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DO NUMERALS NAME NUMBERS?

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The project of discovering ultimate foundation for mathematical knowledge is almost completely abandoned. Instead, a pluralistic view prevails which distinguishes various foundation relations. Logical, semantical, cognitive, epistemological, ontological and methodological foundation relations allow for different ordering: philosophy of mathematics amounts to one of possible orderings. The aim of this paper is to show that treating numerals as singular terms is a semantical mistake which leads to ontologization of numbers.

The Fregean tradition is connected with "syntactic priority thesis": if an expression has the logical role of singular term in a true sentence then it refers to an object. It seems that numerical expressions do function as singular terms especially when flanking identity sign so the numbers must be objects. The objects numerals refer to have a rather curious nature since a) there is a disagreement what numerals name: "classes of all equinumerous classes" (Quine) or "properties of properties which belong to the same number of objects" (Carnap) or ... b) the same name names different objects - in hierarchy of universes (type theory) U0, U1 ,U2 ,... the same numeral refers to different objects i.e. different numbers (four as number of Beatles belongs to class from U2 , four as number of cardinal virtues to class from U3). Therefore, either some singular terms do not refer, or numerals are not names. The neo-Fregean move is to treat singular termhood as relative notion: whether some expression is singular term or not depends on particular theory or discourse domain. Favourite metaphors are those where objects are individuated by their functional role. In explaining the rules of chess game or constitutional government structure we use expressions such as "knight" or "member of parliament" as singular terms. In a different context those same expressions can assume predicate role: "That image on the screen is a knight" or "He was a member of parliament". We should not stretch the metaphor too far: there is property inheritance when an object assumes the role of chess knight, but nothing except ad hoc created abstract entities can take the role of number. It is no oddity that Quine's *de dicto* reading gets false proposition, while *de re* reading comes out true for sentence "The number of planets is necessarily odd" after substitution, for there is no such identity as "number of planets=9".

Neo-Fregean position comes close to structuralist view: numbers are structurally defined, deprived of all properties except those belonging to them as points in structure, therefore numbers are not objects. Category theory exhibits the possibility of conceiving mathematics as investigation of morphisms and not of objects. By conversion of "syntactic priority thesis" we may concede that numerals are not singular terms since numbers are not objects.

The logical role of numbers is connected to quantifiers and predicates, rather than names. If the discourse domain is defined the numbers can function as quantifiers: when we assert that **n** objects from domain of **m** objects satisfy condition **C** then we affirm disjunction where each disjunct is a conjunction with exactly **n** affirmative and **m-n** negative statements in a way that exploits all such combinations of singular statements. In the case when the domain is not fixed understanding of quantified statement, whether numerically refined or not, presupposes knowledge of descriptions under which possible objects may appear or gives a clue how these descriptions may be obtained. The latter is the case with infinity schemata. Consider schema (1) ""x$y"z(Fxy. -Fxx. Fyz ÉFxz)" which requires an infinite universe of objects: if F(a1,a2) then $y(F(a2 ,y)), but that (y) may not be a1 because by transitivity we would arrive at forbidden reflexivity statement F(a1,a1), so there must be a3 for which the same conditions hold, so there must be a4 , and so on *in infinitum*. We must know how to construct an infinite array of ancestors in order to understand sentence "Everybody has ancestors". On the other hand, the number of possible models is infinite as well. Therefore, the key element in understanding the statement of such a kind does not lie only in construction of possible patterns, but also in eliminating some of them. We exclude some patterns when we obtain additional information (such as: everybody has exactly two immediate ancestors, different persons can have the same ancestors ...). Still combinatorial explosion remains and one may wonder whether Darwinism comes as remedy for infinity brought in by biogenesis principle. These cognitive procedures involved in finite and infinite schemata are different from measurement of class/set size and measurement of property intensity.

The other important use of numerals, besides quantifying, lies in creation of measurement space as a predicate part. Measurement of class/set size by natural numbers or measurement of property intensity by real numbers does not gain identity or relational statements. While asserting measurement statements (like these: "size-of-class(planets,9)" or "weight-in-kg(that book, 0.3)") we are not asserting the existence of relation of mentioned objects towards numbers. In these cases the role of numerical expressions is creating predicate multitude. More appropriate approach is to treat measurement predicates as those which assign nomic relations to domain of measured objects. Measurement can be conceived as homomorphic mapping from empirical structure of measured objects and their relations to measuring numerical structure. This mapping is not an isomorphism since some relations may have no counterpart. For instance, the natural number property of having exactly one successor receives no interpretation for empirical sets of objects, except in the relational sense of possibility to define an asymmetric transitive relation on measured structure.

The logical role and corresponding cognitive procedure involved in measurement differ from those exhibited in (numerical) quantifying: the former presupposes use of number structure, while the latter explores either possible states or possible structures. Piagetian analysis convincingly points to specific nature of cognitive procedure of number construction. There is something like generalization, because an abstraction is required, but still it's not a kind of generalization because all the properties are taken away - so that no objects remain. Nevertheless, objects remain thanks to complementary operation which by introducing transitive asymmetric relation permits individuation by position. It is probably true that elementary ways of dealing with objects are equivalence and differential arrangements, but since there are many ways of differential ordering the same explanation given to psychogenesis of understanding of natural number structure can be given for understanding of kinship structure. Numerals as quantifiers have different, combinatorial role and there is no a priori reason why we should say that cognitive complexity involved in understanding quantified sentences surpasses complexity of counting.

It seems that investigation of semantical issues in philosophy of mathematics strongly suggests a version of "cognitive semantics". The use of numerical expressions is in some cases best understood as relation between the language, on one side, and merging cognitive structures, on the other side. Perceptual qualities of colour are mapped into two-dimensional structure in which one axis defines brightness/darkness ordering, while other defines complementarity, weight is represented one-dimensionally , kinship is mapped on complex tree-like structure. The sorites pseudo-paradox shows that we have no reason to suppose that measured system should exploit all properties of measuring structure or that the latter must be unique. The analysis has tried to show that in "talking with numerals" we use numerals as quantifiers or as predicate parts, and not as names. On the other hand, "talking about numbers" is a kind of higher-order language in which we don't quantify over objects, but over functions in order to characterise structures.

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