THE *AHA* PROBLEMS

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The word "algebra" comes from the title of a book written by the ninth-century Arabian mathematician Al-Khowarizmi. In this title, *al-jebr w' almuqabala*, the word *al-jebr* meant transposing a quantity from one side of an equation to another or "rejoining." *Muqabala* meant simplification of the resulting expression. Algebra, then, was the application of a series of techniques, including reductions, simplifications, transpositions, which manipulated mathematical expressions. The reason algebra became so powerful was because the resulting expressions applied to a large number of cases. Arithmetic, on the other hand, dealt with (and applied to) one case at a time. (Cf., Kline, p. 69)

Suppose I wanted to solve for x in the following second-degree equations:

1. $x^2 - 4 = 0$
2. $x^2 - 5x + 6 = 0$
3. $x^2 - 5x + 3 = 0$

By adding 4 to each side of equation (1), we quickly obtain a much clearer expression, one that can be "solved" quite easily:

$x^2 = 4$

So $x = 2$ and $x = -2$

The second equation is not quite that straightforward, but a trained eye would recognize that it is equivalent to the expression

$(x - 3)(x - 2) = 0$

The solutions, or "roots," will then be clear here: when x is 2 or 3, the equation would obtain. Lastly, equation (3) is the hardest of the three because there is no immediate manipulation that can simplify the expression any more. Eventually, mathematicians devised a formula by which to solve *any* equation of the form

$ax^2 + bx + c = 0$

The formula is now called the "quadratic equation" (the actual formulation is not necessary here).

So this is what algebra was (and is) really all about: solving for general cases. In order to do this, it is necessary to substitute some numbers with *any* number, called a "variable." Before this operation, mathematical problems were handled more piecemeal: each particular problem or situation was handled separately.

There are a group of problems, however, called the *aha* problems, that I believe are the precursor to algebra. *Aha* in Egyptian means "something" or "a quantity." Whenever a problem called for an unknown, the Egyptians would simply call it "something" or *aha*. Take for instance example 40 from the Rhind papyrus.(*Cf.*, Wilson, p. 28.) Aahmes writes:

We want to divide 100 loaves between five men in the following way. The second man receives a certain amount more than the first. The third man receives an equal amount more than the second, and so on. Also one seventh of the sum of the three largest shares shall be equal to the sum of the smallest two shares. What is the difference between the shares?

The Egyptians figured out that one could call the unknowns "something," in this case, the unknowns being the *smallest share* and the *difference*. We can write the shares received by the five men as:

Man #1: smallest share

Man #2: smallest share + difference

Man #3: smallest share + 2 differences

Man #4: smallest share + 3 differences

Man #5: smallest share + 4 differences

which adds up to:

(a) 5 x (smallest share) + 10 x (difference) = 100

which summarizes the first condition of the problem. The sum of the three largest shares is "3 x smallest share + 9 differences" and that of the two smallest shares "2 x smallest share + difference." The second condition specifies that the seventh part of the first expression be equal to the second expression, yielding:

b. 1/7 [3(smallest share) + 9(difference)]

   = 2(smallest share) + difference

   Multiplying through by seven, and simplifying terms (doing *al-jebr w' almuqabala*!) we reach:

c. 2 (difference) = 11 (smallest share)

which, together with (a) is a simple system of two equations with two unknowns. The problem is, Egyptians did not know how to solve these *directly*.

Whenever a mathematical problem reached this point, the Egyptians would *guess* at the answer(s). This seems a bit impractical, indeed this is the start of a "trial and error" exercise, but the answer they arrived at was not only *close* to the right answer: it was used to *reach* the right answer. In other words, they would have to "try" just once, and use the "error" to get at the right answer. For the above problem, then, we could guess that the difference was, say (conveniently) 11. From (c), we quickly see that the smallest share has to be two. Plugging these values into (a), we have

5 x 2 + 10 x 11 = 120

which is not equal to 100, but close. To reach the *right* answer, Aahmes instructs: "as many times as 120 must be multiplied to give 100, so many times must 11 be multiplied to give the true difference in shares." The true answer must therefore be given by difference = 100/120 x 11 = 55/6 = 9 1/6

which would put the smallest share value at smallest share = 2/11 x 9 1/6 = 5/3

If we double-check our answers, plugging our values into (a), we see that 5 x (5/3) + 10 x (9 1/6) = 25/3 + 550/6 = 600/6 = 100 which is correct.

The Babylonian *aha* problems are similar. Here is one of the simpler ones: "Find two numbers whose sum is 14 and whose product is 45." (Cf., Wilson, p. 64ff.) If the numbers were equal, then the first condition would be met with the number 7, but 7x7=49, not 45. The Babylonians got around this problem by postulating that there must be *a number* that they could both add to and subtract from 7 such that the *second* condition obtained. Calling this number, appropriately, *aha*, we get

(7 + *aha*) x (7 – *aha*) = 45 …multiplying out…

49 + 7*aha* – 7*aha* – (*aha* x *aha*) = 45 …simplifying…

$49 - aha^2 = 45$

which obviously leads to *aha* being 2. So 5+9=14, and 5x9=45. Notice there was no guessing here.

The Babylonians developed a more useful application of the *aha* problems: they were able to extract the square root of *any* number. Suppose we take the number √ 27 (our *aha*). Since we know that 5 squared is 25, and that 6 squared is 36, we can infer that √ 27 is between 5 and 6. In addition, the Babylonians noticed that since 5 is less than √ 27, 27/5 must be greater than √ 27 since

5 x 27/5 = √ 27 x √ 27 = 27

Therefore, a better approximation of √ 27 is halfway between 5 and 27/5. So √ 27 ≈ ½(5 + 27/5) = 5.2. As it turns out, 5.2 squared is 27.04, a surprisingly close answer. Further, this method could be used again, i.e., knowing that 5.2 is slightly greater than √ 27, then 27/5.2 has to be less than √ 27, so an average of these two numbers would yield an even more accurate answer (better than six significant figures!).

Interestingly (and finally), the Chinese had a different way of solving the exact same *aha* problem. Instead of √ 27 being the *aha*, the *offset* from 5 became the unknown. This is expressed as follows:

(5 + *aha*) x (5 + *aha*) = 27 …multiplying out…

(5 x 5) + 2(5 x *aha*) + (*aha* x *aha*) = 27 …so that…

$10aha + aha^2 = 2$ …or… *aha* (10 + *aha*) = 2

We know that *aha* is less than one, so that replacing 10 + *aha* by 10 gives an error of less than one in ten. Therefore, we can safely make our first iteration (*aha1*) fit the equation like this:

aha1 x 10 = 2, i.e., *aha1* = 0.2

This approximation gives us the same answer to which the Babylonians arrived. Subsequent iterations are produced by making the *aha* in the bracketed term the same as the previous *aha1* resulting above, obtaining a new *aha2*:

aha2 x (10 + *aha1*) = *aha2* x (10.2) = 2

Which gives us an *aha2* of 0.196078, and $5.159078^2 = 26.9992$.

The Chinese method of calculating roots is not as fast as the Babylonian method, but it does have the advantage of being able to be extended to treat higher roots. Suppose we were looking for the cubed root of two. Since the number will be obviously between 1 and 2, we can express the problem as: $(1 + aha)^3 = 2$. Extending the expression out to its simplest form, we obtain that

$$1 + 3aha + 3aha^2 + aha^3 = 2$$

Solving for one of the *aha*s, we get that

$$aha = 1 / (3 + 3aha + aha^2)$$

And setting the *aha*s inside the bracket to 0, we get that *aha1* is 1/3. $1.3333^2 = 2.3704$. Not too good for a first run, but subsequent runs do show a marked improvement.

The Chinese continued to expand these problems to higher orders, which led them to discover that there was a pattern developing. They noticed that there was symmetry to the resulting expanded equation, no matter how high the root. Further, the symmetry seemed to expand in a unique way! Consider the first few iterations:

$$(1 + aha)^0 = 1$$

$$(1 + aha)^1 = 1 + aha$$

$$(1 + aha)^2 = 1 + 2aha + aha^2$$

$$(1 + aha)^3 = 1 + 3aha + 3aha^2 + aha^3$$

$$(1 + aha)^4 = 1 + 4aha + 6aha^2 + 4aha^3 + aha^4$$

$$(1 + aha)^5 = 1 + 5aha + 10aha^2 + 10aha^3 + 5aha^4 + aha^5$$

If we pay attention to the *coefficients*, we notice (as the Chinese evidently did) that they add up in a peculiar way. For example, the 10 in $10aha^2$ is obtained by adding the coefficients directly above and to the left, i.e., 4 (of $4aha$) plus 6 (of $6aha^2$). This is the case for *all* coefficients! What is most remarkable about this pattern is that it is now know as *Pascal's Triangle*, named after French mathematician Blaise Pascal (1623-1662). Clearly, the Chinese knew of this pattern before Pascal did.

We have seen how the Egyptians, the Babylonians and the Chinese worked out mathematical problems by substituting unknowns with a "something," an *aha*. The word *algebra*, however, did not appear in the limelight until the Arabs introduced it. Does this mean that algebra did not exist until then? No, clearly not if by *algebra* we mean the process of reducing and rearranging mathematical terms. It is true that mathematics *for the sake of manipulating numbers* (or for the sake of solving for general cases) did not seem to appear in the cultures we have examined, and it is these theoretical branches of mathematics that we most often find the use of variables instead of numbers. Pure algebra enabled mathematicians to find a *general* solution to specific mathematical problems. In order to find *particular* solutions to many mathematical problems, however, the variables (or *aha*s) used and the algebra-like *techniques* here examined sometimes made things a lot easier to calculate, given that the alternative is a much more inefficient process of trial and error.

Bibliography

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