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MATHEMATICAL ACTIVITY AND RHETORIC: A SEMIOTIC ANALYSIS OF AN EPISODE OF MATHEMATICAL ACTIVITY

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During most of their mathematics learning career from 5 to 16 years and beyond, learners work on textual or symbolically presented tasks. They carry these out, in the main, by writing a sequence of texts (including figures, literal and symbolic inscriptions, etc.), ultimately arriving, if successful, at a terminal text 'the answer'. Sometimes this sequence consists of the elaboration of a *single* piece of text (e.g. the carrying out of 3 digit column addition). Sometimes it involves a *sequence of distinct* inscriptions (e.g. the addition of two fraction names with distinct denominators, such as 1/3 + 2/7 = 1x7/3x7 + 2x3/7x3 = 7/21 + 6/21 = 13/21); or it may combine both activities, as in the example below.

A rough estimate of the magnitude such activity is as follows. A child's compulsory (state) schooling in Britain extends for something over 2000 days. Suppose a typical child attempts a mean of between 5 and 100 mathematical tasks per day (an estimate that is quite plausible). Then a typical British school child will attempt between 10,000 and 200,000 mathematical activities in their statutory school career. The sheer repetitive nature of this activity is under-accommodated in many current accounts of mathematics learning, where the emphasis is more often on the construction of meaning. (Notable exceptions include Christiansen *et al*., 1985, and Mellin-Olsen, 1987.)

In analysing mathematical activity it must be recognised that the classroom is a complex, organised social form of life which includes the following:

(a) persons, interpersonal relationships, patterns of authority, pupil-teacher roles, modes of interaction, etc.

(b) material resources, including writing media, calculators, microcomputers, texts representing school mathematical knowledge, furniture, an institutionalized location and routinized times.

(c) the language and register of school mathematics (and its social regulation), including:

1. the content of school mathematics; the symbols, concepts, conventions, definitions, symbolic procedures, and linguistic presentations of mathematical knowledge;

2. modes of communication: written, iconic and oral modes, modes of representation and rhetorical forms, including rhetorical styles for written and spoken mathematics.

Thus, for example, teacher-pupil dialogue (typically asymmetric in classroom forms) takes place at two levels: spoken and written. In written 'dialogue' pupils submit texts (written work on set tasks) to the teacher, who responds in a stylized way to its content and form (ticks and crosses, marks awarded represented as fractions, crossings out, brief written comments, etc.)

TASKS

Tasks concerning the transformation of mathematical signs are central to this account. Typically a task is a text presented by someone in authority (the teacher), specifying a starting point, intended to elicit a frame (a task in a sequence may assume a frame is in use), and indicating a goal state: where the transformation of signs is meant to lead. The theorization of tasks draws on Activity Theory and semiotic analyses of mathematics (e.g. Rotman, 1988), as well as cognitive science approaches. Mathematics education sources include Christiansen *et al*. (1985), Mellin-Olsen (1987), Davis (1986), Skemp (1982), Ernest (1987a, b). From a semiotic perspective, a completed mathematical task is a sequential transformation of, say, *n* signs ('S_k') inscribed by the learner, implicitly derived by *n-1* transformations ('ð'). This can be shown as the sequence: S_1 ð S_2 ð S_3 ð...ð S_n. S_1 is a representation of the task as initially construed (the text as originally given, curtailed, or some other mode of representation, such as a figure). S_n is representation of the final symbolic state, intended to satisfy the goal requirements as interpreted by the learner. The rhetorical requirements of the social context determine which sign representations ($\boldsymbol{S_k}$) and which steps ($\boldsymbol{S_k}$ ð $\boldsymbol{S_{k+1}}$, for k < n) are acceptable. Indeed, the rhetorical mode of representation of these transformations, with the final goal representation ($\boldsymbol{S_n}$), is the major focus for negotiation between learner and teacher, both during production and after the completion of the transformational sequence.

Each step $\boldsymbol{S_k}$ ð $\boldsymbol{S_{k+1}}$ is a transformation of signs which can be understood on two levels. Drawing on Saussure's analysis, each sign $\boldsymbol{S_k}$ (= S_k / I_k) is a pair made up of a signifier S_k ('S' for *symbol*) and a signified I_k ('I' for *interpretation* or *image*). So the completed task can be analyzed as in the example shown as Figure 1.

Figure 1: A completed mathematical task as a semiotic transformation

Level of Signifiers: S_1 ® S_2 ® S_3 ® S_4 S_5 ® S_6 ® S_7 ® S_8 ® S_9

½ ½ ½ ½ ½ ½ ½ ½ ½

Level of Signifieds: I_1 I_2 I_3 I_4 Þ I_5 Þ I_6 I_7 I_8 I_9

Figure 1 is a schematic representation of the sign transformations in the sample task discussed below. It shows a linear sequence of signifiers with most derived from their predecessor by a symbolic transformation (denoted '®'); it shows each signifier connected vertically to its corresponding signified; it shows a linear sequence of signifieds, two of which are derived from predecessors by a transformation of interpretations (denoted 'Þ'). It parallels Davis' (1986) 'Visually Moderated Sequence', involving symbols and meanings in a goal directed sequence.

To clarify the role of such analyses it should be noted that, first, Figure 1 illustrates that transformations take place on one of the two levels, or both together. Second, signifieds vary with interpreter and context, and are far from unique. The level of signifieds is a private 'math-world' constructed individually, although in a degenerate activity it may be minimal, corresponding to Skemp and Mellin-Olsen's notion of 'instrumental understanding'. Third, signifiers are represented publicly, but to signify for the learner (or teacher), s/he must relate to them (they have to be attended to, perceived, and construed as symbols). Fourth, Figure 1 shows only the structure of a successfully completed task, represented linearly as a text. It does not show the complex process of its genesis. Finally, the levels of signifier and signified are relative; they are all the time in mutual interaction, shifting, reconstructing themselves. What constitutes a sign itself varies: any teacher-set task is itself a sign, with the text as signifier, and its goal (and possibly frame) as signified.

A Case Study

The theory is used to analyze a routine mathematical task carried out by a 14½ year old female 'Nora'. During the Autumn Term, Nora attended a state high school (although absent on a significant number of days). In mathematics class Nora worked from a set mathematical textbook (Cox and Bell, 1986) on a number of topics including trigonometry (first tangent ratios, later sine and cosine ratios). An analysis of Nora's exercise book shows notes taken from two sessions of exposition, including 3 worked examples on Sine & Cosine and the 'tools' indicated below. Based on what is recorded in her exercise book, during the month or so in which Nora was studying trigonometry (and other mathematical topics) she carried out at least 62 trigonometric tasks (26 Tan and 36 Sin & Cos). Almost all were routine but of increasing complexity; a few were non-routine problem tasks. She had feedback via teacher marking on 22 Tan and 15 Sin & Cos tasks, and was marked correct on all but one Tan task. Her exercise book reveals 2 locations where conceptual and symbolic tools for trigonometry were recorded/developed; notes of a lesson of 17 November, and 4 pages of undated rough notes at the rear of the book. The tools involved were: definitions of trigonometric ratios, 2 mnemonics to assist memorization; review of Pythagoras' rule; relabelling of triangle sides 'O', 'A', and 'H' according to a newly designated angle, inverse ratios, calculator use, cross-multiplying to solve e.g. Tan P = O/A for A, and similarity of triangles and ratio.

THE RESEARCH TASK

The research interview took place out of school on 17 December, based on a routine tangent task from the school text. Nora had available her pencil case, calculator, text and exercise book. She was asked to work on plain paper, to think aloud, and was tape recorded.

THE TASK (*Cox and Bell*, 1986: 58)

4 Sketch each of the following triangles PQR, with Ð R = 90°, then calculate both Ð P and Ð Q, correct to the nearest 0.1°.

(a) PR = 7.6m; RQ = 5m (b) ...

Figure 2: The first part of Nora's solution to the experimental task (Q only)

(Figure omitted)

ANALYSIS OF THE WRITTEN SOLUTION

Nora's answer can be analyzed as a sequence of written signs: $\boldsymbol{S_1}$ ð $\boldsymbol{S_2}$ ð ... ð $\boldsymbol{S_9}$, as follows.

$\boldsymbol{S_1}$: Figure with labelled vertices

$\boldsymbol{S_2}$: Figure with lengths of 2 shorter sides marked

$\boldsymbol{S_3}$: Figure with interior angles P & Q marked

$\boldsymbol{S_4}$: Figure with sides labelled 'O', 'A' & 'H'

$\boldsymbol{S_5}$: 'Tan = 7.6/5'

$\boldsymbol{S_6}$: '= 1.52' added

$\boldsymbol{S_7}$: [New line] '56.7°'

$\boldsymbol{S_8}$: '= Q' added

$\boldsymbol{S_9}$: '(56.65929265)' added

This is a transformational sequence of signs, which can be analyzed as follows.

$\boldsymbol{S_1}$ ð $\boldsymbol{S_2}$ ð $\boldsymbol{S_3}$ ð $\boldsymbol{S_4}$ are transformations of the triangle diagram through additions; stages in the elaboration of the iconic plus symbolic representation. The figure is required by the question (presumably for methodological reasons). It enables Nora to cue and build up a simple math-world of triangles and their properties, and construct a representation of the problem situation within it. (The elaboration of a single drawn figure contrasts with the typical justificatory rhetoric of written school mathematics, where repetition of symbols is often required. The rhetoric of diagrams requires the maximum of 'relevant' information be displayed.)

$\boldsymbol{S_4}$ ð $\boldsymbol{S_5}$ is a shift from figure to written text, indicating the choice of the tangent function to express the required angle in terms of the ratio of known lengths. (Having constructed a task-supporting representation $\boldsymbol{S_4}$, both as an iconic symbol S_4 and as a mental image I_4, Nora is thus able to retrieve appropriate conceptual tools, and then to represent the linguistic signs that lead via transformations towards the completion of the task.)

$\boldsymbol{S_5}$ ð $\boldsymbol{S_6}$ is the computation 7.6/5 by calculator, at the level of signifiers, with the answer transformed at the level of signifieds (corrected to 1 decimal place; intermediate answer omitted until S_9) and recorded. This is the only dual-level transformation shown in Fig. 1

S_6 ð S_7 represents the application of the calculator 'inverse tan' function to S_6. (The actual process involved first applying the tan function, and then rejecting it.) The recording of S_7 represents the completion of the main task-goal (the derivation of the answer), but does not yet satisfy the rhetorical requirements of classroom written mathematical language. Thus S_7 ð S_8 is the addition of a label ('Q') to the previous answer (labelling answers is a widespread rhetorical demand). Finally, S_8 ð S_9 is the addition of the omitted earlier answer, to show what was actually derived with the calculator, thus completing a perceived gap in the account. (This satisfies Nora's construal of the rhetoric of mathematics as accurately and completely describing the transformational sequence; whereas the out-of-sequence inclusion of S_9 does not correspond to the usual rhetorical demands of school mathematics.) The signs S_5 to S_9 represent the justificatory rhetorical account of the transformation, recorded after the event.

The final written text (Omitted figure) is in abbreviated form. Earlier work of Nora (school and homework recorded in her exercise book) utilizes a rhetorical form as follows, e.g.: Tan Q = O/A, Tan Q = 7.6/5, Tan Q = 1.52, SHIFT TAN = 56.7° = Q, etc. In Figure 2, the initial definition, the argument of 'Tan', the symbolization of calculator use, etc. are omitted.

Figure 1 shows the sign transformations analyzed into signifiers and signifieds. Most transformations take place on the signifier level of , but in every case these transformations are supported by meaningful interpretations and meaning-relations between them in the math-world.

INTERVIEW PROTOCOL

I: You're going to do number 4a, page 58.

N: [timidly] Read the question?

I: It's over to you, I want you to do all of it yourself.

N: I can't get any help from you?

I: Yes, you can, you can ask me for help, you know, if you need it[1].

N: [Rapidly speaking over the interviewer's last word] I've got to draw it first...PQR, with R angle 90. P, Q, R, [draws triangle] Just doing the triangle...its not to scale. P and Q,..[pause].. oh...alright.[2]

I: Alright what?

N: Oh, are you, it just said, I read the first bit and it said sketch each of the following triangles PQR with angle R 90 degrees and calculate P and, um, Q, but I didn't know how long they were, but its (a) and that[3]. P to R, its 7.6 metres and R to Q its 5 metres [writes in side lengths]

I: Speak out loud, what are you thinking about?

N: I'm thinking about how to do it, hold on...[marks angles P & Q] I know they add up to 90 degrees together.[4] Do I, do I use sine?...tan isn't it? No its not, look see, opposite, hang on, where's the angle I want? I shall want this angle here, if I want Q angle, then it's opposite, which is P to R,[5] I've got that one. Is that opposite over adjacent? so it's O A. Tan I need, opposite divided by the adjacent, that'd be 7.6 divided by 5 metres [uses calculator] that is ... 1.52. Now I press tan [uses calculator]. That's wrong. Maybe it's inverse tan. Tan. [uses calculator] That looks more like it. Is that right?[6] How many significant figures?

I: Did the question say?

N: no,...tenth of a degree. That's Q. 56.7. Hang on, I'm going to write out the whole thing. [writes out working][7] Now I must do P, P angle.

INTERPRETATION OF SIGNIFICANT FEATURES OF THE DIALOGUE

Some of the key features of this dialogue interpreted from the point of view of the present analysis of mathematical activity are as follows (much more could be said):.

1. In the preceding three lines Nora is looking for clues to the nature of the roles/positionings for her and the interviewer, and implicitly acknowledging the dominance of latter. (Is it teacher-learner or tester-examinee?) The context is an unusual out-of-school interview with someone who is not the teacher; and thus pertains to some but not all features of the social context of school mathematics, the source of uncertainty. By the end of this mini-exchange, Nora knows she must do as much as she can unaided, before seeking help.

2. Here Nora has internalized the task (and is subserving herself to the textual commands in the task), is beginning to make the initial symbolic representation (sketch). Finally, Nora has cued a frame to carry out the transformation in.

3. When asked to explain "oh.. alright" Nora firsts searches for a way to begin her account, and then constructs a rhetorical sequence to explain to the interviewer what has just taken place in her thought. Normally (in a school presented task) this rhetoric would not be required, unless interrogated by another person - peer or teacher. It ends with the phrase '(a) and that', referring to part (a) following the stem of Q. [Verified afterwards.]

4. At this point Nora has just employed a conceptual tool/item of knowledge, concerning the angle sum of the two smaller angles in a right angled triangle. It is irrelevant here. It suggests that Nora's frame is a math-world based on triangles and their properties, including (but extending beyond) trigonometric properties.

5. At this point Nora has tentatively chosen the 'Tan' function and managed to ignore (or mentally exchange) the labels 'O', 'A' in the figure to construct the correct ratio 7.6/5 for Tan Q.

6. The whole preceding monologue reflects the uncertainties and doubts, the feints and moves considered and carried out in the math-world, but also involving semiotic representations and tools in the physical world (i.e. keying in the calculations into the calculator). It represents the key thought experiment underpinning the solution and symbolic transformations of the task. Interestingly, it involves self-directed questions, as Nora voices queries and then answers them, regulating her activity meta-cognitively.

7. This represents another shift into rhetorical mode; the representation of the symbolic transformations, after the event, in an acceptable way as required by the teacher in the normal discursive practice of schooling (as construed by Nora). This is followed by a switch of attention to the other part of the question (P) [omitted here].

This analysis reveals some of the multi-levelled complexity involved in a learner carrying out a semi-routine mathematical activity. This included the construction of a math-world, one or more thought experiments or 'journeys' in it, a monological self-commentary on a 'journey', a rhetorical description of thought processes for the interviewer, and he construction of a text addressing the rhetorical demands of written mathematics in Nora's social (school) context. Tools developed by researchers in problem-solving and representation theory in mathematics education could take aspects of this analysis further.

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