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NECESSARY MATHEMATICAL STATEMENTS AND ASPECTS OF KNOWLEDGE IN THE CLASSROOM

Jean-Philippe DROUHARD, Catherine SACKUR, Maryse MAUREL, Yves PAQUELIER & Teresa ASSUDE

DIERF (IUFM, Nice) & CREEM (CNAM, Paris), drouhard@unice.fr

GECO/IREM, Nice, maurel@unice.fr, sackur@unice.fr

GECO, Nice, CREEM (CNAM, Paris) & Lycée de Madrid, paquelie@cnam.fr DIDIREM (Paris 7) & IUFM de Versailles, assude@gauss.math.jussieu.fr

Introduction

In mathematics, most statements may be called *necessary*. They are not just true or false, in the same way as the statement "Osnabrück is the birthplace of Erich Maria Remarque", but they are *necessarily* true or false, like the Pythagora's Theorem.

Until now, few studies in mathematics education have been focusing straightforwardly on the questions of when and how do students become aware of the necessity of such statements.

To address, at least theoretically, this question we were led to introduce into the usual didactic 'models' (as for instance the Brousseau's (1997) theory of didactic situations) some specific elements:

the *Inter subjectivity* of the knowledge (i. e. the central role of the dialogues between the student in the very nature of the mathematical knowledge)

" the subject's *Experience* (of contradiction, for instance)

the crucial role of *Time* to understand the 'construction' of the necessity of the necessary statements,

" the notion of *aspects* of a knowledge

1. The CESAME Research Group

Within the research field of Social Constructivism (Ernest, 1995), we are working on a project called CESAME. This word, "CESAME", is an acronym of the following French words: « "Construction Expériencielle du Savoir" et "Autrui" dans les Mathématiques Enseignées », which could be more or less well translated as: « the Subject's Experience of Dialogues with Interlocutors ("Others") in the Social Construction of Taught Mathematical Knowledge ».

The first outcome of the CESAME group has been a set of questions:

1. What is actually the subject experiencing, whilst constructing his/her mathematical knowledge?
2. What are the various roles that "Others" (other students, the teacher etc.) can play (according to E. Goffman's perspective, 1959) in this subject's experience?

   Is it possible to experience the *necessity* of necessary mathematical statements.

1. If so, what is the very nature of such an experience?
2. And, are actually some students experiencing it, and how?

In this paper, we shall not address the question of the roles of the "Others" (that we previously described in terms of a "double didactic pyramid", see Drouhard 1997a, 1997b). We rather shall focus here on the notions of *necessity* of necessary mathematical statements (part 2: Necessary Statements) and on the *nature* of the related knowledge (part 3: "Aspects of Knowledge").

2. Necessary statements

2.1. Institutionalisation in Mathematical Discussions

Our starting point is the following. What is the teacher supposed to do, at the end of a *Scientific Debate* (Legrand, 1988) or more generally of a *Mathematical Discussion* (Bartolini-Bussi, 1991, see also Hoyles, 1985), in order to turn the many statements yielded by the discussion into "official" mathematical statements, which is "Institutionalisation" in the terms of the Theory of Didactic Situations (Brousseau, 1997, Margolinas 1992, 1993)? And what is s/he actually doing? On the one hand, in a strict Social Constructivist point of view, the group is supposed to " construct " the whole mathematical knowledge by itself, the teacher's role being just to lead the discussion. On the other hand, the teacher actually says something about the new common knowledge. Well, what does s/he says? How what s/he says may change the status of the statements? In most cases, s/he summarises; we think, however, that summarising is not just saying the same things shorter; but has an effect on the *meaning* of the statements. In brief, our question is the following: *what* is institutionalised in a mathematical discussion, and *how? What* is institutionalised must be obviously related with what is specific to mathematics. Then, a first point is to determine what is *specific* to mathematics in a discussion, in other words to what extent a mathematical discussion differs from, say, a political discussion. A first answer could be the following: in a *mathematical* discussion, *individuals are experiencing contradictions*. We use here the words "to experience" and "contradiction" in a strong sense; it is not just to be engaged in some kind of contradictory debate, it is to have the intimate experience that others can be certain of opposite ideas, and cannot be convinced by authority arguments, for instance.

2.2. Resistance

The simple fact that contradictions exist - and are perceived as such by the participants! - involves that in mathematics, some arguments are particularly strong. In other words: in mathematics, statements do *resist*. A wall resists to your efforts to break it, too; but, if you have an axe big enough, you can make a hole in it. On the other hand, there is no axe to allow you to consider true a false mathematical statement. Of course, you *can*, but you cannot avoid the fact that anyway it is an error. This 'resistance' of mathematical statements is, for us, characteristic of their 'necessity'.

Unlike statements like "CERME-1 is held in Osnabrück", most mathematical statements are *necessarily* true or false, as:

« for any a and b belonging to a commutative ring, $a^2-b^2=(a+b)(a-b)$ »

What is the *nature* of this necessity? It is not easy to characterise such an 'epistemic value' (Duval, 1995) of statements. A first (logicist) answer can be found in terms of mathematical logic: a *necessary* statement is the result of a valid inference (Durand-Guerrier, 1995; 199?). From this point of view, the problem of what is institutionalised at the end of a mathematical discussion (and more generally at the end of a lesson) becomes simple and easy to understand: students have just to learn the content of the statements *and* that these statements result from valid inferences; in other words, that they have to be demonstrated.

This approach however remain for us somewhat unsatisfactory. We think, indeed, that it does not take into account the aspect of *resistance* of statements, whereas the experience of this resistance appears to be crucial in mathematical discussions (if it would not be the case, discussions would be just pedagogical 'tricks', to help students to find and refine ideas but not specific of a mathematical content).

2.3. Wittgenstein

We found in Wittgenstein (1978) a more subtle characterisation of necessity, related to resistance, although apparently paradoxical: the idea (expressed here in a very sketchy way) that mathematical objects resist us to the very extent that we *will* that they resist. The mathematical objects do not resist by themselves, as walls resist according to their physical nature. Mathematical objects resist because we *want* them *to resist us*. But, what is the origin of this willpower? It seems that, for Wittgenstein, mathematicians need objects that resist them because *doing mathematics* is precisely *working on such* resisting *objects*. Surely they *could* work with 'weaker' mental objects but then, what they would do, could no longer be called "mathematics" (this is a turning point).

Wittgenstein (1986) uses often a metaphor of games. We could illustrate it as follows. In chess game, a pawn goes forward only. Obviously, you *can* move a pawn backwards; but you *may* not. Well, more precisely, you even *may* decide to authorise this movement indeed; but in this case *you no longer play the same game*. Wittgenstein (speaking here of a "grammar") stressed the importance of the dual notions of arbitrary rules of the game, and of the *definition* of an activity (you may break the rules but in this case you play another game, i.e. a game *defined* in an other way).

This (anti-Platonist) point of view appears often questionable indeed. However, we think that a concrete example may be found in the G. H. Hardy's book of Memoirs. In this book, Hardy speaks about the Indian mathematician Ramanujan (author of incredibly difficult formulas, some of them still used in the very recent computations of p) who learned alone the advanced mathematics in an old-fashioned exercise book (where solutions were given *without any demonstration*). Ramanujan, Hardy said, was a pure genius in mathematics but at the same time he did not really understand some fundamentals, for instance the meaning of a demonstration! We could consider that Ramanujan's mathematics are an example of 'different' mathematics, where most knowledge is similar to usual, except for some high-level touchstones as the role of the demonstration. Likewise, we could address the difficult question of the nature of non-occidental mathematics. Are (classical) Chinese mathematics mathematics? The computations are the same, but likely not the proofs...

In the CESAME group, we called this type of necessity (related to the rules of the mathematics considered as a game) the "epistemic necessity".

2.4. Awareness

From a learning-oriented point of view, an individual has many possibilities to be aware of this epistemic necessity. For a given mathematical statement, s/he may *know* (that the statement is necessarily true), know it *and* know *why*, know it and *remember* that s/he has been knowing why (but ignore why at present); s/he may even know that s/he forgot why precisely the statement is necessarily true, but be persuaded that *s/he could retrieve his/her experience* of the epistemic necessity of the statement, etc. The question of the various ways for a subject to be aware of the epistemic necessity of a statement still remains open. However, it is clear for us now, that one of the specific features of mathematical discussions is to give students a chance to have an experience of the necessity of the statements (in terms of resistance to contradiction) and that this epistemic necessity is a part of what is to be *institutionalised* by the teacher.

3. ASPECTS OF KNOWLEDGE

Let us come back to the initial question: *what* is institutionalised in a mathematical discussion? We just saw that the epistemic necessity is one of the answers. It is clear however that a much larger amount of things may be learnt through such discussions: for instance, in algebra, the know-how and also what we could call the "know-why" of the rules. More generally, something *about* mathematical statements is institutionalised, which is more than their strict content. In order to give a framework to this idea, we propose to consider that a (taught) knowledge present three aspects:

> The *first aspect* is made of the mathematical *content*: of the knowledge, the *semantics* of the related statements, according to the Tarsky logic.

I. The *second aspect* contains the rules of the mathematical game, the know-how, but also that, for instance, algebraic expressions have a denotation (Drouhard, 1995), true statements are necessarily true, definite statements are either true or false, demonstrations are (the) valid proofs etc.

> The *third aspect* contains the most general believes about mathematics, as for instance "mathematics is a matter of understanding" or in the contrary (as many students say), "in mathematics there is nothing to understand".

We chose to call these components of a mathematical knowledge "aspects" to express the idea that these components could hardly be defined and studied separately one from another. A mathematical theorem (whose "content" is its aspect I) is mathematical (aspect III) indeed, inasmuch as it is necessarily true (aspect II)! The distinction between these aspects is just a question of "point of view". We could imagine an optical metaphor: a given mathematical knowledge is like a White light, obtained by superposition of many coloured light; and a colour (corresponding to one aspect in the metaphor) can be viewed only if we observe the light through a coloured filter, which suppresses all the other colours.

Of course, that the expert knows the rules of the mathematical games, and that anybody has general believes about mathematics is in no way original. For us, the interest of focusing on such aspects is that now we are able to address the question of *how* and *when* are these various aspects are *taught* and *learnt*. Actually, we feel that many studies in didactics focus mainly on the teaching of the *aspect I* of the knowledge (its "content") or more precisely that aspects II and III are seldom in the foreground but instead often remain in the shadow.

4. teaching and learning

Let us consider now the second facet of the question: *how* is knowledge, with its three aspects, institutionalised? In particular, what is the role of the teacher related with aspects II and III of a mathematical knowledge? What does s/he do? Clearly, the teacher cannot just say about the epistemic necessity of a statement *S* that "*S* is necessary"! We think that, a priori, this role is rather different than the teacher's role about the aspect I of knowledge. In particular, we think that his/her role is often *performative* (utterances that allow us to do something by the simple fact they are uttered: Austin, 1962).

The teacher puts the label "mathematics" on what the student does and so s/he gives the student a hint on what mathematics are. Let us consider the sentence:

> «Using denotation to invalidate a rule or a result is a mathematical action»

From our point of view such a sentence could illustrate the way we wish to use the aspects of knowledge while teaching:

(a) the rule is invalid - it is false - in the same way that "CERME-1 is held in Berlin" is false,

(b) to see (demonstrate, prove, check) that it is false a mathematician has a method, tries with values of x.

(c) to know that this method is valid and to use it, is "doing mathematics".

We believe that (a), (b) and (c) above do not play the same role in the learning of mathematics and cannot be told in the same way. Similarly, as teachers, we sometimes find work of students which are written with words and symbols commonly used in mathematics. These works look like mathematics and someone who doesn't know mathematics could be mistaken. What we feel then is that "this mess has nothing to do with mathematics": assertions are incoherent, the result is taken for granted and used in the proof, notions are mixed one for the other... not only are the theorems and definitions false (aspect I), but above all we do not recognise the "rules of the mathematical game" (aspect II).

It seemed useful to us to point these three aspects of knowledge so as to keep in mind how learning and consequently teaching mathematics is a complex action. Mathematical statements are mathematical if and only if they actualise the three aspects of knowledge. A consequence of this, is that the students' conceptions about mathematics (aspect III) must be constructed in the process of learning mathematics. It is a sort of paradox: one learn something which is not yet defined and learning it, this person learns what is the nature of what s/he is learning (maybe, this is not specific to mathematics: learning piano is not just learning how to move one's fingers on a keyboard so as to produce the correct sounds).

In brief, we can consider that to learn mathematics (I and II), is also to learn *what are* mathematics (III). On the other hand, to teach that a statement is mathematical (II), is also to teach that mathematics are made (III) of statements like the one which is taught (I)!

5. Provisional conclusion

We are fully aware that in this paper most ideas are mainly tentative and expressed in a quite abrupt way. We tried our best, but the research is still ongoing and to express our ideas in a more detailed and concrete way is our work to come. We tried however to find some empirical data to support these ideas, and this can be found in the paper of Sackur & Assude. An other ongoing research on inequations is precisely addressing the question of to what extent are students aware of the necessity of algebraic statements when they solve inequations.

We would like to stress however the point that our research takes place without ambiguity in the domain of social constructivism, not just for anecdotal reasons as the many references to Wittgenstein but since we think (following this author!) that the question of the epistemic necessity has no sense beside the point of view of the social construction of mathematics: all our keywords (rules of a game, definitions, beliefs...) are *socially* constructed, shared and taught.

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