argues that human reasoning can be formalised more adequately by possibility theory than probability theory. The possibility distribution estimates of the overall system, for each student group, are found using Table 2 and shown in Table 3.

Consider, for example, the overall state (e d c) in Table 3. This overall state indicates that students have achieved complete acquisition of level 1, high acquisition of Level 2 and intermediate acquisition of Level 3 in the geometric task. The membership degree of (e d c), for student group A, is

m_A(e d c)=(1)(6/20)(6/20)

=0.090

where 1 is the membership degree of fuzzy set e: complete acquisition of Level 1, 6/20 is the membership degreee of fuzzy set d: high acquisition of Level 2 and 6/20 is the membership degree of fuzzy set c: intermediate acquisition of Level 3.

Table 3

Possibility distribution estimates derived from *fuzzy* data

$v_1v_2v_3$	m_A	r_A	m_B	r_B	m_C	r_C
s= e e e	0.075	0.500	0	0	0	0
e e a	0.050	0.333	0.017	0.075	0	0
e e b	0.075	0.500	0.012	0.053	0	0
e e c	0.150	1	0.010	0.044	0	0
e e d	0.150	1	0.002	0.009	0	0
e d a	0.030	0.200	0.227	1	0	0
e d b	0.045	0.300	0.150	0.660	0	0
e d c	0.090	0.600	0.126	0.555	0	0
e a a	0.005	0.033	0	0	0.040	0.200
e b a	0	0	0.052	0.229	0.050	0.250
e c a	0.015	0.100	0.070	0.308	0.025	0.125
e c b	0.023	0.153	0.047	0.207	0	0
e b b	0	0	0.035	0.154	0	0
c a a	0	0	0	0	0.150	0.750
c b a	0	0	0.003	0.011	0.200	1
d a a	0	0	0	0	0.074	0.370
d b a	0	0	0.006	0.026	0.100	0.500
b a a	0	0	0	0	0.074	0.370
b b a	0	0	0	0	0.100	0.500
c c a	0	0	0.004	0.018	0.100	0.500
e c c	0.045	0.300	0.038	0.167	0	0
e c d	0.045	0.300	0.008	0.035	0	0
e d d	0.090	0.600	0.025	0.110	0	0
e d e	0.045	0.300	0	0	0	0

d d a	0	0	0.025	0.110	0	0
e b c	0	0	0.029	0.128	0	0
e a b	0.008	0.053	0	0	0	0
e c e	0.023	0.153	0	0	0	0
e a c	0.015	0.100	0	0	0	0
e a d	0.015	0.100	0	0	0	0
e a e	0.008	0.053	0	0	0	0
d b b	0	0	0.004	0.018	0	0
d b c	0	0	0.003	0.013	0	0
d c a	0	0	0.008	0.035	0.049	0.245
d e b	0	0	0.005	0.022	0	0
d e c	0	0	0.004	0.018	0	0
d d b	0	0	0.017	0.075	0	0
d d c	0	0	0.014	0.062	0	0
d d d	0	0	0.003	0.013	0	0
d e a	0	0	0.002	0.009	0	0
d e b	0	0	0.001	0.004	0	0
d e c	0	0	0.001	0.004	0	0
b c a	0	0	0	0	0.050	0.250
e b b	0	0	0.001	0.013	0	0
c b c	0	0	0.002	0.009	0	0
c c b	0	0	0.003	0.011	0	0
c c c	0	0	0.002	0.009	0	0
c d a	0	0	0.011	0.057	0	0
c d b	0	0	0.008	0.035	0	0
c d c	0	0	0.007	0.031	0	0
c d d	0	0	0.001	0.004	0	0

The possibility of (e d c), for student group A, is

R_A(e d c)=0.090/0.150

=0.600

where 0.090 is the membership degree of (e d c) and 0.150 is the maximum membership degree of the overall states of group A. The membership degrees and the corresponding possibilities of (e d c), for student groups B and C, have been found in a similar manner.

The values of student group conflict can be found by using the strife function (Klir & Wierman, 1998)

$$ST(r) = \sum_{i=2}^{n} (r_i - r_{i+1}) \log_2 \frac{i}{\sum_{j=1}^{i} r_j}$$

on the ordered possibility distribution of each student group, $1=r_1 \geq r_2 \geq \ldots \geq r_n$

The ordered possibility distribution for student group A, in Table 3, is

r_A = (1, 1, 0.600, 0.600, 0.500, 0.500, 0.333, 0.300, 0.300, 0.300, 0.200, 0.153, 0.153, 0.100, 0.100, 0.100, 0.053, 0.053, 0.033, 0, ... , 0) where $r_i \geq r_{i+1}$. Then the strife function gives $ST(r_A)=0.490$.

Thus, the conflict of student group A is 0.490 and, in a similar manner, the conflicts of student groups B and C are 0.550 and 0.570 respectively.

Conclusions

Student group A has the least conflict of all student groups and hence the most geometric information. This justifies why student group A had acquired higher van Hiele levels than groups B and C in the experimental results (Gutierrez et al.,1991) shown in Table 1. The same can be said for group B in relation to group C.

This work is the first scientifically based exegesis of the mechanism that prompts the transition from a lower to a higher level in the van Hiele theory of geometric reasoning. It improves van Hiele's transition mechanism by using results from the uncertainty -based information theory (Klir and Wierman, 1998) to control and measure the conflict involved during the transition to higher levels.

The procedure in this paper will improve arguments about transition mechanisms in other models of development, for exampie, the SOLO taxonomy (Biggs and Collis, 1982), the theory of cognitive development (Case, 1980) and the classical Piagetian theory. Besides, future research results, which may appear from applications of the strife measure on developmental models, will probably establish this measure as a viable measure in developmental psychology and education.

It is possible that motivation, cognitive abilities and other variables play a role in the transition to a higher level. However, considering the state of existing knowledge, the best that can be said is that these variables remain fixed in any particular situation or should be considered as a useful background against which the answers to the question of transition are observed.

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