The Algebra Gap Between GCSE and A Level

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Between 1975 and the introduction of the GCSE in 1986, both GCE O Level and CSE examinations were available to 16+ pupils. Together these examinations were designed for the top 60% of the ability range.

With GCSE examinations the expectation was that, as standards of attainment rose, the proportion of candidates obtaining graded results would rise. The examination boards, though were responsible for ensuring that the new GCSE grades A, B and C were linked to O Level grades A, B and C, and that GCSE grades D, E, F and G were linked to CSE grades 2, 3, 4 and 5.

The first GCSE grades were awarded in 1988. Revised courses began in 1992 to link with the National Curriculum, with some restrictions on coursework. The 1995 National Curriculum lead to other, minor changes in the examination, to be first tested in 1998.

The GCSE was introduced mainly in an attempt to make a single examination available to a wider group of 16 year olds. It replaced the O Level (available for the top 20% of the school population) and the CSE (available for a further 40%). It has continued to the present time, although it is now principally an examination for Key Stage 4 of the National Curriculum. In many subjects this may have made A Level more accessible, but in so doing the style of 16+ examinations has changed. In mathematics this change may have widened the gap between the 16+ and A Level examinations.

The language of A Level mathematics is largely algebra, so it would be appropriate to investigate the origins of the algebra content in GCSE for comparisons to be made.

Smith defines algebra as a "science of letters in place of numbers". The algebra content at GCSE has been determined by various factors. Mason indicates the difficulties in algebra as "uninteresting, meaningless and so difficult to remedy" and Kuchemann supports this argument and states that "algebra is often meaningless to children".

Lee and Wheeler have written about algebra needing to be placed into context and used to solve problems, but state that "many of the problems can be trivial and so the algebra used can lose its cutting edge".

Freudenthal indicates that other problems arose when modern maths was introduced, since letters were used both in algebra and also in set notation, causing confusion to pupils To worsen this confusion, letters were used both to indicate the members of a set and also the set itself.

Since the central role of algebra has been reduced, another difficulty is highlighted (and a reason for its devaluation in recent syllabuses), namely the loss of the link with arithmetic, as suggested by Booth and Hart. In the National Curriculum for instance, Hart states that "there is very little explicit mention of the structure of arithmetic and it is unlikely that readers will reconstruct it for themselves". Hart again states that "there is a delay in teaching algebra ... this topic is then underplayed".

The idea of algebra grafted on or added on in the syllabus is again mentioned by Hart. Examples here are the use of symbolic notation in levels 6 and 7 to generalise, "so the algebra is used to express an idea already formulated". In other words the pupils are working with numbers and merely adding a few symbols at a later point, rather than working with algebra throughout.

The same idea is further explained by Walkerdine.

"The crucial moment in understanding lies in the fusing of the signifier and the signified, so in the case of algebra implies that the language of algebra cannot be separated from the processes and added on as a final step".

Hart again talks of this separation of algebra from arithmetic in terms of finding a meaning for it and referring to the "suppression of symbolism" and "the search for referential meaning".

There is also a suspicion that algebra has been added on in the Using and Applying Mathematics attainment target (Ma1) as a means to get a higher mark, perhaps by generalising a result in terms of algebra. In Wolf's view this is both dishonest and muddled and again reinforces the view that in the National Curriculum, algebra has to some extent been added onto the syllabus.

There is a further concern about the content of algebra in the National Curriculum, expressed by Hewitt:

"Algebra is not what we write on paper but it is something that goes on inside us. So as a teacher I realise that notation is only a way of representing algebra, not algebra itself".

For instance at Level 4: "generalise, mainly in words, patters which arise in various situations".

Also at Level 7: "use symbolic notation to express the rules of sequences (mainly linear and quadratic)".

Also at Level 3: "deal with inputs and outputs from simple function machines".

Again at Level 5: "express a simple function symbolically".

In each of these cases the concern is that algebra is being used merely as a symbolic notation to express or generalise an idea. The real language of algebra and the working in algebra is largely missing.

The main influences of the National Curriculum on the algebra content in mathematics are summarised by Hart as follows. First was the suppression of symbolism. The way in which algebra was taught before the advert of the National Curriculum and modern maths was very much as a topic in its own right, as a powerful symbolic system, a language. The advent of modern maths meant some ambiguities arose and there was also a move away from the authoritarian approach, with some dilution of algebra.

Second was the use of algebra largely as a means of expressing generality rather than for symbolism and in an attempt to make algebra more accessible.

Have these changes in algebra content in 16+ examinations resulted in a wider gap between GCSE and A Level? This is a question I have investigated by looking in detail at some recent examination questions.

In my investigation a group of A Level mathematics students were asked questions about the perceived difficulty of selected GCSE 'algebra' questions and also A Level questions. The papers selected were two papers from the higher tier GCSE and a Pure Mathematics One Module paper, all these were selected from the same examination board, ULEAC. Students normally take the Pure Mathematics One paper during the first year of the sixth form, so it often would be the first exam taken after GCSE.

For both the GCSE and A Level papers, the students were asked to indicate the difficulties of the questions on a 5 point scale: A (very hard) to E (very easy). Of the responses to the 19 GCSE questions on the two papers, only 11% were rated very difficult or difficult. These questions comprised graph drawing or sketching (8%), rearranging formulae (2%), solving quadratic equations (0.5%) and finding the nth term of a sequence (0.5%).

On the other hand, the responses rating the GCSE questions either easy or very easy were 62% of total replies. These questions tested the following topics: simultaneous equations, inequalities, expanding brackets, factorising and solving quadratic expressions, and while these appear to be potentially difficult areas in algebra, it was generally thought that the particular questions set were less than demanding.

The same students replied to similar enquiries about A Level questions set on a pure mathematics paper. The very difficult or difficult questions, comprised 35% of all replies and were testing topics such as manipulation of algebraic expressions, integration, determining the range of a function, sketching graphs including inverse functions and determining the number of roots of an equation.

An analysis of these replies indicates that these students found relatively little difficulty with the algebra content of the GCSE questions but much more with the questions set at A Level. The main areas of difficulty at A Level, not surprisingly were with manipulation of expressions, a topic not given great emphasis at GCSE.

Integration was another area of A Level thought to be difficult and the reason given most often was again the algebraic manipulation involved.

There also appears to be a link between GCSE and A Level as far as curve sketching is concerned; it was considered the most difficult area of algebra at GCSE and remained one of the most difficult at A Level.

The same students were asked, in a further part of the questionnaire, to rate various topics at A Level (included in the Pure Mathematics one module syllabus and irrespective of whether they were tested on the particular paper under investigation). Again the 5 point scale was used. The topic rated the hardest was functions (by 78%), for both its algebraic manipulation and also the sketching element. Other topics included here were integration (60%), differentiation (46%) and sequences and series (31%). In these replies it is again apparent that areas of the syllabus with a good deal of algebraic manipulation and/or curve sketching are considered to be the most difficult at A Level, also with the A Level questions regarded as much more difficult than those at GCSE, where algebraic manipulation was given rather less emphasis than at A Level.

Further research has been carried out by SCAA, looking at standards over the period 1975 - 1995 in various subjects, including mathematics. In this research, examination papers were examined to determine the demands made of students both at 16+ and 18+. Also examined were the standards of attainment reached by the students. The research was designed to answer two main questions. First, were syllabuses and their assessment more or less demanding than in earlier years? Second, was the standard of performance required at a given grade equivalent to that of previous years? While these are not the questions central to this particular essay, the research carried out by SCAA does throw some further light as to why students may find a wide gap between the algebra of GCSE and A Level.

In the investigations by SCAA into 16+ examinations in mathematics various changes in the syllabus are indicated over the last few years (e.g. the introduction of coursework and the development of data handling) and a central one has been the reduction of algebra. It is also noted that the more difficult skills and concepts were sometimes under examined in recent years and also that even the highest attainers produced poor answers to questions involving algebra and proof.

Further suggestions are given as to why algebra has been downgraded in recent years in 16+ examinations. Following the publication of the Cockcroft report in 1982, it was suggested that the papers would enable candidates to demonstrate what they know rather than what they do not know! In order to try to teach and test topics used in everyday life, new areas of mathematics such as statistics were introduced. This of necessity resulted in other areas of the subject having a reduction in content, notably algebra.

GCSE mathematics was becoming more a subject meeting the needs of a variety of end-users, a minority wishing to take A Level and those needing GCSE merely as a qualification in itself, perhaps before taking other A Level subjects or before entry into a particular career.

It is also pointed out by SCAA that few questions on present GCSE papers demand the use of multi-step procedures, requiring candidates to select and apply various strategies to solve problems. This may well be a disadvantage for those students wishing to go on to take A Level mathematics, where multi-step problems are more prevalent.

In their conclusions, SCAA state that the range of skills has changed over the past 20 years. There has been a reduction in the emphasis of some skills e.g. numerical and algebraic manipulation. "The reduction in the depth of the study of algebraic manipulation has tended to reduce the opportunity for more able students to demonstrate high-order skills, with a possible effect on A Level performance".

It is worth looking in more depth at the GCSE algebraic questions asked in the papers upon which students had to make judgements in the questionnaire. On June 1994 Paper 5, the first question involved the following pair of simultaneous equations

$2x + 3y = 23$

$x - y = 4$

No particular method is suggested and indeed there are several, but it seems that it is quite possible for a student to arrive at the answer, from the second equation by a trial method or by inspection. If a correct answer is arrived at by this method, and since the solutions are whole numbers it is reasonably straightforward to do so, then it is presumed that full marks would be awarded. The following question, the solution of $2x \leq 10$, requires a very simple division by 2.

In the next question, the formula F = + 32 is given and the value of F is required when R = -20.

In the following question, a further formula, a = is given, and candidates are asked to write down numbers in order to estimate the value of the variable "a" without using a calculator. Then, in the second part of the question, this estimate has to be worked out. Both the latter two questions are very much arithmetical questions, although they are put into an algebraic context. In the next question the graphs of $y = x^3 - 10x$ was given, and candidates were asked to draw the graph of $y = 2x^2$ and hence solve the equation: $x^3 - 2x^2 - 10x = 0$ using the graphs. In the final part of the question, the same equation was to be solved using an iterative process. This was a question that candidates in the questionnaire rated difficult, essentially because of its graphical nature.

In the final question on this paper involving algebra, it was asked that the candidates could write down three inequalities, based on the information given, where x represented the number of children and y represented the number of adults, based on a problem involving people attending a cinema.

These inequalities then had to be represented on graph paper and the region indicated within which x and y must lie to satisfy the three inequalities. Further parts of the question requested candidates to find the maximum possible income from ticket sales, given the individual prices of adult and child tickets. This was another question which was rated difficult, again largely because of its graphical content.

The second GCSE paper (Paper 6, June 1994) had two questions requiring the nth term of a sequence. Both of these were not rated difficult by the majority. In the first of these the question was in context, connecting the number of posts and enclosures. The second question merely gave the first few terms of a sequence. There is a good deal of emphasis on this type of question in the National Curriculum, but it is not quite the algebra of previous years and seems to be the ‘added on’ type, rather like that found in some recent coursework tasks. The question remains whether those are really algebraic or more number based questions.

The expansion of $(2x +1) (x +4)$ was the first part of the next algebraic question. The second part of this question was to factorise completely $4x^2 - 6x$. This question appears rather isolated, preceded and followed by totally numerical questions, and again was not rated particularly difficult. In view of the algebraic manipulation which is required at some stages of A Level, it is a fairly straightforward question which could be answered well by many 15 year olds.

The formula $V = \frac{1}{2} LW (E+R)$ had to be rearranged in the next question, to make L the subject of the formula. In the next part, given also $A=2GL+2EL+W(E+R)$ and also $V=500$, $A=300$, $E=6$ and $G=4$, candidates had to show that $L^2 - 15L + 50=0$.

The questionnaire revealed that those asked did not rate this as particularly difficult, although my own feeling was that it did require a fair amount of algebraic manipulation. The fact that that the required formula was given, may perhaps have influenced the candidates responses.

The final part of this question required the solution of the quadratic equation $L^2-15L+50=0$, and no particular method was suggested, so both factorisation and using the quadratic formula were possible methods.

The final question on this GCSE paper involved the formula $T^3W^{-2} = K$, and required the candidates to substitute T=140 and W=8, to find the value of K. In the second part, the formula had to be rearranged into the form of $T=aW^n$, which did require a little algebraic manipulation, but the question is based rather more in arithmetic nevertheless.

To summarise these two GCSE papers, the algebraic questions which caused most problems were those involving graphical work and algebraic manipulation, although they were few quantity. Other questions involving sequences caused fewer problems however and are good examples of the newer type of National Curriculum questions.

In the P1 paper used in the questionnaire, among the most difficult questions was one involving the functions $f(x) = 3x -5$ and $g(x) = e^{-2x}$. Part (a) required the range of g, part (b) the sketches of the inverse functions and (c) required the number of roots of the equation $f^{-1}(x) = g^{-1}(x)$, following on from part (b). Finally, in part (d), it was required that candidates calculate fg ($^1/_3$). It was part (b) and (c) which were regarded as the most difficult and, rather like the GCSE questions, these involved graphical work. It is assumed here that the inverse graphs could be obtained by reflecting the basic graphs in the line $y = x$. Nevertheless students do find it hard to sketch graphs of these types of functions. Alternatively the inverse functions could be worked out algebraically and then their graphs drawn, again this was regarded as a difficult task.

Also rated as a difficult question was the following. In part (a) the function, f(x) =

was defined and it was asked that f(x) should be expressed in the form $A+Bx^{-½}+Cx^{-1}$, giving the values of A, B and C. This can be regarded as an algebraic manipulation question, requiring both the expansion of the brackets and also dividing each term by x . Part (b) required students to integrate f(x), using the result from Part (a). This too was regarded by the students as a difficult question. Finally in Part (c), the area under the curve between $x = 4$ and $x = 9$, was to be calculated. In this question the greatest stumbling block is the algebra, both the manipulation in Part (a) and the integration in Part (b). Part (c) was generally thought to be less difficult, involving mostly numerical calculations. Part (a) is also a good example of a multi-step problem with no clue as to the methods required, and this type of question is not usually associated with GCSE, which may well explain part of the relative difficulty of this particular question.

The relative difficulties of algebraic techniques have been analysed in some detail by Kuchemann on the research programme "Concepts in Secondary Mathematics and Science" based at Chelsea College, University of London, between 1974 and 1979. In this research a group of just under 1,000 children were given various questions based on different topics such as fractions, decimals and graphs. One of the sections was based on algebra. Kuchemann found that in the answers given by these children, they interpreted letters used in algebra in six different ways, in order of difficulty as follows.

i. the letter assigned to a numerical value from the outset;

ii. the letter ignored or at best acknowledged but without giving is a meaning;

iii. the letter regarded as a shorthand for an object in it's own right;

iv. the letter regarded as a specific but unknown number, which can be operated on

directly;

v. the letter representing or at least able to take several values;

vi. the letter representing a range of unspecified values.

The children's responses were then classified into four levels of understanding, numbered 1 to 4 in degrees of difficulty.

Level 1 responses are classified as purely numerical or with a simple structure.

Level 2 is of a greater complexity and indicates the beginning of an acceptance of answers which are incomplete or ambiguous.

At Level 3, children are able to use letters as specific unknowns, although only when the structure is simple.

At Level 4 specific unknowns are coped with and which have a complex structure. How do the GCSE papers analysed previously fit into Kuchemann's 4 levels of algebra?

On the GCSE Paper 5, the first question involved the solution of the simultaneous equations:

$2x + 3y = 23$

$x - y = 4$

This appears at face value to be at Level 4, but as previously stated, the answers may be obtained by mere inspection, especially from the latter equation and this may place the level lower on the scale.

In No 2(b) the inequality is $2x \leq 10$. I would suggest this cannot be placed higher than level 3, since there is a single unknown and a very straightforward method of solution.

In No. 4 the formula F= + 32, and the substitution R=-20 is required. This may well be placed no higher than Level 1, with a numerical method of solution.

In No. 7 the formula a= , has a similar type of substitution, although there is an element of estimation required too.

In No. 10(c), the iteration $x_{n+1}=(2x+10)^{½}$ is used, and I would suggest this would be at Level 3/4, using specific unknowns, although there are various stages in the solution.

In No 12 (a), three inequalities are required based on information given in the question. This again, I feel is at Level 3 of complexity with letters used as specific values.

In the final two algebraic questions on this paper, graphs are involved. First in No. 10(a) the graph of $y = 2x^2$ is to be drawn on a grid already containing the graph of $y = x^3 - 10x$.

In No. 10(b) the solution of the equation $x^3 - 2x^2 - 10x = 0$ is required from the intersection of the two graphs.

In No 12(b) straight lines have to be drawn and regions have to be shaded. To analyse the difficulty of these latter two questions requiring graphical work, it is appropriate to look at a more graphical rather than solely algebraic scale of assessment.

Kerslake has suggested three levels of competence for graphical ability. At Level 1, an ability to plot points and a recognition that a straight line represents a constant rate are among the indicators.

At Level 2, simple interpolation and a recognition of the connection between the rate of growth and the gradient are among the indicators.

Finally at Level 3, an understanding of the relation between the shape of a graph and its algebraic expression is required.

In the two GCSE questions, No 10(b) and No 12(b) as mentioned earlier, there is an algebraic expression involved in both cases, placing each question at Level 3 of the Kerslake scale, although the graph of $y = 2x^2$ in No 10(b) may well be drawn largely from memory.

Moving on to Paper 6 of the GCSE, No 2 required the nth term of a sequence. I would suggest this is at Level 3, on the algebraic (Kuchemann) scale, since a linear expression is required.

No. 10 also requires the nth term of a sequence and again the answer is linear and so again is at Level 3.

No. 13(a) asks for the expansion of the brackets $(2x+1)(x+4)$, which according to Kuchemann is Level 4.

No. 13(b) required factorisation of $4x^2 - 6x$, again at Level 4.

No. 17(a) has a given formula $V=½LW(E+R)$ and requires a rearrangement to make L the subject.

No. 17(b) has a further formula $A=2GL+2EL+W(E+R)$ and in conjunction with No. 17(a) has to be rearranged to give $L^2-15L+50 = 0$.

Both these parts of No. 17 would be of Level 4 difficulty, with part (b) especially hard since it requires several parts of rearranging with the various letters.

In No. 17(c), the equation $L^2-15L+50=0$ has to be solved, by any method, the two most likely of which are by factors or by using the quadratic formula. These both require the manipulation of a single unknown, so they are likely to be at Level 3.

In No. 19(a) a substitution of values into the equation $T^3 W^{-2}=K$ to find the constant K, was required; here a Level 3 I feel is appropriate.

In No. 19(b), a rearrangement of the above formula to give T in the form $T=aW^n$ is required and since two variable are involved, this would be at Level 4.

In summary, the two GCSE papers looked at in detail contain about 20% of the total marks on algebra, in line with the National Curriculum requirement. Of these algebraic questions, about half are at a Level below 4 on Kuchemann's scale, and the other half at Level 4. Some difficulty in the questions is quite apparent, but the algebraic manipulation required does not reach a consistently high standard throughout the two papers. However the algebraic graph-type questions, which do appear to cause difficulty at GCSE, are placed consistently at the highest of the three levels of Kerslake's scale.

Bearing in mind the algebraic rigour of much of the mathematics required at A Level, there does appear to be both a lack of depth and a scarcity of algebraic manipulation at GCSE reflected in these two papers. There does also appear to be a good deal of misconception regarding algebraic graphical work which continues into A Level.

There is therefore an apparent gap in the algebra content between GCSE and A Level, due in some part to syllabus changes at 16+ over the past few years. The level of algebraic manipulation required at GCSE is not of a consistently high standard when compared with A Level and fails to prepare students fully for the rigours of A Level.

These are observations are made by SCAA:-

"Compared with 1975.... the contexts of questions were mostly familiar and the algebraic manipulation required was of a more basic level than in earlier years".

There does need to be an urgent review of the algebra content in 16+ syllabuses, as suggested by SCAA: "to ensure that in the new GCSE syllabuses the required increased emphasis given to the skills of algebraic manipulation is sufficient".

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