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An Optimal Auction for Complements

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This paper considers the optimal selling mechanism for complementary items. When buyers are perfectly symmetric, the optimal procedure is to bundle the items and run a standard auction. In general, however, bundling the items is not necessarily desirable, and the standard auctions do not maximize revenue. Moreover, the optimal auction allocation may not be socially efficient since the auction must discriminate against bidders who have strong incentives to misrepresent their true preferences. *Journal of Economic Literature* Classification Number: D44.

1. INTRODUCTION

A key issue faced by the designers of the recent Federal Communications Commission (FCC) auction was how to account for likely complementarities between spectrum licenses (McMillan, 1994). Should the licenses be bundled or sold separately? What format would allow for aggregation? How would complementarities affect the efficiency and revenue potential of the auction? These issues were informally discussed during the design process (McMillan, 1994; Cramton, 1995; Chakravorti *et al.*, 1995), but only recently has formal analysis begun to look at the importance of complementarities for multiobject auctions.

In fact, complementarities between items at auction are not unusual. Firms bidding for franchises or distributorships in adjacent territories may find it cheaper to run them as a single unit rather than individually. Bidders at estate sales may see value in keeping parts of a collection intact. Producers bidding for contracts may realize that there are certain learning costs or specific investments that must be incurred regardless of the number of contracts they are awarded. Rassenti *et al.* (1982) consider the

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case of airport landing slot allocation, an extreme example of interdependence, where bidders want to submit combinatorial bids. In all these examples, bidders' desires to aggregate items can play a crucial role.

Recent papers motivated by the FCC auction and these other examples have focused on the strategic effects and potential inefficiencies caused by complementarities. Krishna (1993) gives an example where sequential auctions lead to the benefits of aggregation not being realized. She then goes on to study a monopolist bidding for capacity blocks against a stream of entrants. Levin (1996) explores further the potential for inefficiency in a sequential auction model where both parties can benefit from combining the items. And Krishna and Rosenthal (1995) examine simultaneous sealed bid auctions where bidders may either be "local," desiring one item, or "global."

Here, we characterize the auction format that will generate the most revenue for the seller. Maximizing revenue is clearly of paramount importance in many of the examples given above (although efficiency was the principle objective in the FCC auction). Gale (1990) has shown that a seller of identical goods selling to bidders with superadditive values will maximize revenue by bundling. We seek to find the revenue-maximizing auction for the sale of complements in a significantly more general case.[1]

The surprising result of Myerson (1981) and Riley and Samuelson (1981) is that, in many cases, the standard first price and second price auctions not only generate the same revenue on average, but, given the correctly chosen reserve price, are optimal ways to sell a single object.[2] These papers make a number of stringent assumptions. They consider sales of a single item, and look at symmetric risk-neutral bidders with independent valuations. For optimality, it is also necessary to impose a regularity assumption on the distribution of values. A major research program in the 15 years since the publication of these papers has been to relax these assumptions and consider the consequences. I am primarily concerned here with the assumptions of symmetry and of a single item.[3]

Some work has already been done towards examining these assumptions. Harris and Raviv (1981) demonstrate revenue equivalence for the case of

[1] In independent work, Branco (1995) pursues a similar goal, establishing a model where bidders are either local or global. He explores the efficiency of the optimal auction for this case and identifies a condition on the distributions of valuations which guarantees the allocation will be efficient.

[2] These papers greatly strengthen Vickrey's (1961) "revenue equivalence theorem" by proving optimality as well as equivalence.

[3] Myerson discusses the regularity assumption in his original paper. On the effects of risk-aversion, see Maskin and Riley (1984). When buyers are risk-averse, the seller benefits from making the losers pay as well as the winner. The optimal auction for correlated values, derived by Cremer and McLean (1985), doesn't resemble any standard procedure, but does allow the seller to extract *all* the surplus.

identical goods and a uniform distribution of values. They assume that each buyer demands only a single item, i.e., has unit demand. Maskin and Riley (1989) and Bulow and Roberts (1989), working again with unit demands, generalize Harris and Raviv's result to allow for general distributions and show that, with the correct reserve prices, the standard auctions are optimal. But Maskin and Riley show that the standard auctions are no longer optimal when buyers have downward sloping demands. In that case, the seller uses a nonlinear pricing scheme.

This paper takes the opposite tack from Maskin and Riley by requiring that goods be complements instead of substitutes. It also relaxes the assumption that the objects are identical. It turns out that the optimal auction for complements takes a form that is fairly different from the standard first price or second price sealed bid auction. In particular, bidders' values for the bundle of items (which includes their value for the complementarity) are considered explicitly rather than just entering through bids on the individual items. While the informational requirements to run a true optimal auction are high, this result suggests that bidding procedures which allow for combinatorial bids (e.g., Banks *et al.*, 1989) may be likely to generate more revenue than standard simultaneous or sequential procedures.[4]

In the optimal auction for complements, unlike in second price and symmetric first price auctions, it is also the case that the bidders with the highest values do not necessarily win. Because I allow for several potential asymmetries, this is hardly surprising. Myerson notes in his original paper that if bidder valuations are not identically distributed, the auction that maximizes revenue does not necessarily assign the items to the bidders with the highest valuations. The standard auctions cannot be optimal.[5] And just as Myerson's auction can discriminate against bidders with ex ante higher distributions for the item,[6] preventing them from winning even if they have the highest valuation, the mechanism described here can do the same. The model below allows for further complexity, however, since a buyer might for example have a high distribution for one item, a low distribution for a second, and a particularly high distribution for the two items together.

[4] Chakravorti *et al.* (1995) make the case that the FCC should have adopted a combinatorial auction. Branco (1995) also emphasizes the importance of allowing combinational bids.

[5] It is possible to construct examples of asymmetric first price auctions where the bidder with the highest value does not always win. But the allocation is still not the same as the optimal auction.

[6] More precisely, Myerson's auction discriminates against those bidders with the greatest incentive to report their preferences deceitfully. If bidders have similarly shaped and sufficiently regular value distributions, the auction relatively favors bidders with lower ex ante expected valuations. See also Example 4 below.

The key assumption here is that bidders' valuations can be parametrized by a single "type." While this is somewhat unsatisfactory, the general multiobject problem involves solving a multidimensional mechanism design problem. There have been some advances on this theoretical front,[7] but the general multidimensional problem awaits resolution. One way to defend the approach here is to assume that the seller knows bidders' preferences but not, say, their incomes. Or correspondingly, their cost structures but not their underlying cost parameter.

2. PREFERENCES

For ease of exposition, we consider the case of a seller with two indivisible objects.[8] There are N potential buyers, $i = 1, \dots, N$, who have valuations of $g_{i1}(\theta_i)$ for the first good, $g_{i2}(\theta_i)$ for the second good, and $g_{i1}(\theta_i) + g_{i2}(\theta_i) + c_i(\theta_i)$ for both goods, where θ_i is bidder i's "type." Note that $c_i(\cdot)$ is additional value due to owning both objects together. Formally,[9] bidder i has preferences:

$$\begin{aligned} u_i &= V_i(x_i, \theta_i) + t_i \\ &= g_{i1}(\theta_i)x_{i1} + g_{i2}(\theta_i)x_{i2} + c_i(\theta_i)x_{i1}x_{i2} + t_i, \end{aligned}$$

where x_i is the vector (x_{i1}, x_{i2}), x_{ij} is the amount of good j won by i, $-t_i$ is his payment in the auction, and θ_i is distributed on $[\underline{\theta}_i, \overline{\theta}_i]$ with cumulative density function $F_i(\cdot)$. Assume that the density functions $f_i(\cdot)$ are positive and bounded everywhere on $[\underline{\theta}_i, \overline{\theta}_i]$. Since the items are indivisible, x_{ij} will take the value zero or one ex post, but we can alternatively interpret x_{ij} as the probability that bidder i receives good j. There is no resale. We make the following assumptions:

Assumption 1 (Preferences). For all i, $g_{i1}(\cdot)$, $g_{i2}(\cdot)$, and $c_i(\cdot)$ are non-negative and differentiable, with $g'_{i1}, g'_{i2} > 0$, $c'_i \geq 0$, and $g''_{i1}, g''_{i2}, c''_i \leq 0$.

[7] McAfee and McMillan (1988), and more recently Armstrong (1996), have looked at monopoly selling strategies with multidimensional type spaces.

[8] The case with M objects is essentially the same, only with more notation.

[9] The notation and presentation follows Fudenberg and Tirole (1991, Chap. 7.5).

Assumption 2 (Monotone hazard-rate condition). For all i, $F_i(\cdot)$ satisfies the monotone hazard-rate condition:[10]

$$\frac{d}{d\theta_i}\left[\frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right] \leq 0.$$

We also make a final assumption about the nature of the complementarities between the goods at auction. This condition will be necessary to guarantee that bidders never get a worse allocation of objects by revealing a higher type.

Assumption 3 (Strict complementarity). For all i,

$$c_i(\theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} c_i'(\theta_i) \geq 0.$$

Note that we allow $c_i(\theta_i)$ to equal a nonnegative constant. If $c_i(\theta_i)$ is increasing, we require that the value of the complementarity not increase too fast in θ_i and that $c_i(\underline{\theta}_i)$ is sufficiently positive.

3. THE SELLER'S PROBLEM

From the Revelation Principle, we know the seller can restrict herself to direct revelation mechanisms, which means she must maximize her expected profits subject to the constraint that bidders reveal their true preferences. We say the procedure must be "incentive compatible" in that it must provide bidders with the right inducements to be honest. In addition, we require that given his type, each bidder must anticipate nonnegative gains from the auction. This is the "voluntary participation" or "individual rationality" constraint, which guarantees that bidders will not choose to opt out of the auction.

Suppose the seller has zero valuation for all bundles (if she has positive valuation we need to adjust the reservation price, but this poses no additional problems) and introduces the following notation. Allow θ to

[10] A discussion of optimal auctions when the monotone hazard-rate condition is not satisfied can be found in, among other places, Bulow and Roberts (1989). For the purposes here, the monotone hazard-rate condition and the assumptions that g_{ij} and c_i have nonpositive second derivatives are unnecessarily strong; but they are the most obvious way to get the necessary regularity conditions.

represent the profile of bidder types, $\theta = (\theta_1, \dots, \theta_N)$. The seller maximizes

$$R = \mathrm{E}_\theta \sum_{i=1}^{N} - t_i(\theta), \tag{1}$$

subject to incentive compatibility,

$$\forall\big(i, \theta_i, \hat{\theta}_i\big), \qquad \begin{aligned} &\mathrm{E}_{\theta_{-i}}\big[V_i\big(x_i(\theta_i, \theta_{-i}), \theta_i\big) + t_i(\theta_i, \theta_{-i})\big] \\ &\quad \geq \mathrm{E}_{\theta_{-i}}\Big[V_i\Big(x_i\Big(\hat{\theta}_i, \theta_{-i}\Big), \theta_i\Big) + t_i\Big(\hat{\theta}_i, \theta_{-i}\Big)\Big], \end{aligned} \tag{2}$$

individual rationality,

$$\forall(i, \theta_i), \qquad \mathrm{E}_{\theta_{-i}}\big[V_i\big(x_i(\theta_i, \theta_{-i}), \theta_i\big) + t_i(\theta_i, \theta_{-i})\big] \geq 0, \tag{3}$$

and the obvious quantity constraints,

$$\forall(i, \theta), \qquad x_{i1}(\theta), x_{i2}(\theta) \geq 0, \tag{4}$$

$$\forall(j, \theta), \qquad \sum_{i=1}^{N} x_{ij}(\theta) \leq 1. \tag{5}$$

If we think of the auction mechanism as a game of imperfect information, the incentive compatibility constraint guarantees the existence of a Bayesian Nash equilibrium in which all bidders reveal the truth.

As stated, this is simply the standard optimal auction problem. Following the familiar procedure from Fudenberg and Tirole (1991), we can reformulate it in terms of the surplus that bidders expect from the auction. Given type θ_i, bidder i's expected surplus is

$$U_i(\theta_i) = \mathrm{E}_{\theta_{-i}}\big[V_i\big(x_i(\theta_i, \theta_{-i}), \theta_i\big) + t_i(\theta_i, \theta_{-i})\big]. \tag{6}$$

This allows us to rewrite the equation for expected revenue:

$$R = \mathrm{E}_\theta\left[\sum_{i=1}^{N} V_i\big(x_i(\theta), \theta_i\big)\right] - \sum_{i=1}^{N} \mathrm{E}_{\theta_i} U_i(\theta_i). \tag{7}$$

From the envelope theorem, we know

$$\frac{dU_i}{d\theta_i} = \mathrm{E}_{\theta_{-i}} \frac{\partial V_i}{\partial \theta_i}\big(x_i(\theta_i, \theta_{-i}), \theta_i\big),$$

which implies

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \mathrm{E}_{\theta_{-i}} \frac{\partial V_i}{\partial \theta_i}\Big(x_i\Big(\tilde{\theta}_i, \theta_{-i}\Big), \tilde{\theta}_i\Big)\, d\tilde{\theta}_i. \tag{8}$$

If the seller is to maximize revenue, it must be true that $U_i(\underline{\theta}_i) = 0$, since the seller should never leave any surplus to a buyer with the lowest type. We now note that the expected surplus to bidder i can be written as

$$\begin{aligned} \mathrm{E}_{\theta_i} U_i(\theta_i) &= \mathrm{E}_{\theta_i} \int_{\underline{\theta}_i}^{\theta_i} \mathrm{E}_{\theta_{-i}} \frac{\partial V_i}{\partial \theta_i}\left(x_i\left(\tilde{\theta}_i, \theta_{-i}\right), \tilde{\theta}_i\right) d\tilde{\theta}_i \\ &= \mathrm{E}_\theta \left[\frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial V_i}{\partial \theta_i}(x_i(\theta), \theta_i) \right]. \end{aligned} \tag{9}$$

Substituting (9) into (7), we can rewrite the revenue equation again as

$$R = \mathrm{E}_\theta \sum_{i=1}^{N} \left(V_i(x_i(\theta), \theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial V_i}{\partial \theta_i}(x_i(\theta), \theta_i) \right). \tag{10}$$

The optimal auction maximizes (10) with respect to (2), (3), and the two quantity constraints.

4. SOLVING THE COMPLEMENTS PROBLEM

So far the analysis is standard, but now we substitute in bidders' preferences. The seller's objective function is thus

$$\begin{aligned} \mathrm{E}_\theta \sum_{i=1}^{N} \Bigg[& g_{i1}(\theta_i) x_{i1} + g_{i2}(\theta_i) x_{i2} + c_i(\theta_i) x_{i1} x_{i2} \\ & - \left(\frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) \left(g'_{i1}(\theta_i) x_{i1} + g'_{i2}(\theta_i) x_{i2} + c'_i(\theta_i) x_{i1} x_{i2} \right) \Bigg]. \end{aligned} \tag{11}$$

To proceed, we introduce the notation

$$MR_{i1}(\theta_i) = g_{i1}(\theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} g'_{i1}(\theta_i),$$

$$MR_{i2}(\theta_i) = g_{i2}(\theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} g'_{i2}(\theta_i),$$

$$MR_{iC}(\theta_i) = c_i(\theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} c'_i(\theta_i),$$

where $MR_{i1}(\theta_i)$ is the "marginal revenue"[11] generated by giving item 1 to bidder i, $MR_{i2}(\theta_i)$ is the marginal revenue from allotting him good 2, and

[11] Bulow and Roberts (1989) introduce the "marginal revenue" interpretation in order to exploit the links between monopoly theory and auctions. The marginal revenue interpretation proves very useful; see Bulow and Klemperer (1996) for an example of its application.

$MR_{iC}(\theta_i)$ is the extra marginal revenue generated by allotting him the two items as a bundle. The seller, who we treat simply as an additional bidder, has by definition $MR_{s1} = MR_{s2} = MR_{sC} = 0$.

Notice that by the monotone hazard-rate condition (Assumption 2) and the fact that g_{ij} and c_i are increasing and (weakly) concave (Assumption 1), MR_{i1}, MR_{i2}, and MR_{iC} are all increasing in θ_i.[12] Moreover, by Assumption 3, MR_{iC} is always nonnegative. These prove to be the key conditions for sustaining incentive compatibility.

PROPOSITION 1. *The seller's problem is to maximize*

$$\mathrm{E}_{\theta} \sum_{i=1}^{N} MR_{i1}(\theta_i) x_{i1} + MR_{i2}(\theta_i) x_{i2} + MR_{iC}(\theta_i) x_{i1} x_{i2}, \tag{12}$$

subject to incentive compatibility, individual rationality, and the quantity constraints, and using a payment scheme that awards zero surplus to bidders of the lowest possible type.

Since the seller knows the type of each bidder when she makes the allocation, we can ignore the expectation operator and take θ as given. So the seller's problem is to choose x_{ij}'s to maximize (12) subject to the quantity constraints (4) and (5).

LEMMA 1. *If the seller chooses x_{ij}'s to maximize* (12) *subject to* (4) *and* (5), *it will be the case that at the maximum, $x_{j1} = 1$, $x_{k2} = 1$ for some j, k possibly equal, with $x_{i1} = 0$ $\forall i \neq j$ and $x_{i2} = 0$ $\forall i \neq k$.*

Proof. See the Appendix.

This implies that the seller would never want to resort to a random mechanism to distribute the items.

Implementing the allocation is easy. The seller must simply check the value of (12) for each of $(N+1)^2$ possible allocations—with M objects, there are $(N+1)^M$ allocations.[13]

What are the implications of this procedure? As is the case for a single object where bidders have asymmetric distributions, the revenue-maximizing and socially efficient allocations do not necessarily coincide. There are three factors that can prevent high-valued bidders from being assigned an item. First, $(1 - F_i(\theta_i))/f_i(\theta_i)$ may be particularly large. This could work against a bidder whose type distribution has a high support, but who has a particularly low value from this distribution. Second, $g'_{ij}(\theta_i)$ might be large.

[12] These are really the three regularity conditions we require. Assuming weak concavity and the monotone hazard-rate condition is just an easy way to guarantee regularity.

[13] Notice that the seller may end up keeping one or both of the items. That is, she essentially sets a reserve marginal revenue of zero, below which she does not sell.

Last, a large $c_i'(\theta_i)$ works against a bidder who might potentially win both items. An essential point concerning the last two: these are the bidders who have the most incentive at the margin to misrepresent their true preferences. In order to ensure that they reveal the truth, the seller must trade off allocative efficiency.

Our remaining task is to address the problems of incentive compatibility and voluntary participation, which we have thus far neglected. To guarantee that these are satisfied, we need to specify an appropriate payment rule. Before doing that, however, it is helpful to prove the following important lemma.

LEMMA 2 (Allocation monotonicity). *If a bidder reporting $\tilde{\theta}_i$ receives some item, he receives that item for any report $\theta_i \geq \tilde{\theta}_i$.*

Proof. We prove this first for a bidder who receives exactly one item and then for a bidder who receives two items.

Suppose bidder i reports $\tilde{\theta}_i$, and receives one item. Without loss of generality, let this be item 1. And suppose that bidder j receives item 2. Then

$$MR_{i1}(\tilde{\theta}_i) \geq MR_{k1}(\theta_k), \qquad k \neq i, \tag{13}$$

and furthermore,

$$\begin{aligned}
&MR_{i1}(\tilde{\theta}_i) + MR_{j2}(\theta_j) \\
&\quad = \max_{x_1, x_2} \sum_{k=1}^{N} \left[MR_{k1}(\theta_k) x_{k1} + MR_{k2}(\theta_k) x_{k2} + MR_{kC}(\theta_k) x_{k1} x_{k2} \right] \\
&\quad \geq \max_{\substack{x_1, x_2 \\ \text{s.t. } x_{i1} = x_{i2} = 0}} \sum_{k=1}^{N} \left[MR_{k1}(\theta_k) x_{k1} + MR_{k2}(\theta_k) x_{k2} + MR_{kC}(\theta_k) x_{k1} x_{k2} \right] \\
&\quad = \text{max marginal revenue} \mid i \text{ wins nothing.}
\end{aligned}$$

Since $MR_{i1}(\theta_i)$ is increasing in θ_i, we know that for all $\theta_i > \tilde{\theta}_i$,

$$MR_{i1}(\theta_i) + MR_{j2}(\theta_j) > \text{max marginal revenue} \mid i \text{ wins nothing,}$$

so it could not happen that by reporting $\theta_i \geq \tilde{\theta}_i$ bidder i would get *neither* item 1 nor item 2. Could i overreport and win item 2 and *not* item 1? This would imply that there exists some $\hat{\theta}_i > \tilde{\theta}_i$ and $k \neq i$ such that

$$MR_{k1}(\theta_k) + MR_{i2}(\hat{\theta}_i) \geq MR_{i1}(\hat{\theta}_i) + MR_{i2}(\hat{\theta}_i) + MR_{iC}(\hat{\theta}_i),$$

which would imply $MR_{k1}(\theta_k) \geq MR_{i1}(\hat{\theta}_i) + MR_{iC}(\hat{\theta}_i) > MR_{i1}(\tilde{\theta}_i)$ since $MR_{iC}(\hat{\theta}_i) \geq 0$. But this of course contradicts (13). So if bidder i gets only item j with report $\tilde{\theta}_i$, he will receive item j for any report $\theta_i \geq \tilde{\theta}_i$.

Now suppose bidder i reports $\tilde{\theta}_i$ and receives both items. Then it must be the case that the maximized value of (12) is $MR_{i1}(\tilde{\theta}_i) + MR_{i2}(\tilde{\theta}_i) + MR_{iC}(\tilde{\theta}_i)$. All three terms are increasing functions, so it is clearly the case that for any $\theta_i \geq \tilde{\theta}_i$, (12) will again be maximized by awarding both items to bidder i. Q.E.D.

COROLLARY 1. *A bidder who reports* $\hat{\theta}_i$ *and does not receive a particular item can never receive that item by reporting* $\theta_i \leq \hat{\theta}_i$.

We now specify the payment scheme for the auction.

Payment Rule. 1. Bidders who receive no items pay nothing.

2. For a bidder i who gets only item j, let θ_i^0 denote the lowest value he could have announced and been awarded the item. Then he pays $g_{ij}(\theta_i^0)$.

3. For a bidder i who gets two items, let θ_i^1 be the lowest value he could have announced and been awarded two items, and let θ_i^0 be the lowest value he could have announced and been awarded one item. Suppose announcing θ_i^0 corresponds to winning item 1. Then this bidder pays $g_{i1}(\theta_i^0) + g_{i2}(\theta_i^1) + c_i(\theta_i^1)$. If announcing θ_i^0 corresponds to winning item 2, he pays $g_{i1}(\theta_i^1) + g_{i2}(\theta_i^0) + c_i(\theta_i^1)$.

The intuition behind the payment scheme is that winning bidders should not pay more than the benefits they receive from having the lowest possible type that would guarantee them their bundle. Thus the price paid by a winner depends on the reports of the other bidders.

PROPOSITION 2. *Given the preferences described above, and under Assumptions* 1–3, *the allocation and payment rules above fully describe the optimal auction. Furthermore, in equilibrium all bidders reveal the truth*.

To prove this proposition, we need to verify (a) that bidders of lowest type receive zero surplus, (b) that bidders expect nonnegative surplus and will participate voluntarily, and (c) that revealing the truth is an equilibrium strategy for the bidders. Assuming truthful revelation, the first two are easily checked. Observe that if a bidder reveals a type lower than his lowest possible type, it is the case that $MR_{i1} = MR_{i2} = MR_{iC} = -\infty$. So no bidder could ever receive an item by revealing a type below their lowest possible type.

Then if bidder i has type $\underline{\theta}_i$, and receives some bundle x_i, he gains benefits $V_i(x_i, \underline{\theta}_i) = g_{i1}(\underline{\theta}_i)x_{i1} + g_{i2}(\underline{\theta}_i)x_{i2} + c_i(\underline{\theta}_i)x_{i1}x_{i2}$ and pays the same amount since he could not have announced less and won. So his total surplus is zero and we have checked (a). Now notice that winning bidders are never required to pay more than they gain from the auction—since in fact the maximum amount they could pay corresponds to their true benefit. Because losing bidders pay nothing, it is obvious that the voluntary participation constraint is satisfied. No bidder can ever end up with negative returns from the auction.

Finally, we prove by way of three lemmas that truthful revelation is a (weakly) dominant strategy.

LEMMA 3. *A bidder who wins no items by truth-telling can never do better by lying.*

Proof. Consider a bidder i of type θ_i who will receive nothing if he tells the truth; because i also pays nothing, his total surplus is zero. If he reports $\theta_i^{\text{low}} < \theta_i$, he still gets nothing (by allocation monotonicity) and pays nothing. So there is no benefit to underreporting. If i reports $\theta_i^{\text{high}} > \theta_i$, it is possible that he will now receive one or more items. If he receives one item, say item 1, he will pay $g_{i1}(\theta_i^0)$, where $\theta_i^{\text{high}} \geq \theta_i^0 > \theta_i$. But $g_{i1}(\theta_i^0) > g_{i1}(\theta_i)$, so he gets negative surplus from the auction. If i wins two items, by a similar argument it is easy to see that he also gets negative surplus. Thus truth-telling must be at least as good as any other strategy. Q.E.D.

LEMMA 4. *A bidder who wins one item by truth-telling can never do better by lying.*

Proof. Suppose bidder i gets item 1 but not item 2 if he reveals the truth, and is required to pay $g_{i1}(\theta_i^0)$, where θ_i^0 is the minimum he could report and still win item 1. If i overreports, he will still win item 1. If he only wins 1, he pays the same amount and gets the same surplus. However, he may win item 2 as well. In this case, he pays $-t_i^{\text{lie}} = g_{i1}(\theta_i^0) + g_{i2}(\theta_i^1) + c_i(\theta_i^1)$, where $\theta_i^1 > \theta_i$. But this is not optimal, since his total surplus from lying will be less than his total surplus from telling the truth:

$$g_{i1}(\theta_i) + g_{i2}(\theta_i) + c_i(\theta_i) + t_i^{\text{lie}} < g_{i1}(\theta_i) - g_{i1}(\theta_i^0).$$

What if i underreports? From Corollary 1, i cannot underreport and win item 2. Either he wins nothing and gets zero surplus, or he still wins item 1 and pays the same amount as before, $g_{i1}(\theta_i^0)$. In neither event does he do better than if he reveals truthfully. Q.E.D.

LEMMA 5. *A bidder who wins two items by truth-telling can never do better by lying.*

Proof. Suppose if i reveals the truth, θ_i, he gets two items. If i overreports, he still gets two items and pays the same amount. So i cannot improve his payoff by overreporting. On the other hand, if i underreports, he could get 2, 1, or 0 items. Let θ_i^1 be the minimum type i could report and still win both items, and let θ_i^0 be the minimum type he could report and still win one item. Without loss of generality, let item 1 be the item i will win by revealing θ_i^0. Note that $\theta_i \geq \theta_i^1 \geq \theta_i^0$.

Suppose i reveals $\tilde{\theta}_i \epsilon [\theta_i^1, \theta_i]$. Then i gets payoff

$$g_{i1}(\theta_i) + g_{i2}(\theta_i) + c_i(\theta_i) - g_{i1}(\theta_i^0) - g_{i2}(\theta_i^1) - c(\theta_i^1),$$

regardless of whether he reveals $\tilde{\theta}_i$ or θ_i. So nothing is gained by lying in this case.

Now suppose i reveals $\breve{\theta}_i \epsilon [\theta_i^0, \theta_i^1]$. Then i must get item 1 but not item 2. His payoff will be

$$\begin{aligned} &g_{i1}(\theta_i) - g_{i1}(\theta_i^0) \\ &\quad \leq g_{i1}(\theta_i) + g_{i2}(\theta_i) + c_i(\theta_i) - g_{i1}(\theta_i^0) - g_{i2}(\theta_i^1) - c_i(\theta_i^1) \\ &\quad = \text{payoff from truth-telling}. \end{aligned}$$

Finally, if bidder i reports $\hat{\theta}_i < \theta_i^0$, he receives no items and pays nothing, which cannot be better than his truth-telling payoff. So bidder i never can do better by lying. Q.E.D.

Lemmas 3–5 show that for every bidder telling the truth is at least as good as any other strategy. It follows immediately that there is an equilibrium where all bidders reveal their true types.

5. EXAMPLES

In this section, we consider several examples to demonstrate the implications of the optimal auction for complements. In all examples, we impose Assumptions 1–3 on bidder preferences.

EXAMPLE 1. Let all bidders, $i = 1, \ldots, N$, have preferences $g_{i1}(\theta_i) = \theta_i$, $g_{i2}(\theta_i) = 0$, $c_i(\theta_i) = 0$, for all θ_i. That is, their type is simply their value for the first item. They have zero value for the second item and no extra value for the bundle. As we would expect, the mechanism described above is exactly the standard optimal auction for a single item.

EXAMPLE 2. Let bidders have preferences $c_i(\theta_i) = 0 \ \forall \theta_i$. Then the seller's problem is

$$\max_{x_{i1}, x_{i2}} \sum_{i=1}^{N} [MR_{i1} x_{i1} + MR_{i2} x_{i2}],$$

which is equivalent to

$$\max_{x_{i1}} \sum_{i=1}^{N} MR_{i1} x_{i1} + \max_{x_{i2}} \sum_{i=1}^{N} MR_{i2} x_{i2}.$$

Since there is truthful revelation, this is equivalent to running two independent optimal auctions—one for each item.

EXAMPLE 3 (Symmetry). Suppose that bidders' valuations are symmetrically distributed. That is, allow $F_i(\cdot) = F(\cdot)$, $g_{i1}(\cdot) = g_1(\cdot)$, $g_{i2}(\cdot) = g_2(\cdot)$, and $c_i(\cdot) = c(\cdot)$ for $i = 1, \ldots, N$. Then the optimal way to auction items 1 and 2 is simply to bundle them together and run an optimal auction for the pair. But this, as we know, can be done with a standard first or second price auction given the correct reserve price. Thus, we have

PROPOSITION 3. *Under perfect symmetry, the optimal auction for complementary items is simply a standard auction for the bundle.*

EXAMPLE 4. To highlight how the mechanism differs from a more standard auction procedure, we consider the set-up from Levin (1996). Suppose there are two bidders who have valuations:

$$\text{Bidder 1:} \qquad g_{11}(\theta_1) = \theta_1, \, g_{12}(\theta_1) = k_1\theta_1, \, c_1(\theta_1) = c,$$

$$\text{Bidder 2:} \qquad g_{21}(\theta_2) = \theta_2, \, g_{22}(\theta_2) = k_2\theta_2, \, c_2(\theta_2) = c.$$

Suppose that $k_2 > k_1$ and that $F_1(\cdot) = F_2(\cdot) = F(\cdot)$. The seller may either allocate the bundle to a single bidder or split the items between bidders. Since $k_2 > k_1$, if the seller splits the items she will always want to give item 1 to bidder 1 and item 2 to bidder 2. We can calculate the marginal revenue generated by each allocation:

$$MR(\text{Both to bidder 1}) = c + \theta_1 + k_1\theta_1 - \frac{1 - F(\theta_1)}{f(\theta_1)}(1 + k_1), \tag{14}$$

$$MR(\text{Both to bidder 2}) = c + \theta_2 + k_2\theta_2 - \frac{1 - F(\theta_2)}{f(\theta_2)}(1 + k_2), \tag{15}$$

$$MR(\text{Bidder } i \text{ gets } i) = \theta_1 + k_2\theta_2 - \frac{1 - F(\theta_1)}{f(\theta_1)} - \frac{1 - F(\theta_2)}{f(\theta_2)}k_2. \tag{16}$$

Suppose that c is large so that (14), (15) > (16). That is, bundling the items will always be optimal. Levin (1996) examines the outcome when the seller auctions the goods sequentially, with good 1 being sold first. He shows that bidder 2 may receive the bundle despite having a lower total valuation.

The sequential format gives a disproportionate strategic advantage to the bidder with the higher second period valuation. The optimal auction has very different implications. Now bidder 1 may win both items despite having lower total value for the goods. To see this most clearly, suppose that θ_i is distributed uniformly on $[0, 1]$ for $i = 1, 2$. Then bidder 1 has higher total valuation if

$$\theta_1 + k_1\theta_1 > \theta_2 + k_2\theta_2,$$

receives both items in a sequential auction if

$$\theta_1 + 2k_1\theta_1 > \theta_2 + 2k_2\theta_2,$$

and receives both items in the optimal auction if

$$\theta_1 + k_1\theta_1 - \tfrac{1}{2}k_1 > \theta_2 + k_2\theta_2 - \tfrac{1}{2}k_2.$$

The optimal auction favors bidder 1 even more than is socially optimal. Why? Because we need to give sufficient incentives for the bidders with ex ante higher distributions to reveal the truth. The cost of these incentives is that sometimes we have to give up social efficiency.

EXAMPLE 5. Finally, we look at an example that explores further the potential for allocative inefficiency. We consider a case with two bidders: one who values item 2 only if he already has item 1, another who views the objects as identical and without complementarities,

$$\begin{array}{llll} \text{Bidder 1:} & g_{11}(\theta_1) = \theta_1, & g_{12}(\theta_1) = 0, & c_1(\theta_1) = c, \\ \text{Bidder 2:} & g_{21}(\theta_2) = \theta_2, & g_{22}(\theta_2) = \theta_2, & c_2(\theta_2) = 0. \end{array}$$

Assume that θ_i is distributed uniformly on $[1, 2]$ for $i = 1, 2$. Depending on the values of θ_1, θ_2, and c, the optimal auction will either allocate the goods as a bundle or give item 1 to bidder 1 and item 2 to bidder 2. The marginal revenue generated by these allocations is

$$MR(\text{Both to bidder 1}) = (2\theta_1 - 2) + c \tag{17}$$

$$MR(\text{Both to bidder 2}) = (2\theta_2 - 2) + (2\theta_2 - 2) \tag{18}$$

$$MR(\text{Bidder } i \text{ gets } i) = (2\theta_1 - 2) + (2\theta_2 - 2). \tag{19}$$

The resulting outcomes are determined by the following inequalities:

$$\begin{array}{rlcl} \text{Both to Bidder 1:} & \theta_2 < 1 + \tfrac{1}{2}c & \text{and} & 2\theta_2 < \theta_1 + 1 + \tfrac{1}{2}c, \\ \text{Both to Bidder 2:} & \theta_2 > \theta_1 & \text{and} & 2\theta_2 > \theta_1 + 1 + \tfrac{1}{2}c, \\ \text{Split Items:} & \theta_1 > \theta_2 & \text{and} & \theta_2 > 1 + \tfrac{1}{2}c, \end{array}$$

We can contrast this with the socially efficient allocation:

$$\begin{array}{rlcl} \text{Both to Bidder 1:} & \theta_2 < c & \text{and} & 2\theta_2 < \theta_1 + c, \\ \text{Both to Bidder 2:} & \theta_2 > \theta_1 & \text{and} & 2\theta_2 > \theta_1 + c, \\ \text{Split Items:} & \theta_1 > \theta_2 & \text{and} & \theta_2 > c. \end{array}$$

If c equals 2, the allocations are identical. If c is less than 2, the optimal auction will tend to discriminate in favor of bidder 1. Bidder 1 will win more than socially optimal, bidder 2 will win less than socially optimal, and the goods will be divided between bidders less than is socially efficient. On the other hand, if c is greater than 2, bidder 1 will win too little, bidder 2 will win too much, and the items will be split up more often than they should be. The intuition here is that when the complementarity is large, bidder 1 has a bigger incentive to misrepresent his true preferences and the auctioneer must provide a disincentive. This leads both to bidder 2 winning the bundle more often *and* to less bundling than society would desire.

PROPOSITION 4. *The optimal auction may award the goods as a bundle either more or less often than is socially efficient*.

6. CONCLUSION

This paper has addressed the question of how a seller holding complementary goods might best arrange a sale of those items. We have found that when bidders are perfectly symmetric, the optimal auction is equivalent to a standard first or second price auction for the bundle of goods. In general, however, the standard auctions will not be optimal. And depending on bidder valuations, the optimal auction may or may not allocate the items as a bundle. Moreover, when there are asymmetries, the optimal revenue-maximizing auction does not necessarily lead to a socially efficient outcome. This inefficiency can take several forms: too much or too little bundling and/or discrimination against particular bidders. Finally, we have seen that if bidders' value distributions are sufficiently regular and have sufficiently similar shapes the optimal auction discriminates against those buyers who have an intrinsic ex ante advantage. While the mechanism described here requires too much information to be feasibly implemented except in special cases, the fact that it differs markedly from the standard formats suggests that we should seek procedures which explicitly account for the presence of complementarities between items. The present mechanism provides a benchmark against which proposed multiobject auction rules might be measured.

APPENDIX

Proof of Lemma 3.1. Consider assigning a portion of good 1 to both bidder j and bidder k. That is, set $x_{j1} = \alpha_j$, $x_{k1} = \alpha_k$, where $0 < \alpha_j$, $\alpha_k < 1$, and $\alpha_j + \alpha_k \leq 1$. Suppose we do not give j or k any part of item 2. If $MR_{j1} \leq MR_{k1}$, (12) is at least as high if we give k's portion of good 1 to j, and vice versa with $MR_{k1} > MR_{j1}$. So it cannot be strictly optimal to split an item between two bidders who receive none of the other item.

Now let j receive a portion β_j of item 2, with $0 < \beta_j \leq 1$, and k receive none of item 2. Then either $MR_{j1} + \beta_j MR_{jC} \geq MR_{k1}$, in which case we can give k's portion of the first item to j, or $MR_{k1} > MR_{j1} + \beta_j MR_{jC}$, and we can raise (12) by giving the joint portion of good 1 to k.

Finally, if j receives a portion β_j of item 2 and k receives a portion β_k, then either $MR_{j1} + \beta_j MR_{iC} \geq MR_{k1} + \beta_k MR_{kC}$ or vice versa, so it cannot be strictly optimal to divide item 1 between them. Q.E.D.

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