IS MATHEMATICS DISCOVERED OR INVENTED?

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Recently a heated debate between realists and relativists in science has erupted. The conflict is between those who see science as a rational description of the world converging on the truth, and those who argue that it is a socially constructed account of the world, and just one of many possible accounts. Typically scientists and philosophers of science are realists, arguing that science is approaching a true and accurate description of the real world, whereas social and cultural theorists support a relativist view of science, and argue that all knowledge of the world is socially constructed.

What has gone unnoticed in this debate is that there is a parallel and equally fundamental dispute over whether mathematics is discovered or invented. The absolutist view of mathematics sees it as universal, objective and certain, with mathematical truths being discovered through the intuition of the mathematician and then being established by proof. Many modern writers on mathematics share this view, including Roger Penrose in *The Emperor's New Mind*, and John Barrow in *Pi in the Sky*, as indeed do most mathematicians. The absolutists support a 'discovery' view and argue that mathematical 'objects' and knowledge are necessary, perfect and eternal, and remark on the 'unreasonable effectiveness' of mathematics in providing the conceptual framework for science. They claim that mathematics must be woven into the very fabric of the world, for since it is a pure endeavour removed from everyday experience how else could it describe so perfectly the patterns found in nature?

The opposing is view often called 'fallibilist' and this sees mathematics as an incomplete and everlasting 'work-in-progress'. It is corrigible, revisable, changing, with new mathematical truths being invented, or emerging as the by-products of inventions, rather than discovered. So who are the fallibilists? Many mathematicians and philosophers have contributed to this perspective and I will just mention a few recent contributions. First of all, the philosopher Wittgenstein in his later works such as *Remarks on the Foundations of Mathematics* contributes to fallibilism with his claim that mathematics consists of a motley of overlapping and interlocking language games. These are not games in the trivial sense, but the rule -governed traditional practices of mathematicians, providing meanings for mathematical symbolism and ideas. Wittgenstein argues that we often follow rules in mathematical reasoning because of well-tried custom, not because of logical necessity. So Wittgenstein's contribution is to point out that it is what mathematicians do in practice, and not what logical theories tell us, which is the engine driving the development of mathematical knowledge.

Imre Lakatos is another fallibilist, and he argues that the history of mathematics must always be given pride of place in any philosophical account. His major work *Proofs and Refutations* traces the historical development of a result in topology, the Euler Relation, concerning the number of faces (F), edges (E) and vertices (V) of mathematical solids. For simple flat-sided solids, the relationship is F+V=E+2. However, proving this fact took over a hundred years as the definitions of mathematical solids, faces, edges and vertices were refined and tightened up, and as different proofs were invented, published, shown to have loopholes, and modified. Lakatos argues that as in this example, no definitions or proofs in mathematics are ever absolutely final and beyond revision.

Philip Kitcher offers a further refinement of fallibilism in his book *The Nature of Mathematical Knowledge*. He argues that much mathematical knowledge is accepted on the authority of the mathematician, and not based on rational proof. Furthermore, even when mathematical results are proved much of the argument is tacit and draws on unspoken mathematical knowledge learned through practice, as opposed to being completely written down explicitly. Since the informal and tacit knowledge of mathematics of each generation varies, mathematical proof cannot be described as absolute.

In my book *Social Constructivism as a Philosophy of Mathematics*, I argue that not only is mathematics fallible, but it is created by groups of persons who must both formulate and critique new knowledge in a formal 'conversation' before it counts as accepted mathematics. These conversations embody the process that Lakatos describes in the evolution of the Euler Relation, as well as what goes on in Wittgenstein's mathematical language games. Knowledge creation is part of a larger overall cycle in which mathematical knowledge is presented to learners in teaching and testing 'conversations' in schools and universities, before they themselves can become mathematicians and participate in the creation of new knowledge. This perspective offers a middle path between the horns of the traditional objective/subjective dilemma in knowledge. According to social constructivism, mathematics is more than a collection of subjective beliefs, but less than a body of absolute objective knowledge, floating above all human activity. Instead it occupies an intermediate position. Mathematics is cultural knowledge, like the rest of human knowledge. It transcends any particular individual, but not all of humankind, like art, music, literature, religion, philosophy and science.

Although fallibilist views vary, they all try to account for mathematics naturalistically, that is in a way that is true to real world practices. Unfortunately, fallibilism is too often caricatured by opponents as claiming that mathematics may be part or all wrong; that since mathematics is not absolutely necessary it is arbitrary or whimsical; that a relativist mathematics, by relinquishing absolutism, amounts to 'anything goes' or 'anybody's opinion in mathematics is as good as anybody else's'; that an invented mathematics can be based on whim or spur of the moment impulse; and that if social forces are what moulds mathematics then it must be shaped by the prevailing ideology and prejudices of the day, and not by its inner logic.

However these claims and conclusions are caricatures, and no fallibilist I know would subscribe to them. Fallibilism does not mean that some or all of mathematics may be false (although Gödel's incompleteness results mean that we cannot eliminate the possibility that mathematics may generate a contradiction). Instead, fallibilists deny that there is such a thing as absolute truth, which explains why mathematics cannot attain it. For example, 1+1=2 is not absolutely true, although it is true under the normal interpretation of arithmetic. However in the systems of Boolean algebra or Base 2 modular arithmetic 1+1=1 and 1+1=0 are true, respectively. As this simple example shows, truths in mathematics are never absolute, but must always be understood as relative to a background system. Unlike in physics, in which there is just one world to determine what is true or false, mathematics allows the existence of many different interpretations. So an assumption like Euclid's Parallel Postulate and its denial can both be true, but in different mathematical interpretations (in the systems of Euclidean and non-Euclidean geometries). Mathematicians are all the time inventing new imagined worlds without needing to discard or reject the old ones.

A second criticism levelled at fallibilism is that if mathematics is not absolutely necessary then it must be arbitrary or whimsical. Relativist mathematics, the criticism goes, by relinquishing absolutism amounts to 'anything goes'. Therefore an invented mathematics is based on whims or spur of the moment impulse. For example, Roger Penrose asks, are the objects and truths of mathematics "mere arbitrary constructions of the human mind?" His answer is in the negative and he concludes that mathematics is already there, to be discovered, not invented.

Plausible as this view seems at first, it is often argued on mistaken grounds. Mathematicians like Penrose often contrast necessity with arbitrariness, and argue that if relativist mathematics has no absolute necessity and essential characteristics to it, then it must be arbitrary. Consequently, they argue, anarchy prevails and anything goes in mathematics. However as the philosopher Richard Rorty has made clear, contingency, not arbitrariness, is the opposite of necessity. Since to be arbitrary is to be determined by chance or whim rather than judgement or reason, the opposite of this notion is that of being selected or chosen. I wish to argue that mathematical knowledge is based on contingency, due to its historical development and the inevitable impact of external forces on the resourcing and direction of mathematics, but is also based on the deliberate choices and endeavours of mathematicians, elaborated through extensive reasoning. Both contingencies and choices are at work in mathematics, so it cannot be claimed that the overall development is either necessary or arbitrary. Much of mathematics follows by logical necessity from its assumptions and adopted rules of reasoning, just as moves do in the game of chess. This does not contradict fallibilism for none of the rules of reasoning and logic in mathematics are themselves absolute. Mathematics consists of language games with deeply entrenched rules and patterns that are very stable and enduring, but which always remain open to the possibility of change, and in the long term, do change.

The criticism that relativism in mathematics means that "anything goes" and that "anybody's opinion is as good as anybody else's" can be countered by using William Perry's distinction between the positions of Multiplicity and Contextual Relativism. *Multiplicity* is the view that anyone's opinion is valid, with the implication that no judgements or rational choices among opinions can be made. This is the crude form of relativism in which the opposite of necessity is taken as arbitrariness, and which frequently figures in 'knockdown' critiques of relativism. It is a weak and insupportable 'straw person' position and does not represent fallibilism. *Contextual Relativism* comprises a plurality of points of view and frames of reference in which the properties of contexts allow various sorts of comparison and evaluation to be made. So rational choices can be made, but they always depend on the underlying contexts or systems. Fallibilists adopt a parallel position in which mathematical knowledge is always understood relative to the context, and is evaluated or justified within principled or rule governed systems. According to this view there is an underlying basis for knowledge and rational choice, but that basis is context-relative and not absolute.

This position weakens the criticism from absolutists that an invented mathematics must be based on whims or spur of the moment impulses, and that the social forces moulding mathematics mean it can blow hither and thither to be reshaped accorded to the prevailing ideology of the day. The fallibilist view is more subtle and accepts that social forces do partly mould mathematics. However there is also a largely autonomous internal momentum at work in mathematics, in terms of the problems to be solved and the concepts and methods to be applied. The argument is that these are the products of tradition, not of some externally imposed necessity. Some of the external forces working on mathematics are the applied problems that need to be solved, which have had an impact on mathematics right from the beginning. Many examples can be given, such as the following. Originally written arithmetic was first developed to support taxation and commerce in Egypt, Mesopotamia, India and China. Contrary to popular opinion, the oldest profession in recorded history is that of scribe and tax collector! Trigonometry and spherical geometry were developed to aid astronomy and navigational needs. Later mechanics (and calculus) were developed to improve ballistics and military science. Statistics was initially developed to support insurance needs, to compute actuarial tables, and subsequently extended for agricultural, biological and medical purposes. Most recently, modern computational mathematics was developed to support the needs of the military, in cryptography, and then missile guidance and information systems. These examples illustrate how whole branches of mathematics have developed out of the impetus given by external needs and resources, and only afterwards maintained this momentum by systematising methods and pursuing internal problems.

This historical view of fallibilism also partly answers the challenge that John Barrow issues to ‘inventionism’. He asks if mathematics is invented how can it account for the amazing utility and effectiveness of pure mathematics as the language of science? But if mathematics is seen as invented in response to external forces and problems, as well as to internal ones, its utility is to be expected. Since mathematics studies pure structures at ever increasing levels of abstraction, but which originate in practical problems, it is not surprising that its concepts help to organise our understanding of the world and the patterns within it.

The controversy between those who think mathematics is discovered and those who think it is invented may run and run, like many perennial problems of philosophy. Controversies such as those between idealists and realists, and between dogmatists and sceptics, have already lasted more than two and a half thousand years. I do not expect to be able to convert those committed to the discovery view of mathematics to the inventionist view. However what I have shown is that a better case can be put for mathematics being invented than our critics sometimes allow. Just as realists often caricature the relativist views of social constructivists in science, so too the strengths of the fallibilist views are not given enough credit. For although fallibilists believe that mathematics has a contingent, fallible and historically shifting character, they also argue that mathematical knowledge is to a large extent necessary, stable and autonomous. Once humans have invented something by laying down the rules for its existence, like chess, the theory of numbers, or the Mandelbrot set, the implications and patterns that emerge from the underlying constellation of rules may continue to surprise us. But this does not change the fact that we invented the ‘game’ in the first place. It just shows what a rich invention it was. As the great eighteenth century philosopher Giambattista Vico said, the only truths we can know for certain are those we have invented ourselves. Mathematics is surely the greatest of such inventions.

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NOTE. An abridged version of this article was published as ‘New angles on old rules’, *Times Higher Educational Supplement*, 6 September 1996.