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An Analysis of Thought Processes during Simplification of an Algebraic Expression

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Introduction

The purpose of this research is to try to determine, through individual interview sessions, the reasoning behind the adoption of certain strategies by several pupils as they are engaged in simplifying a complex algebraic term.

The simplification of expressions was chosen as an area of study for two reasons. First, this researcher has witnessed many instances of errors in the cancelling of terms. Second, he has found much less literature on the simplification of complex algebraic terms than on the solution of equations.
Mathematically, complex algebraic terms differ from equations in many ways. But they also differ in several psychological ways as well; perhaps even at the metacognitive level of thought associated with choosing a strategy. For instance, equations have a solution as a goal. When the line beginning "variable =" is reached even the beginning algebra pupil will know that this signifies the end of the problem. In other words, there is a definite indication that the end is reached, albeit produced by the pupil himself. On the other hand, expressions may have no such a trigger to signal an ending. Indeed, depending on the particular expression, there may be several mathematically satisfactory places to stop the simplifying process e.g. is it more useful to leave a term in factorised or unfactorised form. As will be seen from the interview transcripts in this investigation, there is significant thought by several pupils on devising a plan and evaluating the answer, albeit with mal-rules of algebra. Hence pupils' metacognition may come into play while determining the number of steps in the problem (i.e. planning and deciding when to stop) when following the instruction "simplify".

One of the main aims of this investigation will be to show that some pupils overshoot an acceptable correct answer (and get the final answer wrong) in an attempt to simplify. A psychological effect, namely fear of lack of closure, may be experienced by pupils as they attempt to simplify expressions: pupils may be unwilling to stop before reaching an answer with which they are comfortable e.g. a numerical answer. In other words, if there are few clues as to when an acceptable answer is reached, the pupil may continue simplifying until she arrives at one acceptable to her. At the point of oversimplification, certain errors may be identified which might have remained undetected had the pupil simply stopped earlier. Such errors will be analysed with reference to the cognitive and metacognitive domains.
Research into pupils' thoughts while simplifying expressions gains in significance when it is appreciated that algebraic simplification can be viewed in several ways: as being a skill in its own right and as a skill useful in the solution of equations.

Further, pupils' thoughts on the subject can provide a window into other areas such as how variables are perceived and the level of meaning algebraic expressions may have for pupils. Indeed, according to recent research by Tirosh et. al. (1998), there has been evidence of success in making teachers knowledgeable about research findings regarding specific student conceptions and developing new ways to teach that take such knowledge into account.

Review of Literature

Hiebert & Carpenter (1992) have pointed out that error analysis in algebra is of great value in influencing instruction in a positive way. However, it may be worth noting that researchers such as Bell-Gredler (1986), Kieran (1989), and Ernest & Bayliss (1995) are well aware of the possibility that too much can sometimes be inferred by the teacher from an analysis of errors. Thus, one of the main objectives of this author's research design was to try to minimise such inference by asking the pupil to explain each step of her reasoning in an interview.

The mathematical topic "simplifying an algebraic fraction" was used to probe pupils' thinking. In this topic, the process of cancelling may be done in the same way as it is in arithmetic. However, confusion in going from arithmetic to algebra may arise at these stages. Demby (1997) reports that the traditional emphasis in the curriculum on 'finding the answer' allows learners to get by with informal and intuitive procedures in arithmetic, but that in algebra they are required to recognize the structure that they have been able to avoid in arithmetic. Matz (1982) argues that, for instance, as 3¾ is to be interpreted as 3 + 3/4 it is not unreasonable that the student should interpret the algebraic expression, 3x as 3 + x. Thus there may be room for confusion and misinterpretation in the initial stages of simplifying an expression.

Bloom's (1956) taxonomy of the cognitive domain identified different levels of intellectual functioning. If one of the goals of mathematics education is to foster higher intellectual functioning, it may be interesting to note here that none of the pupils interviewed suggested a method for checking the algebraic simplification i.e. performing at the highest cognitive level of evaluation, as identified by Bloom (1956). This may be in alignment with findings from Poland on transforming algebraic expressions which is, according to Demby (1997, p46), concerned with 'degenerate formalism' characterised by thoughtless, 'slapdash' manipulation of symbols. It may be more difficult for pupils to operate at the higher levels of Bloom's hierarchy within the context of simplification of an algebraic expression than, say, solving a linear equation. One reason for such difficulty may be explained by Tirosh et. al. (1998), who describe the dual nature of mathematical notations: process and object e.g. 5x + 8 can be viewed as the process 'add five times x and eight' or as an object in its own right. In other words, we may be expecting pupils to finish simplifying (i.e. end a question) at a stage where there may exist this process/object dichotomy. Obviously, it will be difficult to demonstrate this by relying on pupils' articulation of their difficulties at this stage near the end of the problem. However, the researcher will attempt to encourage further reflection on the thought process used by pupils, in an effort to examine this possibility.

Fernandez et. al. (1993) characterise metacognition as the use of a set of managerial processes during problem solving. In the interviews, several pupils expressed dissatisfaction with certain answers. Given that they were not blindly accepting the logical conclusions of their work, it could be argued that they are operating at a metacognitive level. As will be shown, the precise motives for their thoughts at the metacognitive level stem largely from algebraic misconceptions and mal-rules.
One of the most important precursors to this study is that of Erlwanger (1973), in which a pupil successfully completed a self-paced course by applying 'mal-rules' which happened to lead to a preponderance of correct final answers. Mal-rules are rules, perhaps invented by the pupil, which appear effective but in fact work only under certain conditions. Several pieces of literature inform us that just because a correct answer is given, it cannot be assumed that it is based on the desired understanding. As Steinberg et al. (1990) observed in their study, many students who are able to solve equations and perform other algebraic tasks correctly may have misconceptions or lack good understanding of algebra concepts.

The literature points to many complex psychological processes involved in gaining an understanding (and avoiding a misunderstanding) of the rules of algebra, and being able to operate correctly in accordance with them. For example, Kieran (1992) reports that only a very small percentage of 13- to 15- year old pupils is able to consider the letter as a generalised number. Also, Küchemann (1978, 1981) identified six levels of interpretation of letters:
a) Letter evaluated: The letter is assigned a numerical value from the outset;
b) Letter not considered: The letter is ignored or its existence is acknowledged without giving it a meaning;
c) Letter considered as a concrete object: the letter is regarded as a shorthand for a concrete object or as a concrete object in its own right;
d) Letter considered as a specific unknown: The letter is regarded as a specific but unknown number;
e) Letter considered as a generalized number: The letter is seen as representing, or at least as being able to take on, several values rather than just one;
f) Letter considered as a variable: The letter is seen as representing a range of unspecified values and a systematic relationship is seen to exist between two such sets of values.

This understanding is important in the process of the simplification of algebraic fractions, in that both the question and the answer can involve variables. Further, it will be seen that this understanding of a letter as a generalised number has implications in the checking of work, especially as relatively few 13-15 year olds in the study appeared to reach (e) above. Also, a big difference exists between checking (i.e. performing at the highest level of Bloom's taxonomy) the end result of an equation and an expression. This arises out of the fact that, if the solution of an equation is x = 6, then 6 is to be substituted for x in the original. Following Kieran's (1992) and Küchemann's (1978, 1981) view that many pupils have difficulty viewing a letter as a generalised number, does it follow that most pupils will not realise that they may check their answer to a simplification by substituting almost any number? Checking a single specified number (as in an equation) could be viewed as easier than checking any/every number (as in an expression).

That this link (of non-checking because of non-specificity of the 'answer' to a simplification) is not made in the literature (as far as this researcher can determine) may be to do with the fact that it may be difficult to prove, or simply that the task of inserting a suitable number for checking purposes is itself a difficult one e.g. whereas most numbers, when substituted, will suffice to show any simplification of
x2 - 9
x2
to be impossible, the numbers 3, -3 and 0 will present problems to the pupil if used in such a check. However, there appears to be no

mention in the literature of a conjecture to this effect: namely that one factor contributing to why simplifying expressions may be found by pupils to be more difficult than solving equations may be that checking a generalised number is more difficult than checking a specified number, and is therefore done less frequently and/or with less success. This conjecture springs from two seemingly disparate strands of literature: namely Bloom (1956) and Kieran (1992) / Küchemann (1978, 1981), and is supported, albeit in a small way in this experiment, by the fact that none of the pupils interviewed mentioned checking the finished simplification, or even alluded to the fact that the checking of a simplification is possible.

Pupils frequently attempt to "solve" expressions, i.e. according to Wagner et. al. (1984), many algebra students tried to add " = 0 " to expressions they were asked to simplify". Again, one explanation may lie in the unwillingness of pupils to accept 'lack of closure' as suggested by Hoyles and Sutherland (1992):

Previous studies have found that many pupils cannot accept that an unclosed algebraic expression is an algebraic object. So, for example, pupils are unable to accept that an expression of the form x + 3 could possibly be the solution of a problem.

e.g. 2a + a + 3
= 3a + 3 = 0
= 3a = -3
= a = -1

and x2 + 5x + 6
= (x + 3)(x + 2) = 0
x = -3 or -2

This kind of error may indicate an absence of knowledge of the difference in meaning of an expression and an equation. Such a "lack of closure" experienced by pupils may be found to be a contributing factor to the production of errors, or at least a misunderstanding of the very objective of trying to simplify an expression. As Tirosh et. al. (1998) point out, students frequently face cognitive difficulty in accepting lack of closure: they conceive open expressions as incomplete and tend to 'finish' them, mimicking the final number answer found in arithmetic.

Methodology and Results

The main focus of the investigation was to try to determine, by individual interviews, why certain errors are made during the simplification of a complex algebraic fraction.

The particular problem: Simplify x2 + 3x - 10
x2 + 2x - 8

was given to all 180 secondary pupils from Years 9, 10 and 11 in a selective Bermuda school. No instruction, other than 'simplify', was given. Ten minutes was taken out of a normal mathematics lesson for the pupils to work individually on the problem.

This particular problem was chosen because the researcher had, over the course of twenty five years, noticed that pupils' approaches to cancelling was often, to quote Demby (1997, p46), "thoughtless and slapdash". Further, it was thought to be a suitable problem to expose weaknesses in cancelling because of the many points during the problem where "slapdash" or mal-rule cancelling might be possible. A certain proportion of the pupils cancelled (incorrectly) on the first line, but, although interesting, this was not the focus of the experiment. The pupils who were selected for the follow-up interview were those who did not cancel incorrectly throughout the problem, arrived at the correctly simplified expression

x + 5
x + 4
then proceeded to cancel (incorrectly) the x's.
To reiterate, it is beyond the scope of this report to identify and categorise all the specific errors made, but out of the 180 pupils' work:
70 accomplished few correct steps,
44 simply cancelled the original numerator and denominator x2, and stopped.
41 found (and stopped at) the correct, i.e. simplified, answer of x + 5 ,
x + 4

20 factorised correctly, but left the final answer as (x + 5) (x - 2) , but
(x + 4) (x - 2)
5 pupils overshot the correct answer and then cancelled the x's at the end, having successfully not cancelled at the beginning. It is these 5 pupils who were interviewed as the main focus of the experiment.

In the 5 individual interviews the next day, the researcher presented the pupils with their answer sheets, lead the pupils through their simplification of the original expression, and gradually elicited from them any thought processes occurring at each step. This was done by using a series of standard questions (listed later). The main thrust of these interviews was to determine why the pupils had resisted the temptation to cancel near the beginning of the problem, yet yielded to it later. This begs two important questions. First, is there a trigger for the use of the cancelling mal-rule and, second, if there is no trigger (i.e. the mal-rule is being applied at random), why does the pupil feel that rules of algebra may be applied at random?

For the answers to these questions, as given by the pupils in this experiment, it is necessary to look at excerpts of the interview transcripts.
Each of the five interviewees wrote
x2 + 3x - 10
x2 + 2x - 8

= (x + 5) (x - 2)
(x + 4) (x - 2)

= x + 5
x + 4

= 5
4

Each of the five interviewees was asked the same sequence of questions while looking at their own work from the previous day.
1) Why did you factorise initially?
2) Why did you cancel the (x-2)'s ?
3) Why did you cancel the x's ?
4) If you felt it was sensible to cancel the x's, why not cancel the x2 at the beginning?
5) Do you now want to cancel the x2 at the beginning?
6) Free discussion based on responses.

The interviewer then gave a short individualised lesson on cancelling to each pupil to try to clear up any misunderstandings surrounding cancelling which may have been generated by the interview.
For purposes of clarity and anonymity, I shall call the five interviewees Allan, Bert, Cecil, Daphne, and Erica. The interviewer's part of the dialogue is verbatim. The interviewees' responses and comments will be in their own words as far as possible, but edited when verbatim transcription would have lacked clarity.

Allan's interview
x2 + 3x - 10 = (x + 5) (x - 2) = x + 5 = 5
x2 + 2x - 8 (x + 4) (x - 2) x + 4 4

1) Why did you factorise initially? I factorise when I see x-squareds.
2) Why did you cancel the (x-2)'s ? When two things are the same, you cancel.
3) Why did you cancel the x's ? Because they are two the same.
4) If you felt it was sensible to cancel the x's, why not cancel the x2 at the beginning? I was taught to factorise each equation, then cancel.
5) Do you now want to cancel the x2 at the beginning? Yes, it's a good idea. Yes.
6) Interviewer - If you are allowed to cancel the x's in x + 5 then why did you x + 4 not cancel the x-squareds at the beginning?

Allan - Because I wouldn't know what to do at the next step. I would be left with
3x-10 etc. It's just easier to factorise and cancel twice then I know what 5 over 4 means.

Interviewer - So you are saying you know what to do with 5 over 4 at the end, and that is why you cancelled to get 5 over 4. Are you still sure you are allowed to cancel the x's at the end?
Allan - Yes

Interviewer - Did we do that kind of cancelling anywhere else in the question?
Allan - No.

Interviewer - Are you allowed to cancel the x's?
Allan - No, because it is only part of the bracket (x + 5).

The interviewer then explains to Allan about when cancelling is appropriate, and this concludes the interview.

Several observations can be made about Allan's approach to this simplification question.
His reply to Q1 shows that x-squared acts as a mechanical trigger for factorising. There appears to be little thought about the merits or rewards of factorising, but factorising is done merely because it can be done. There may be some value in this approach, in that it gets the pupil to the next line. However, Allan's lack of forward thinking and understanding is evident from this response.
His reply to Q2 shows an oversimplification of the cancelling process.
His reply to Q3 is a restatement of his oversimplified rule.
Q4 echoes Q1.

Now sure that his oversimplified rule appears to work, Allan changes his mind in accordance with his rule, and decides he may cancel the original x-squareds.
In view of this change of heart, the interviewer decides to find out why Allan did not cancel the original x-squareds. It is at this point that it becomes clear that Allan did indeed contemplate cancelling the x-squareds, but did not do so because he would be left with 3x - 10 over 2x - 8, which he admits he did not know how to simplify. Part of the reason for this may be due to the dual nature of expressions such as 3x - 10, as described by Tirosh et. al. (1998). Allan elects to factorise and cancel correctly, but continues to cancel because the result is 5 over 4, with which he is comfortable i.e. he now has the closure suggested by Hoyles and Sutherland (1992) and Tirosh et. al. (1998).

Bert's interview
x2 + 3x - 10 = (x + 5) (x - 2) = x + 5 = 5
x2 + 2x - 8 (x + 4) (x - 2) x + 4 4

1) Why did you factorise initially? It's a quadratic equation, so you need to get it into brackets. The x-squared tells me to do this.
2) Why did you cancel the (x-2)'s ? Because they are the same.
3) Why did you cancel the x's ? Just trying to simplify, but I now think this is wrong because it is too simple - and my teacher has a poster on the wall which says simplify, but don't make it too simple.
4) If you felt it was sensible to cancel the x's, why not cancel the x2 at the beginning? I could get rid of more stuff by factorising than by cancelling.
5) Do you now want to cancel the x2 at the beginning? You could cancel the x-squareds at the beginning, but it would be harder to simplify.
Interviewer - So you now say you may cancel the x-squareds at the beginning.
Bert -Yes.

6) Interviewer - Do you think cancelling the x's at the end is different from cancelling the x-squareds at the beginning?
Bert - Yes

Interviewer - Why?
Bert - Because if you cancel the x's at the end you won't have enough x's to make the equation.

Several points are worth noting in this interview. First, Bert receives the same cue as Allan from the x-squared on the first line, and hence factorises, and continues with a similar type of reasoning as Allan.

Second, there is evidence in Bert's response to Q6 of the equation / expression confusion as documented by Wagner et. al. (1984):

his use of the word "equation" is wrong. Bert's response to Q5 shows him changing his mind about when it is possible to cancel, and in doing so, giving an example to his assertion in Q4. Bert final statement (that if you cancel, you won't have enough x's for the equation) goes against what he wrote initially. This change of mind is similar to Allan's in Q5, and may be indicative of Demby's (1997) 'degenerate formalism'. Bert displays good forward thinking in Q4, and he has chosen the correct route.

Cecil's interview

x2 + 3x - 10 = (x + 5) (x - 2) = x + 5 = 5
x2 + 2x - 8 (x + 4) (x - 2) x + 4 4

1) Why did you factorise initially? Because it's so long - if you factorise it gets shorter in the long run.
2) Why did you cancel the (x-2)'s ? It's killing two birds with one stone by doing the same to the top and bottom.
3) Why did you cancel the x's ? For the same reason.
4) If you felt it was sensible to cancel the x's, why not cancel the x2 at the beginning? If you cancel the x-squareds at the beginning you don't get to factorise.
5) Do you now want to cancel the x2 at the beginning? Yes, if it came out the same.
6) Interviewer - Do you now want to change anything?
Cecil - Yes, I don't want to cancel the x's at the end.

Interviewer - Why not?
Cecil - Because x is the number you put in later on to find the value.

Interviewer - Are you saying that because we start with x's, we should finish with x's?
Cecil - Yes.

Interviewer - Now you say you may not cancel the x's. So why did you cancel the x's when you did this on paper?
Cecil - I had already done some cancelling and I guess I got into a routine, especially when the problem said "simplify".

It is clear that Cecil has the variable confusion encountered before. This is evident when he states he is unwilling to lose the last two x's by cancelling because he will have nowhere to substitute the number. Also, since Cecil has said that he sees x as representing a particular number, he may be part of Kieran's (1992) large group who has difficulty viewing x as a generalised number. Cecil admits getting into a cancelling routine, and from his comment that the problem said to simplify, it would appear that the problem is not closed for him on the second last line. Such indiscriminate cancelling may be evidence that either he has no understanding of its meaning (he seems unaware of the difference between the operations of addition and multiplication as they relate to cancelling) or that he is caught up in the procedural aspect of the problem, not paying attention to the meaning of the expression.

Daphne's interview

x2 + 3x - 10 = (x + 5) (x - 2) = x + 5 = 5
x2 + 2x - 8 (x + 4) (x - 2) x + 4 4

1) Why did you factorise initially? I tried to cancel the x-squareds, but it didn't seem to work, because I couldn't get the right answer.
2) Why did you cancel the (x-2)'s ? and 3) Why did you cancel the x's ? I didn't. I cancelled the x's then the -2's then the x's. This was because the same things were on the top and bottom.
4) If you felt it was sensible to cancel the x's, why not cancel the x2 at the beginning? You can cancel the x's and the -2's and then the x's because they are the same, but you can't cancel the x-squareds because they might be different. You don't know how any times the x's are squared. Also, you don't want to cancel the x-squareds because then you can't factorise then cancel.
5) Do you now want to cancel the x2 at the beginning? No, because I came out with two different answers, and I know that five over four is the correct answer.
6) Interviewer - What if I tell you that five over four is the wrong answer?
Daphne - I would go back to the beginning and find out what x is, and build it up from there.

Interviewer - But you still would not cancel the x-squareds at the beginning?
Daphne - Well, I tried that but when I did all my cancelling on the first line I got a negative answer. I don't like negative answers, so I preferred my method which got a positive answer.

The first point is that Daphne did something different from the other four pupils in that she cancelled parts of a bracket in (x - 2). This is interesting because evidence of this mal-rule shows up in the interview, but not in the written work: though time-consuming, an interview may lead to the pinpointing an error undetected by analysis of an answer paper. Daphne also appears confused when she says she does not know how many times the x is squared, and because of this 'fact', says the x-squareds might be different and therefore cannot be cancelled.

Daphne, like Cecil, probably believes x to be a particular number when she says "find out what x is". This means she does not understand that x is a variable, probably identifying her (and perhaps Cecil) as operating at level (d) in Küchemann's classification.

Also interesting is the fact that Daphne is certain that five over four is the correct answer, and cites this as her reason for not cancelling the original x-squareds - because it yields a different answer. However, careful probing leads to the real reasoning: when she cancelled the original x-squareds and x's on the first line, she was left with -7 over -6. Her self-confessed aversion to negative numbers (and, by extension, a negative quantity divided by a negative quantity) made her retry the problem. Either she may have been operating at a metacognitive level in evaluating her answer in terms of her own sense of reasonableness, or it might appear that, again, there is a lack of closure, but this time with an answer of -7 over -6.

Erica's interview.
x2 + 3x - 10 = (x + 5) (x - 2) = x + 5 = 5
x2 + 2x - 8 (x + 4) (x - 2) x + 4 4

1) Why did you factorise initially? I remembered that factorising leads to cancelling.
2) Why did you cancel the (x-2)'s ? The (x - 2)'s are the same, and this would leave the positive numbers.
3) Why did you cancel the x's ? Again, the x's are the same, so they cancel.
4) If you felt it was sensible to cancel the x's, why not cancel the x2 at the beginning? It would have been right to cancel the x-squareds, but I couldn't have gone any further in the problem.
5) Do you now want to cancel the x2 at the beginning? No. I still can't go any further. If I cancel the x-squareds, I can't do any more, and I know the problem is longer from other problems I have done.
6) Free discussion - none.

The first point here is that the researcher missed a valuable opportunity to find out exactly why Erica was unwilling to stop after simply cancelling the x-squareds. It may have been because of what she said i.e. the context clue that these problems are usually longer, or it could have had something to do with the closure issue, in which non-numerical answers are viewed as incomplete. However, accepting Erica's reason at face value, it is interesting to observe that convenience, or inconvenience in this case, played a major part in her strategy of simplification of this expression. Convenience (and context) may override purely mathematical considerations. This is understandable given the weak grasp of the mathematical concepts behind cancelling displayed by Erica: maybe something must fill the void, and she has chosen the contextual clue of believing this type of problem to be of a certain length. Hence (correctly) she does not cancel at the beginning, but for contextual rather than mathematical reasons. Unfortunately for Erica, as she progresses through the simplification, the argument of excessive brevity weakens. This leads to almost indiscriminate cancelling. This might appear to be an example of Demby's (1997) 'degenerate formalism', and closer analysis through this interview may have unearthed a more subtle process involving a contextual clue. However, from Erica's verbal responses in the interview, the interviewer surmised that Erica's mal-rule for cancelling would distil to "cancel when the result is simple (or perceived to be easily simplified), otherwise find another method, always bearing in mind that the problem not be too short".

Lack of closure is normally characterised by, and is well-documented in the literature as, unwillingness to accept a variable as an answer or in an answer. The five pupils wrote that
x + 5 = 5
x + 4 4

Allan said he continued to cancel because he knew what 5 over 4 meant, and Erica said she had to make the problem longer from experience. Both pupils' remarks could be interpreted as those of pupils trying to achieve closure.

Conclusion
Perhaps the most obvious observation that can be made is that behind the five identical pieces of written work lay a variety of different patterns of reasoning.
The main similarity lay in the fact that the x-squared acted as a factorising cue. Unfortunately this was a cue followed with little understanding of the later effects.

All five recalled that they cancelled the x - 2's simply because they were "the same" i.e. the same factor. However, it is not to be expected that they would wish to elucidate on this by mentioning the precise rule or thought behind the rule. In a follow-up study, a direct question to the effect "What is the reasoning behind cancelling the x - 2's?" should be included i.e. question 2 is too vague. Naturally, almost the same answer was given by the pupils to question 3, which was again too open-ended to elicit an accurate description of their own cancelling rules.

As has been stressed before, question 4 was anticipated to yield the most revealing answers since the investigation was designed to focus on idiosyncratic methods and reasoning applied by pupils at this point. The consensus was that one simply did not cancel x-squareds at the beginning, albeit for a variety of reasons which had more to do with closure and ease of simplifying than with the meaning of the algebraic terms.

Part of the design of the original expression (in this case, the use of negative signs) was borrowed from the design of searching examination questions. This investigation was devised to probe thinking during cancelling. The use of negative signs may have, as in this case, unnecessarily confused the issue by increasing the number of opportunities to go wrong, but not in the area under investigation i.e. cancelling. In a follow-up study it would be important to give pupils access to such reasoning in the interview stage, and this might be helped by the removal of negative signs.

Great difficulty was found in classifying some thoughts: were they examples of functioning at the higher end of the cognitive domain or in the metacognitive domain? And where does the lack of closure, even at the rudimentary level, fit in? Answers might be found in a further study if it included more searching interview questions concerned with reasons for ending the simplification process where they did. It is acknowledged that such a study would have a slightly different focus from this present study.

Allowing again for the self-selection of the interviewees, there was direct corroboration of Kieran's (1992) and Küchemann's (1978, 1981) assertion that many pupils have difficulty in viewing a variable as a generalised number. This evidence lay in the fact that none of the pupils mentioned a means of checking by replacing x by any number in both the question and the answer to see if there was agreement. Although this was not the focus of the investigation, an additional question probing this generalised number effect might prove worthwhile in a follow-up study. Also in an expanded study, it would be interesting to ask the twenty who 'stopped' at the correct point in the problem why they did not proceed any further.

Certainly, the results of this investigation have shown that identical written answers may mask an array of disparate ideas.

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