AGENTS' STRATEGIC BEHAVIOR AND RISK-SHARING INEFFICIENCY

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Abstract. We consider the market of n financial agents who aim to increase their expected utilities by sharing their random incomes. Given the optimal sharing rules, we address the situation where agents do not share their true random endowments, but instead they report as endowments the random quantities that maximize their expected utility when the sharing rules are applied. It is shown that this strategic behavior results in a Nash-equilibrium type of agreement among the agents, which implies an inefficient risk sharing. Under quadric utility functionals, we give closed form solutions for this Nash equilibrium and discuss the associated findings. The effect of a similar agents' strategic behavior is studied in the oligopoly over-the-counter market of finite financial securities, whose equilibrium prices are determined by the equality of demand and supply. The resulting risk sharing inefficiency is even more intense if the agents' participation in the market becomes an endogenous problem. Regarding this issue, we give conditions under which the participation of an extra agent is beneficial for all the existed ones. This discussion naturally leads to the problem of sub-group formation in the market, which is addressed for the first time in a financial risk sharing literature. A related example under quadratic utility functionals is extensively analyzed.

Keywords: Optimal risk sharing, equilibrium pricing, risk sharing games, formation of financial markets, Nash equilibrium in risk sharing, oligopoly financial markets.

JEL classification: G11, G13, G32, D53

1. Introduction

The concept of risk sharing is central in many economic applications ranging from rural and other goods' production to insurance and structured finance. This is because producers, insurers and financial agents often find that the sharing of their productions, insurance portfolios and investments is mutually beneficial in the sense that such transaction improves their decision criteria regarding risk. The majority of the literature in economics and finance models the participants' risk preferences by imposing utility functions (see among others [23]) or risk measures (see [9]). Then the optimal sharing of random incomes is the one that maximizes (minimizes) the sum of the utilities (risk measures) or their certainty equivalences in a Pareto optimal way. The question then is whether such optimal risk sharing (also called risk sharing rule) exists under different models and which factors determine its characterization.

In more practical situations though the optimal risk sharing is not always possible both for exogenous and endogenous reasons. In financial risk sharing markets for example the recent regulation in over-the-counter (OTC) markets and the induced transaction costs may make the transaction of the optimal risk sharing contracts very expensive. In these cases the agents can indirectly share their risks not by designing and pricing new (optimal) contracts, but instead by trading a vector of standardized securities (such as derivatives, preferred stocks and bonds, standard reinsurance contracts etc) and hence exploit their correlation with their risk exposures. This results in an equilibrium trading on the given vector of securities, which equilibrium price is the one that clears out the market (that is when the sum of agents' demand functions is equal to zero). The first task of this paper (Section 3) is to model such equilibrium transaction by assuming that the agents' criteria are based on a concave utility functional (see Definition 2.1) and when each agent's risk exposure is incorporated in a random variable written on a common probability space (similar model has been introduced in [5], see also [14] and [26]). We focus on transactions among financial agents where each one can be the seller (short position) or the buyer (long position) of the traded securities. Sufficient conditions for the existence and uniqueness of such equilibrium for a large family of utility functionals are stated in Theorem 3.1. We call this (partially) optimal risk sharing transaction, a *constrained risk sharing* to emphasize that agents share their random incomes only by trading a given vector of financial securities and not their incomes themselves. The case of mean-variance preferences (quadratic utility function) is analyzed extensively. For this special case of utility, we provide analytic formulas for the equilibrium price and allocation. These results can be considered as generalizations of the standard results in mean-variance equilibrium pricing (see [15] for an overview) under the presence of agent's random incomes and when there is no budget constrain (in the sense there is no bound for agent's borrowing cash). We then calculate the utility of each individual agent when the equilibrium transaction is applied (see Proposition 3.2). The induced formula implies that an agent's gain (in terms of utility) is an increasing function of the correlation between the vector of traded securities and the contract that she would have purchased at the optimal risk sharing transaction. We also conclude that the utility of each agent at the optimal risk sharing is always higher than the associated utility after the equilibrium transaction of any vector of securities which is different than the vector of agents' random incomes (i.e. all agents suffer a lower utility when the risk sharing is constrained, see Proposition 3.3).

The main concept of this paper however is to study the situation where the optimal risk sharing is not applied because of endogenous reasons and in particular the agents' strategic behavior. When the optimal risk sharing rules are designing, each agent reports the endowment that is going to share. The application of the optimal sharing rules is going to increase the utility of each of the agents. We consider then an individual agent and suppose that she knows the endowments that the other agents are willing to share. Since her criterion is the maximization of her utility function, it is reasonable to assume that she is not going to share her true endowment but rather she is going to report as endowment the random variable that maximizes her utility after the optimal rules are applied. We call this reported random variable the agent's *best endowment response*, which in principle depends on the mechanism of the optimal sharing rules and the endowments of the rest of the participants in the market.

In Section 4, we model this agent's strategic behavior at the optimal risk sharing setting in two different situations: (i) when agent's set of choices as reported endowment is all measurable to the given filtration random variables and (ii) when the set of choices is only a percentage of her true endowment. In the case where agent's criterion is based on quadratic utility, we are able to give closed form solutions of agents' best responses (see Propositions 4.1 and 4.2). In case (i), agent should report a fraction of her true endowment (which approaches to $1/2$ as her risk aversion goes to zero and to 1 when her risk aversion goes infinity) and a factor of the rest of the agents' aggregate endowment. That is, her optimal response is to report that she is exposed to less of her true risk and also to some risk similar to the sum of the other agents' risk. In case (ii), the percentage of her true endowment that should be shared can well be zero or even higher than one and according to Proposition 4.2 the main factor that determines the exact percentage is the correlation of her true endowment and the other agents' aggregate endowment (higher the correlation implies higher percentage). In both situations however, it is clear that it is almost never optimal for the agent to report as endowment her true one. Also, this strategic behavior is more intense for the agents with low risk aversion coefficient, which is an expected outcome since the implication of such strategy implies a speculation attitude of the agent.

A modified version of the above agent's strategic behavior can also be applied when the risk sharing is achieved by the equilibrium trading of a given vector of securities. In such situations, we adapt arguments similar to the ones in the classical oligopoly good markets. More precisely, we suppose that an agents knows the demand functions of the rest of the agents on the traded securities and her best response is the demand function that makes the market equilibrate at the price at which her utility function is maximized when the transaction is proceeded (see subsection 4.3 for the exact formulation). Under mean-variance preferences we are able to give the exact equilibrium price and agent's demand function (see Proposition 4.3 and Remark 4.4 for a related discussion).

The next question after modelling individual agent's strategic behavior is whether the market equilibrates when all the agents follow similar strategy. In other words, we address the problem of the existence (and the uniqueness) of the Nash-equilibrium in the case where agents negotiate the sharing of their endowments or the sharing of the percentage of their endowments or their demands on a given vector of securities. This problem is solved and discussed in subsection 4.4 for these three risk sharing settings (see Propositions 4.4, 4.5 and 4.6). Equilibria in risk sharing settings when agents follow a type of strategic behavior has recently been studied (see for example [34] and in [24]). However, the present paper is up to our knowledge, the first one which deals with Nash equilibria in the financial risk sharing where agents' best responses refer to their shared-to-be endowments.

These Nash-type equilibria in the risk sharing markets imply risk sharing rules that are not the optimal ones, i.e. inefficient risk sharing. We define the risk sharing inefficiency as the difference between the aggregate utility at the optimal risk sharing and the aggregate utility at the realized risk sharing (see equations (17) and (18) for the exact definition). For each of these inefficient risk sharing markets we measure their inefficiency and determine the factors that affect it. A summarized list of findings is the following: (a) the inefficiency of risk sharing is always positive and equals to zero if and only if the optimal risk sharing rule is a simple cash-only transaction among agents (i.e. constant contracts), (b) as the number of agents increases, the inefficiency approaches zero and (c) the inefficiency increases when the agents' endowment are more different (in the sense that the variance of their difference is relatively high).

Although the Nash equilibria reduce the aggregate utility, for individual agent's utility the transaction of Nash equilibrium may result higher utility when compared to the associated transaction of the Pareto equilibrium. As it is shown in Section 4, when an agent's risk aversion coefficient is sufficiently low, the Nash equilibrium is more preferable. This implies that Nash-type of games played among agents (i.e. when agents apply a speculation attitude when sharing risk) increase the utility of the least risk averse ones.

Section 5 deals with another kind of agents' behavior in risk sharing market, which addresses the question: *with whom should an agent share her risk?* Although, this kind of questions are common in structured finance, they have not been studied in financial economics literature (for related studies in rural markets we refer the reader to [10] and [27] and their references). In this paper, we consider risk sharing markets where the participation of agents is not given, but instead is considered as an endogenous issue. We introduce this kind of problem in the financial risk sharing literature by addressing two related questions under quadratic utility functions: First we give conditions under which the participation of an extra agent in the equilibrium transaction of a given vector of securities is beneficial for all the other agents; and second, we give an example of subgroup market formation which indicates that the increased risk sharing inefficiency can well be a result of the agents' choices to form subgroups in order to share their risk exposures. In the first question, it is shown (Proposition 5.2) that under any type of utility function, when there is only one traded security, an extra agent's participation is beneficial for the rest of the agents only if this extra agent is the only seller or the only buyer at the equilibrium. This means that a group of financial agents is very difficult to mutually accept an additional participant in their group. In fact, this happens only if the latter is so desperate to buy (or desperate to sell) the traded security that drives the equilibrium price at very high (or very low) levels. However, this situation is not generalized in the case where the number of traded securities is more than 2 (see counterexample 5.1). Finally, in subsection 5.2 we give a simple example for the risk sharing market formation in order to show how the agents' option to select the group of agents with whom they are going to share their risk results a significant risk sharing inefficiency.

The paper proceeds as follows: Section 2 describes the market model and the agents' decision criteria, which imply their demand functions for any vector of securities and the optimal sharing rules when this vector consists of agents' endowments. In Section 3, we deal with the equilibrium trading of a given vector of securities, i.e. the so-called constrained risk sharing. The main ideas of this paper are presented in Section 4, where three different cases of agent's behavior is modelled and the induced risk sharing Nash equilibria are defined and discussed. Finally, in Section 5, we address the market formation problem. For the reader's convenience, some of the proof are omitted from the main body of the paper and given at the Appendix A.

2. Market Set Up and Agents' Demand

We consider a two period market model of n financial agents aiming to reduce their risk exposures by proceeding to a transaction. It is assumed that there exists an exogenously priced numéraire (also called currency) in terms of which all the mentioned financial quantities are measured and discounted.

Each agent has an already undertaken risk exposure, which will be called *random endowment* (also called *random income*, *labor income* or *business income*). This quantity incorporates the net discounted payoffs of all the unhedgeable financial positions with maturity up to a given future time horizon T, that each agent has taken (such as positions on derivative products, insurance contracts, investments in real estates, illiquid stocks etc). We will denote these endowments by the random variables $\mathcal{E}_i$, $i \in \{1, 2, ..., n\}$ which are defined in a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$[1]. The sum of random endowments is called the *aggregate endowment* and is denoted by $\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}_i$. Below, $\mathcal{X}$ denotes the set of discounted payoffs of all possible financial positions and we assume that it is a linear subspace of $\mathbb{L}^0(\mathcal{F})$ (standard examples are $\mathbb{L}^2(\mathcal{F}, \mathbb{P})$ and $\mathbb{L}^\infty(\mathcal{F}, \mathbb{P})$). In words, $\mathcal{X}$ stands for a set of (regular) payoffs of financial positions that can be produced by the information incorporated in $\mathcal{F}$.

The agents' valuation criteria are modelled by *concave utility functionals*. The exact definition is given below.

Definition 2.1. *A map $\mathbb{U} : \mathcal{X} \to \mathbb{R}$ is called concave utility functional if it is concave, that is for every $\lambda \in [0, 1]$ and $X, Y \in \mathcal{X}$*

$$\mathbb{U}(\lambda X + (1 - \lambda)Y) \geq \lambda \mathbb{U}(X) + (1 - \lambda)\mathbb{U}(Y) \tag{1}$$

and cash invariant, that is for any $X \in \mathcal{X}$ and constant $c \in \mathbb{R}$

$$\mathbb{U}(X + c) = \mathbb{U}(X) + c. \tag{2}$$

The property of concavity stems from the classical theory in utility functions and in the concept of risk measurement refers to the principle that diversified portfolios do not increase risk. The additional property of cash invariance (also called translation invariance) means that $-\mathbb{U}(X)$ can be thought as the price of X that makes its purchase an investment of zero utility. This property also implies that $\mathbb{U}(X)$ is measured in currency units (i.e. in terms of the numéraire). In fact, $-\mathbb{U}(X)$ is a convex risk measure[2]. Concave utility functionals (or convex risk measures) of the above type have been recently used by many authors for modelling financial agents' risk preferences regarding a number of problems ranging from risk management (e.g [3] and [20]) and investment optimal strategies (e.g. [33]) to risk sharing (e.g. [7], [8], [13], [28]) and equilibrium pricing ([4], [14], [18]).

A standard example of such utility functionals is the *quadratic utility functional* (or *mean-variance preference*) that is given by

$$\mathbb{U}(X) = \mathbb{E}[X] - \gamma\, \mathbb{V}\mathrm{ar}[X] \tag{3}$$

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[1]Probability measure $\mathbb{P}$ is the so-called "*subjective*" probability measure and is assumed to be common for each agent. The withdrawal of this assumption does not change the general concept of this paper. For the effect of ambiguity in risk sharing, we refer the reader to [32].

[2]A convex risk measure is map from the space of payoffs to the real line, with the properties of convexity, cash invariance and monotonicity. In this paper, the assumption of monotonicity is generally not imposed.

where the constant $\gamma > 0$ and is called *risk aversion coefficient* of the associated agent and $\mathbb{E}[\cdot]$ and $\mathbb{V}\mathrm{ar}[\cdot]$ are the expectation and variance operations under probability measure $\mathbb{P}$. In this case, the set of security payoffs $\mathcal{X}$ is a subspace of $\mathbb{L}^2(\mathcal{F},\mathbb{P})$ (hereafter denoted simply by $\mathbb{L}^2(\mathcal{F})$).

Another well-known example of concave utility functional is the *certainty equivalent* of the exponential utility $u(x) = -e^{-\gamma x}$, that is the solution p of the equation $\mathbb{E}[u(X)] = u(p)$, where X is a given payoff. The resulting solution, $\mathbb{U}(X) = -\frac{1}{\gamma}\ln \mathbb{E}[e^{-\gamma X}]$, is called *entropic risk measure.*

In what follows, $\mathbb{U}_i$ stands for the utility functional of agent-i, for $i \in \{1, 2, ..., n\}$. Given her random endowment $\mathcal{E}_i$, agent-i evaluates any financial position $X \in \mathcal{X}$ using her (adapted to endowment) utility functional

$$\mathbb{U}_i(X + \mathcal{E}_i).$$

This function of X still satisfies the properties of the concave utility functional definition, but it also considers the already undertaken risky exposure of agent-i. Given her utility functional and the participation of the rest of the agents in the market, agent-i is going to take short or long positions on securities with payoff in $\mathcal{X}$ that increase her utility

$$\max_{X\in\mathcal{X}} \mathbb{U}_i(X + \mathcal{E}_i) \tag{4}$$

This is a version of utility maximization problem that has been studied by many authors in a variety of market models (see [31] for an extensive overview). A special example is the case where there exist k financial securities, $\mathbf{C} = (C_1, C_2, ..., C_k) \in \mathcal{X}^k$ in the market whose prices are given by a vector $\mathbf{p} \in \mathbb{R}^k$. Then problem (4) becomes

$$\sup_{\mathbf{a}\in\bar{\mathbb{R}}^k} \{\mathbb{U}_i(\mathbf{a}\cdot\mathbf{C} - \mathbf{a}\cdot\mathbf{p} + \mathcal{E}_i)\} = \sup_{\mathbf{a}\in\bar{\mathbb{R}}^k} \{\mathbb{U}_i(\mathbf{a}\cdot\mathbf{C} + \mathcal{E}_i) - \mathbf{a}\cdot\mathbf{p}\}. \tag{5}$$

The set of vectors $\mathbf{a} \in \mathbb{R}^k$ that maximize (5) for a given price vector $\mathbf{p}$ is the *demand* of agent-i on $\mathbf{C}$ at price $\mathbf{p}$ (hereafter denoted by $Z_i(\mathbf{p})$), i.e.

$$Z_i(\mathbf{p}) = \operatorname*{argmax}_{\mathbf{a}\in\bar{\mathbb{R}}^k} \{\mathbb{U}_i(\mathbf{a}\cdot\mathbf{C} + \mathcal{E}_i) - \mathbf{a}\cdot\mathbf{p}\} \tag{6}$$

The decision criterion (6) implies a number of assumptions, namely: (i) there is no budget constraints, i.e. agents are able to borrow units of numéraire without changing their utility status, (ii) there is no short selling constraints on securities in $\mathbf{C}$ and (iii) there is no further transaction costs. Although the last assumption is standard in the literature, the first two need some defense. We should mention that these assumptions are taken in the view of risk sharing arguments, which means that any transfer of cash is summed up to zero. Furthermore, one may think that the absence of short selling constrains implies unbounded supply, however this can well be avoided by imposing regularity constraints on the set of admissible positions $\mathcal{X}$. Unbounded supply is indeed the case if the price is extremely high, however the equilibrium arguments that follow will endogenously exclude this possibility.

Optimization problem (6) is a fundamental optimal portfolio choice problem that has been well-studied and analyzed by number of authors (see among others [15] for an overview in the case of quadratic utility and budget constraints and [4] and [14] under concave utility functionals).

Sufficient conditions under which the set $Z_i(\mathbf{p})$ is a singleton for any price vector $\mathbf{p}$, are that the function $f_i(\mathbf{a}) = \mathbb{U}_i(\mathbf{a} \cdot \mathbf{C} + \mathcal{E}_i)$ is continuous and strictly concave and that elements of $\mathbf{C}$ are linearly independent (i.e. the variance-covariance matric $\mathbb{V}\text{ar}[\mathbf{C}]$ is invertible). We summarized the above requirements in the following assumption which shall be imposed in the rest of this paper.

Assumption 2.1. *For every considered vector of securities* $\mathbf{C}$*, matrix* $\mathbb{V}\text{ar}[\mathbf{C}]$ *is invertible and the function* $f_i(\mathbf{a}) = \mathbb{U}_i(\mathbf{a} \cdot \mathbf{C} + \mathcal{E}_i)$ *is continuous and strictly concave for each* $i \in \{1, 2, ..., n\}$*.*

Under Assumption 2.1, $Z_i(\mathbf{p})$ is a function from $\mathbb{R}^k$ to $\mathbb{R}^k$ which shall be called the *demand function* of agent-i of securities $\mathbf{C}$ (for the proof see Lemma A.3 and Remark A.4. in [4]). This assumption also guarantees that $f_i(\cdot)$ is continuously differentiable and function $v_i : \mathbb{R}^k \to \mathbb{R}$ defined by

$$v_i(\mathbf{p}) = \max_{\mathbf{a} \in \mathbb{R}^k} \{f_i(\mathbf{a}) - \mathbf{a} \cdot \mathbf{p}\} = \mathbb{U}_i(Z_i(\mathbf{p}) \cdot \mathbf{C} + \mathcal{E}_i) - Z_i(\mathbf{p}) \cdot \mathbf{p}, \tag{7}$$

is the conjugate (dual) of the convex function $-f_i(-\mathbf{a})$. Function $v_i(\mathbf{p})$ has a particular financial meaning. It represents the utility that the agent-i would enjoy if she gets the desired short/long positions on $\mathbf{C}$ at a given price $\mathbf{p}$. Furthermore, smoothness of f_i implies (see Chapter 3 of [17]) that

$$f_i(\mathbf{a}) = \min_{\mathbf{p} \in \mathbb{R}^k} \{v_i(\mathbf{p}) + \mathbf{a} \cdot \mathbf{p}\}.$$

Hence $\mathbb{U}_i(\mathcal{E}) = \min_{\mathbf{p} \in \mathbb{R}^k} \{v_i(\mathbf{p})\}$. The minimizer $\mathbf{p}_{0i}$ of $v_i(\mathbf{p})$ is usually called the *reservation price* of agent-i (see Chapter 2 of [30] for an analogous definition), that is the price of $\mathbf{C}$ at which agent-i is unwilling to participate in any transaction of $\mathbf{C}$. For any other price however, the agent is willing to long or short some securities in $\mathbf{C}$, since this will result an increase in her utility.

For the quadratic utility functional, where $\mathcal{X} = \mathbb{L}^2(\mathcal{F})$, the demand function of agent-i at price $\mathbf{p}$ is simply the vector $\mathbf{a}$ that maximizes the quantity

$$\mathbf{a} \cdot (\mathbb{E}[\mathbf{C}] - \mathbf{p}) - 2\gamma_i \mathbf{a} \cdot \text{Cov}(\mathbf{C}, \mathcal{E}_i) - \gamma_i \mathbf{a} \cdot \Sigma(\mathbf{C}) \cdot \mathbf{a} + \mathbb{U}_i(\mathcal{E}_i).$$

Hence, if agent-i uses quadratic utility, her demand function of $\mathbf{C}$ is given by

$$Z_i(\mathbf{p}) = \left(\frac{\mathbb{E}[\mathbf{C}] - \mathbf{p}}{2\gamma_i} - \text{Cov}(\mathbf{C}, \mathcal{E}_i) \right) \cdot \mathbb{V}\text{ar}^{-1}[\mathbf{C}] \tag{8}$$

where $\mathbb{E}[\mathbf{C}]$ stands for the vector $(\mathbb{E}[C_1], \mathbb{E}[C_2], ..., \mathbb{E}[C_k])$ and for any payoff $X \in \mathbb{L}^2(\mathcal{F})$, $\mathrm{Cov}(\mathbf{C}, X)$ denotes the vector $(\mathrm{Cov}(C_1, X), \mathrm{Cov}(C_2, X), ..., \mathrm{Cov}(C_k, X))$. Note that the demand has two distinguished sources, namely the risk premium scaled by agent's risk aversion, $\frac{\mathbb{E}[\mathbf{C}]-\mathbf{p}}{2\gamma_i} \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]$ and the correlation of the tradeable securities and the agent's endowment, $\mathrm{Cov}(\mathbf{C}, \mathcal{E}_i) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]$. It is clear that absolute size of the demand function is a decreasing function of the risk aversion coefficient, γ_i. This means that the more risk averse the agent becomes the less willing she is to participate in a transaction on $\mathbf{C}$. Also, the demand for a particular security C_j (for $j \in \{1, 2, ..., k\}$), is a decreasing function of the covariance of C_j with agent's endowment. This supports the use of the quadratic evaluation for risk management purposes. Indeed, we expect that when the covariance of particular security is a large negative number, the agent would be willing to take long position on this security, since this would decrease the risk exposure the random endowment creates. Finally, on the same nature it is the observation that $|Z_{i,j}(\mathbf{p})|$ is decreasing with respect to $\mathbb{V}\mathrm{ar}[C_j]$, i.e. the riskier the security becomes, the smaller the desired position on it is.

Remark 2.1. *For the case of the certainty equivalent of expected exponential utility function (i.e. for the entropic risk measure), we have that* $f_i(\mathbf{a}) = -\frac{1}{\gamma_i} \ln \mathbb{E}[e^{-\gamma_i(\mathbf{a}\cdot\mathbf{C}+\mathcal{E}_i)}] - \mathbf{a} \cdot \mathbf{p}$. *If we set* $\mathcal{X} = \mathbb{L}^\infty(\mathcal{F}, \mathbb{P})$, *then the demand function is the solution* $\mathbf{a}$ *of the following equation*

$$\frac{\mathbb{E}_{\mathbb{P}_{-\gamma_i \mathcal{E}_i}}[\mathbf{C} e^{-\gamma_i(\mathbf{a}\cdot\mathbf{C})}]}{\mathbb{E}_{\mathbb{P}_{-\gamma_i \mathcal{E}_i}}[e^{-\gamma_i(\mathbf{a}\cdot\mathbf{C})}]} = \mathbf{p}$$

where the probability measure $\mathbb{P}_{-\gamma_i \mathcal{E}_i}$ *is defined through its Radon-Nikodym derivative* $\frac{d\mathbb{P}_{-\gamma_i \mathcal{E}_i}}{d\mathbb{P}} = \frac{e^{-\gamma_i \mathcal{E}_i}}{\mathbb{E}[e^{-\gamma_i \mathcal{E}_i}]}$.

2.1. **Optimal Risk Sharing.** In this subsection, we address the main subject of this manuscript, that is how agents can mutually reduce their risk exposure by proceeding to certain transactions. We recall that endowment $\mathcal{E}_i$ denotes the unhedgable part of agent-i's risky exposure, the part of her financial position that can not be hedged out by trading to any other market at which agents are price-takers. In our over-the-counter transaction, we assume that each agent shares her random endowment with the other agents aiming to reduce her risk or equivalently increase her expected utility. This is a general picture of the risk sharing problem and in the case where there is no exogenous constrain or additional transaction costs, agents freely negotiate the design of contracts that "*optimally*" share their risk exposures. The word optimally refers to a Pareto-type of transaction, that is the transaction that maximizes the *sum* of the agents' utility functionals. More precisely, the set $\mathcal{A} = \{\mathbf{C} \in \mathcal{X}^n : \sum_{i=1}^n C_i = 0\}$ contains all the possible sharing rules of agents' endowments and the optimal risk sharing problem is finding $\mathbf{C} \in \mathcal{A}$ that solves the following maximization problem

$$\max_{\mathbf{C}\in\mathcal{A}} \sum_{i=1}^{n} \mathbb{U}_i(\mathcal{E}_i + C_i) \tag{9}$$

In the above problem, C_i is the payoff of the contract that agent-i is going to receive at the terminal time. The requirement that the sum of these contracts is zero implies that risk sharing does not add further risk in the market. Solutions of problem (9) (if they exist) shall be called *optimal sharing rules*.

This setting of risk sharing problem was first introduced by [11] in reinsurance market and more recently developed in more general risk preferences modelling in [8], [13], [19], [28] (see also [1] for the cases without the monotonicity of utility functional).

2.2. **Optimal Span of Endowments.** In every practical situation, the optimal contracts do not take complex forms, that is the solution $\mathbf{C}^*$ of (9) is not a complicated function of $\mathcal{E}_i$'s. In fact, for practical purposes, we may only consider the sharing of endowments through linear combinations. More precisely, instead of looking at problem (9), a more practical perspective suggests the following one

$$\max_{A\in\mathbb{A}_n} \sum_{i=1}^{n} \mathbb{U}_i(\mathcal{E}_i + \mathbf{a}_i \cdot \mathbf{E}) \tag{10}$$

where $\mathbf{E} = (\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$, $\mathbb{A}_n$ is the set of square matrices of dimension n that represent the allocations of securities in $\mathcal{X}^n$ and $\mathbf{a}_i \in \mathbb{R}^n$ are their rows. In other words, for each allocation $A \in \mathbb{A}_n$, the element a_{ij} denotes the number of security $\mathcal{E}_j$ that agent-i holds (negative a_{ij} means short position). Hence, vector $\mathbf{a}_i \in \mathbb{R}^n$ stands for vector $(a_{i1}, a_{i2}, ..., a_{in})$ and indicates the number of each security in $\mathbf{E}$ agent-i holds (and hence $\sum_{i=1}^{n} \mathbf{a}_i = \mathbf{0}$).

Problems (9) and (10) are not necessary equivalent and the existence of a solution of one does not imply the solution of the other. However, in the case of quadratic utility functional both have a common solution (see also [7] and [1]).

Proposition 2.1. *Let all agents have quadratic utility functionals. The unique (up to constant additions) optimal risk sharing rule is given by the vector of securities* $\mathbf{C}^* = (C_1^*, C_2^*, ..., C_n^*)$ *with*

$$C_i^* = \mathbf{a}_i^* \cdot \mathcal{E} \tag{11}$$

where elements of vector $\mathbf{a}_i^* = (a_{i1}^*, a_{i2}^*, ..., a_{in}^*)$ *are given by*

$$a_{ii}^* = \frac{\gamma - \gamma_i}{\gamma_i} \qquad \text{and} \qquad a_{i\zeta}^* = \frac{\gamma}{\gamma_i}, \text{ for } \zeta \neq i, \tag{12}$$

where, γ *is the aggregate risk aversion coefficient, i.e.* $\gamma = \left(\sum_{i=1}^{n} \frac{1}{\gamma_i}\right)^{-1}$. *Furthermore, the allocation* $A^* = (a_{ij}^*) \in \mathbb{A}_n$ *is the optimal allocation for vector* $\mathbf{E}$*, i.e.* A^* *solves problem* (10).

Proof. We have that for every $\mathbf{C} = (C_1, C_2, ..., C_n) \in \mathcal{A}$

$$\sum_{i=1}^{n} \mathbb{U}_i(\mathcal{E}_i + C_i) = \sum_{i=1}^{n} \mathbb{E}[\mathcal{E}_i] - \sum_{i=1}^{n} \gamma_i \, \mathbb{V}\mathrm{ar}[C_i + \mathcal{E}_i].$$

Hence, it is enough to find $\mathbf{C}^* \in \left(\mathbb{L}^2(\mathcal{F})\right)^n$ that minimizes the sum $\sum_{i=1}^{n} \gamma_i \, \mathbb{V}\mathrm{ar}[C_i + \mathcal{E}_i]$. Note the for each $i = 1, 2, ..., n$,

$$C_i^* + \mathcal{E}_i = \frac{\gamma}{\gamma_i} \mathcal{E}$$

and

$$\gamma_i \, \mathbb{V}\mathrm{ar}[C_i^* + \mathcal{E}_i] = \frac{\gamma^2}{\gamma_i} \, \mathbb{V}\mathrm{ar}[\mathcal{E}],$$

where we recall that $\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}_i$. Therefore,

$$\sum_{i=1}^{n} \gamma_i \, \mathbb{V}\mathrm{ar}[C_i^* + \mathcal{E}_i] = \sum_{i=1}^{n} \frac{\gamma^2}{\gamma_i} \, \mathbb{V}\mathrm{ar}[\mathcal{E}_i] = \gamma \, \mathbb{V}\mathrm{ar}[\mathcal{E}].$$

It is then enough to show that

$$\sum_{i=1}^{n} \gamma_i \, \mathbb{V}\mathrm{ar}[C_i + \mathcal{E}_i] \geq \gamma \, \mathbb{V}\mathrm{ar}[\mathcal{E}], \quad \text{for every } \mathbf{C} \in \mathcal{A}. \tag{13}$$

or equivalently

$$\sum_{i=1}^{n} \gamma \gamma_i \, \mathbb{V}\mathrm{ar}[C_i + \mathcal{E}_i] \geq \gamma^2 \, \mathbb{V}\mathrm{ar}[\mathcal{E}], \quad \text{for every } \mathbf{C} \in \mathcal{A}. \tag{14}$$

The left hand side of (14) equals to

$$\sum_{i=1}^{n} \frac{\gamma}{\gamma_i} \gamma_i^2 \, \mathbb{V}\mathrm{ar}[C_i + \mathcal{E}_i] = \sum_{i=1}^{n} \frac{\gamma}{\gamma_i} \, \mathbb{V}\mathrm{ar}[\gamma_i (C_i + \mathcal{E}_i)]$$

and inequality (14) follows by the convexity of $\mathbb{V}\mathrm{ar}[\cdot]$ and the fact that $\sum_{i=1}^{n} C_i = 0$.

The last part of the proposition follows directly. □

The interpretation of the sharing rule (11) and (12) is clear. Agent-i is going to short (sell) a part of her endowment and long (buy) equal part of the other agents' random endowments. The prices of these transactions however can not be induced by the problem (9) (or problem (10)). Note also that this optimal sharing rule is independent on the distribution of the endowments which simply reflects the fact that agents have the same belief (that is common probability measure) and evaluate payoffs following similar lines (that is common type of utility)[3].

The aggregate maximized utility of the optimal risk transaction is given by

$$\sum_{i=1}^{n} \mathbb{U}_i(\mathcal{E}_i + C_i^*) = \mathbb{E}[\mathcal{E}] - \gamma \, \mathbb{V}\mathrm{ar}[\mathcal{E}]$$

[3] By following the same lines of the above proof, we can show that the equality of problems (9) and (10) holds in the case of entropic measures too (see [7] and [11]).

which is sometimes called the risk measure of the *representative agent*, whose utility remains quadratic and her endowment and risk aversion coefficient are the aggregate ones. Furthermore, the difference

$$\sum_{i=1}^{n} \mathbb{U}_i(\mathcal{E}_i + C_i^*) - \sum_{i=1}^{n} \mathbb{U}_i(\mathcal{E}_i) = \sum_{i=1}^{n} \gamma_i \, \mathbb{V}\mathrm{ar}[\mathcal{E}_i] - \gamma \, \mathbb{V}\mathrm{ar}[\mathcal{E}] \tag{15}$$

is the maximized aggregate *gain* of utility that the market (or the representative agent) enjoys by the optimal risk sharing trading.

3. Constrained Risk Sharing and Equilibrium Pricing

As discussed in the introduction, the construction of the contracts that optimally share the risky endowments among the agents is not always possible. There are a number of exogenous reasons for that: (i) the new regulation on the OTC transactions stating new restrictions on credit levels, which can not be met after the optimal risk sharing transaction. The reason for this is that risk measurement that is used by agents and investors is usually different than those used by the regulators (e.g. we have not imposed any credit constrain or margin account requirement in problems (9) and (10)). This makes the optimal risk sharing very expensive in the sense that the agents should commit part of their capital as additional margin accounts. (ii) Some types of random incomes simply can not be divided due to their nature and the difficulties in their modelling as random variables, e.g. real estate investments, revenues of a running business, dividends from illiquid shares etc. (iii) In some cases, sharing of endowment are against the general policy of an agent, for example selling part of her endowment may imply a message of lack of trust to her clients on which the endowment payoffs are based. (iv) Further liquidity concerns and transaction costs may make the optimal risk sharing trading disadvantageous.

3.1. **Price-Allocation Equilibrium.** Under these realistic conditions, agents can reduce their unhedgeable risk exposures through trading some of the standardized contracts whose payoffs are correlated to their endowments. These securities could be any OTC derivative products, standard reinsurance contracts, premium stocks and bonds etc. We assume that there is a finite number of those and all of them are measured with respect to $\mathcal{F}$ or in other words they belong to the set of possible financial positions $\mathcal{X}$. Their payoffs shall be denoted by the random vector $\mathbf{C} = (C_1, C_2, ..., C_k) \in \mathcal{X}^k$. In these situations, agents do not negotiate the design of the optimal contracts but rather the prices and the quantities of the given securities in $\mathbf{C}$ they will trade. We in fact follow the lines of problem (10) where instead of $\mathbf{E}$, agents maximize the sum of their utilities by trading vector $\mathbf{C}$. The set of allocations of $\mathbf{C}$ among agents will be denoted by $\mathbb{A}_{n\times k}$, that is

$A \in \mathbb{A}_{n\times k}$ is a matrix in $\mathbb{R}^{n\times k}$, where its element a_{ij} denotes the number of security C_j that agent-i holds ($i \in \{1,2,...,n\}$ and $j \in \{1,2,...,k\}$) and $\sum_{i=1}^{n} \mathbf{a}_i = \mathbf{0}$, where $\mathbf{a}_i$ is the i-th row of A.

The problem of risk sharing through vector of securities $\mathbf{C}$ is

$$\sup_{A\in\mathbb{A}_{n\times k},\mathbf{p}\in\mathbb{R}^k}\left\{\sum_{i=1}^{n}\mathbb{U}_i(\mathbf{a}_i\cdot\mathbf{C}+\mathcal{E}_i)-\sum_{i=1}^{n}\mathbf{a}_i\cdot\mathbf{p}\right\}=\sup_{A\in\mathbb{A}_{n\times k}}\left\{\sum_{i=1}^{n}\mathbb{U}_i(\mathbf{a}_i\cdot\mathbf{C}+\mathcal{E}_i)\right\} \tag{16}$$

It is clear that the above argument of optimal sharing provides only the allocation of $\mathbf{C}$ that maximizes the sum of agents utilities (or minimizes the associated risk measures). In other words, due to the cash invariance property, maximization of sum of utilities does not provide the prices at which the optimal transaction on $\mathbf{C}$ will take place. Getting the consistent price vector for $\mathbf{C}$ (which is a crucial part of transaction) needs another argument. This can be based on the demand function for each of the agents. More precisely, we suppose that the transaction prices of elements of $\mathbf{C}$ will be those that make the sum of agents' demand functions equal to zero, or in other words the price that makes the risk sharing consistent are the ones that makes supply of $\mathbf{C}$ equal to its demand. More formally, we have the following definition of price-allocation equilibrium.

Definition 3.1. *For a given vector of securities $\mathbf{C} \in \mathcal{X}^k$, the pair $(\mathbf{p}^*, A^*) \in \mathbb{R}^k \times \mathbb{A}_{n\times k}$ is called a price-allocation equilibrium if $Z_i(\mathbf{p}^*) = \mathbf{a}_i^*$ for each $i \in \{1,2,...,n\}$.*

This equilibrium setting for risk sharing has been established in [4] under the presence of an exogenously priced financial market (see also [5] for the special case of exponential utility and $n = 2$) and also studied in [14] in a discrete time dynamic model. Similar equilibrium arguments has recently been supported by a number of paper in option and future markets [16], [25] and [26]. Below, we give sufficient conditions for the existence and the uniqueness of price-allocation equilibrium in the special case of bounded payoffs, i.e. when $\mathcal{X} = \mathbb{L}^\infty(\mathcal{F},\mathbb{P})$. For this we need to give another definition.

Definition 3.2. *We say that a concave utility functional $\mathbb{U} : \mathbb{L}^\infty(\mathcal{F},\mathbb{P}) \to \mathbb{R}$ satisfies the Lebesque property if for any bounded sequence in $X_n \in \mathbb{L}^\infty(\mathcal{F},\mathbb{P})$ that converges in $\mathbb{P}$ to some $X \in \mathbb{L}^\infty(\mathcal{F},\mathbb{P})$, then $\lim_{n\to\infty}\mathbb{U}(X_n) = \mathbb{U}(X)$.*

The following theorem which follows the arguments of Theorem 4.10 in [4] states the existence and the uniqueness of the equilibrium price-allocation and its connection with problem (16).

Theorem 3.1. *Let $\mathcal{X} = \mathbb{L}^\infty(\mathcal{F},\mathbb{P})$ and $\mathbf{C}$ satisfies Assumption 2.1. If we further assume that $\mathbb{U}_i$ is non-decreasing and satisfies the Lebesque property for all $i \in \{1,2,...,n\}$, there exists a unique price-allocation equilibrium $(\mathbf{p}^*, A^*) \in \mathbb{R}^k \times \mathbb{A}_{n\times k}$. Furthermore, A^* is the allocation that solves the risk sharing problem* (16).

Proof. See Appendix A. □

At equilibrium, each agent enjoys a level of utility that is higher or at least equal to her initial utility, that is $v_i(\mathbf{p}^*) \geq \mathbb{U}_i(\mathcal{E}_i)$ where the equality holds if and only if the equilibrium price vector coincides with agent's reservation price vector, $\mathbf{p}_i^0$. Given the vector of securities $\mathbf{C}$, we call the quantity $v_i(\mathbf{p}^*)$ the *utility level* at equilibrium of agent-i. The difference $v_i(\mathbf{p}^*) - \mathbb{U}_i(\mathcal{E}_i)$ measures (in terms of utility) the gain that agent-i enjoys by entering into the equilibrium transaction of $\mathbf{C}$. The advantage of using $v_i(\mathbf{p}^*)$ is that it is measured in currency units and hence can be used for the comparison among different equilibria and agents.

The fact that the optimal risk sharing rule can not be applied means that any other sub-optimal sharing transaction reduces the efficiency of this OTC market of n agents. The difference between the aggregate utility of the optimal risk sharing and the aggregate utility after the equilibrium sharing of $\mathbf{C}$ can be considered as a measure of the *risk sharing inefficiency.*

$$\text{Risk Sharing Inefficiency} = \text{Optimal Aggregate Utility} \; - \; \text{Realized Aggregate Utility} \tag{17}$$

In the case where the realized sharing is done through trading a vector of securities $\mathbf{C}$

$$\text{Risk Sharing Inefficiency} \; = \sup_{\mathbf{X} \in \mathcal{A}} \left\{ \sum_{i=1}^{n} \mathbb{U}_i(\mathcal{E}_i + X_i) \right\} - \sup_{A \in \mathbb{A}_{n \times k}} \left\{ \sum_{i=1}^{n} \mathbb{U}_i(\mathbf{a}_i \cdot \mathbf{C} + \mathcal{E}_i) \right\} \tag{18}$$

We shall analyze these quantities in the tractable case of quadratic utility functional below.

3.2. **Equilibrium with Quadratic Utilities.** In the subsection we consider the special case where all agents' risk preferences are modelled by quadratic utilities (see (3)). Although, the quadratic utility functionals do not satisfy the conditions of Theorem 3.1 (they are not monotonic), its specific form allows us to get a closed form solution of the pair of price and allocation that optimally shares the agents' endowments through trading a given vector of securities $\mathbf{C}$. Namely, assuming that $\mathbb{V}\text{ar}[\mathbf{C}]$ is invertible and taking into account equation (8) we get the following result.

Proposition 3.1. *Let all agents have quadratic utility functional with risk aversion coefficients* γ_i, $i \in \{1, 2, ..., n\}$ *and assume that* $\mathbb{V}\text{ar}[\mathbf{C}]$ *is invertible. Then, the equilibrium price of a vector of securities* $\mathbf{C}$ *is given by*

$$\mathbf{p}^* = \mathbb{E}[\mathbf{C}] - 2\gamma \, \text{Cov}(\mathbf{C}, \mathcal{E}) \tag{19}$$

and the equilibrium allocation A^* *is the matrix whose rows* $\mathbf{a}_i^*$ *are given by*

$$\mathbf{a}_i^* = \text{Cov}\left(\mathbf{C}, C_i^*\right) \cdot \mathbb{V}\text{ar}^{-1}[\mathbf{C}] \tag{20}$$

for each $i \in \{1, 2, ..., n\}$, *where* C_i^* *is given by* (11) *and* (12), $\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}_i$ *and* $\gamma = \left(\sum_{i=1}^{n} \frac{1}{\gamma_i} \right)^{-1}$.

Remark 3.1. *A first observation on Proposition 3.1 is that the equilibrium prices do not depend on the covariance matrix of securities, but only on their expectations and their covariances with the aggregate endowment. The equilibrium price of a security C_j increases as its covariance with the aggregate endowment decreases, which simply reflects the higher demand of this security in this case. Also, the vector of quantities that agent-i is going to short/long at equilibrium has a clear link with the covariances of the traded securities and the contract that she would have got if the optimal risk sharing were possible. The higher (in absolute terms) the covariance of the tradable security* $\mathbf{C}$ *and her optimal contract is, the higher is her demand (i.e. her participation in the market).*

Proposition 3.2. *Let all agents have quadratic utility functional. Then, the utility level at equilibrium of agent-i is given by*

$$v_i(\mathbf{p}^*) = \gamma_i \operatorname{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \operatorname{Cov}(\mathbf{C}, C_i^*) + \mathbb{U}_i(\mathcal{E}_i). \tag{21}$$

Proof. We simply apply in (7), equations (19) and (20). Indeed, for each $i \in \{1, 2, ..., n\}$

$$\begin{aligned}
v_i(\mathbf{p}^*) &= \mathbb{E}[Z_i(\mathbf{p}^*) \cdot \mathbf{C}] - \gamma_i \mathbb{V}\mathrm{ar}[Z_i(\mathbf{p}^*) \cdot \mathbf{C}] - 2\gamma_i \operatorname{Cov}(Z_i(\mathbf{p}^*) \cdot \mathbf{C}, \mathcal{E}_i) - Z_i(\mathbf{p}^*) \cdot \mathbf{p}^* + \mathbb{U}_i(\mathcal{E}_i) \\
&= -\gamma_i Z_i(\mathbf{p}^*) \cdot \mathbb{V}\mathrm{ar}[\mathbf{C}] \cdot Z_i(\mathbf{p}^*) - 2\gamma_i Z_i(\mathbf{p}^*) \cdot \operatorname{Cov}(\mathbf{C}, \mathcal{E}_i) + \mathbb{U}_i(\mathcal{E}_i) + 2\gamma Z_i(\mathbf{p}^*) \cdot \sum_{j=1}^{n} \operatorname{Cov}(\mathbf{C}, \mathcal{E}_j) \\
&= -\gamma_i Z_i(\mathbf{p}^*) \cdot \mathbb{V}\mathrm{ar}(\mathbf{C}) \cdot Z_i(\mathbf{p}^*) + 2\gamma_i Z_i(\mathbf{p}^*) \cdot \left(\frac{\gamma}{\gamma_i} \sum_{j=1}^{n} \operatorname{Cov}(\mathbf{C}, \mathcal{E}_j) - \operatorname{Cov}(\mathbf{C}, \mathcal{E}_i) \right) + \mathbb{U}_i(\mathcal{E}_i) \\
&= \gamma_i Z_i(\mathbf{p}^*) \cdot \mathbb{V}\mathrm{ar}[\mathbf{C}] \cdot (Z_i(\mathbf{p}^*) + \mathbb{U}_i(\mathcal{E}_i) \\
&= \gamma_i \operatorname{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \operatorname{Cov}(\mathbf{C}, C_i^*) + \mathbb{U}_i(\mathcal{E}_i).
\end{aligned}$$

□

For example, if $k = 1$ (i.e. there is only one security to be traded) and $n = 2$ we have

$$\begin{aligned}
v_1(p^*) &= \gamma_1 \mathbb{V}\mathrm{ar}[C] Z_1^2(p^*) + \mathbb{U}_1(\mathcal{E}_1) = \frac{\gamma_1}{\mathbb{V}\mathrm{ar}[C]} \operatorname{Cov}\left(C, \frac{\gamma_2 \mathcal{E}_2 - \gamma_1 \mathcal{E}_1}{\gamma_1 + \gamma_2}\right) + \mathbb{U}_1(\mathcal{E}_1) \\
v_2(p^*) &= \gamma_2 \mathbb{V}\mathrm{ar}[C] Z_1^2(p^*) + \mathbb{U}_2(\mathcal{E}_2) = \frac{\gamma_2}{\mathbb{V}\mathrm{ar}[C]} \operatorname{Cov}\left(C, \frac{\gamma_2 \mathcal{E}_2 - \gamma_1 \mathcal{E}_1}{\gamma_1 + \gamma_2}\right) + \mathbb{U}_2(\mathcal{E}_2)
\end{aligned}$$

since $Z_1(p^*) = -Z_2(p^*)$.

Remark 3.2. *From the above calculations, it is implied that the gain of equilibrium transaction of agent-i, i.e. the difference* $v_i(\mathbf{p}^*) - \mathbb{U}_i(\mathcal{E}_i)$ *is increasing with respect to the absolute size of her positions on* $\mathbf{C}$*. Moreover, the smaller the variance of the* $\mathbf{C}$ *is, the higher is the volume of the equilibrium transaction and therefore the gain of each agent.*

Remark 3.3. *The term* $\mathrm{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \mathrm{Cov}(\mathbf{C}, C_i^*)$ *has a nice interpretation in the case where* $\mathbf{C}$ *and* $\mathbf{E}$ *follow normal distribution. Under this assumption, we have that (see e.g.* [29]*)*

$$\mathrm{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \mathrm{Cov}(\mathbf{C}, C_i^*) = \mathbb{V}\mathrm{ar}[C_i^*] - \mathbb{V}\mathrm{ar}[C_i^* | \mathbf{C} = c], \quad \textit{for every vector } c \in \mathbb{R}^k.$$

This implies that

$$v_i(\mathbf{p}^*) = \gamma_i \, \mathbb{V}\mathrm{ar}[\mathbb{E}[C_i^* | \mathbf{C}]] + \mathbb{U}_i(\mathbb{E}_i).$$

3.3. **Equilibrium Prices of Endowments.** A special case of problem (16) is when the securities available for trading are the agents' endowments, i.e. when $\mathbf{C} = \mathbf{E}$. As shown in Proposition 3.1, when utilities are quadratic, problem (16) is equivalent to the optimal risk sharing problem (9). Hence, solution of (16) provides not only the optimal sharing contracts but also the equilibrium prices of the agents' endowments. By applying Proposition 3.1 for the vector of endowments $\mathbf{E}$, we get that

$$\mathbf{p}_{\mathcal{E}}^* = \mathbb{E}[\mathbf{E}] - 2\gamma \mathbf{1} \cdot \mathbb{V}\mathrm{ar}[\mathbf{E}] \tag{22}$$

where $\mathbf{1}$ is the vector of dimension n with all elements equal to 1. Straightforward calculations also give that the utility level of agent-i at equilibrium after the optimal risk sharing is given by

$$v_i^o(\mathbf{p}_{\mathcal{E}}^*) = \gamma_i \, \mathbb{V}\mathrm{ar}[C_i^*] + \mathbb{U}_i(\mathcal{E}_i) \tag{23}$$

where, $v_i^o(\cdot)$ is utility level of agent-i when the tradable security vector is $\mathbf{E}$. The term $\gamma_i \, \mathbb{V}\mathrm{ar}(C_i^*)$ is the highest possible gain (in terms of utility) of agent-i that she could get by trading with all the rest $n-1$ agents. Hence, the risk sharing inefficiency caused by equilibrium trading of vector of securities $\mathbf{C}$ is equal to

$$\sum_{i=1}^{n} \left(v_i^o(\mathbf{p}_{\mathcal{E}}^*) - v_i(\mathbf{p}^*) \right) = \gamma \, \mathbb{V}\mathrm{ar}[\mathcal{E}] - \sum_{i=1}^{n} \gamma_i \, \mathrm{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \mathrm{Cov}(\mathbf{C}, C_i^*) \tag{24}$$

Thanks to the above formulas, we are able to give explicit answers to questions like: *do all agents suffer a loss of utility when optimal sharing is not possible*? or *which is the agent which suffers the highest loss*?. Namely, for the first question the answer is affirmative, whereas the size of agent's loss is low when her risk aversion is low and when the correlation between her optimal contract and the securities in $\mathbf{C}$ she gets at equilibrium trading is high.

More precisely, for agent-i, this utility "loss" is given by

$$v_i^o(\mathbf{p}_{\mathcal{E}}^*) - v_i(\mathbf{p}^*) = \gamma_i \left(\mathbb{V}\mathrm{ar}[C_i^*] - \mathrm{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \mathrm{Cov}(\mathbf{C}, C_i^*) \right). \tag{25}$$

Note also that (25) is always non-negative and equal to zero if and only if the optimal contract of agent-i can be written as a linear combination of elements of $\mathbf{C}$. We state this fact in the following proposition whose proof follows from standard arguments.

Proposition 3.3. *Let all agents have quadratic utility functionals. When the risk sharing is done through trading a vector of securities* $\mathbf{C} \neq \mathbf{E}$*, then each of them suffers a loss of utility. This loss is zero for agent-i if and only if there exist* $b \in \mathbb{R}$ *and* $\mathbf{a} \in \mathbb{R}^k$ *such that* $C_i^* = b + \mathbf{a} \cdot \mathbf{C}$*,* $\mathbb{P}$*-a.s., where* C_i^* *is given by* (11) *and* (12)*.*

Remark 3.4. *From* (25) *we conclude that the agent that suffers the highest loss of utility is the one with high risk aversion and for which the correlation between the tradable assets and her optimal contract is low. In particular, in the case of only two agents, their losses are proportional to each other, i.e.*

$$\textit{Loss of agent-}1 = \tfrac{\gamma_1}{\gamma_2}\ \textit{Loss of agent-}2.$$

4. Nash Equilibrium on Risk Sharing Settings

In this section we address situations where the strategic behavior of agents, i.e. an endogenous reason, results in sub-optimal risk sharing transaction. Assuming that agents follow a certain type of strategy means that the induced risk sharing is not likely to be the optimal (we replace Pareto to Nash-type equilibrium game).

The majority of the financial OTC markets are oligopolies, which means that the decisions of one agent will affect the equilibrium price-allocation and hence the level of utility of all the other participants. Therefore, when modelling the equilibrium at such markets, we should model not only the mechanism that leads to equilibrium price allocation but also the strategy followed by the participants. The implication of such strategic behavior of agents in risk sharing setting gives an additional explanation why the realized risk sharing in the market is not the efficient one. In what follows, we introduce a model of an individual agent's strategic behavior which is based on the equilibrium arguments of the previous discussion.

Namely, we assume that the n agents form a market in which they report their endowments and then (according to the induced by problem (9) sharing rules) the optimal-sharing contracts are designed, priced and traded among them. However, an agent may not report her true endowment but rather choose to report as her "shared-to-be" endowment a random variable that (if involved in the optimal trading procedure) would result higher expected utility. This is a misreporting (or hidden information) problem that can be considered as an example of a *moral hazard* in such a market. Below, we take a closer look on the formulation of this problem and establish the existence of a Nash-type equilibrium when each agent follows similar strategy, under the quadratic utility functional. We divide these game theoretic problems into three cases: when agents negotiate the whole randomness of the endowment that are going to share, when they negotiate only a percentage of their true endowments and when they manipulate the price of a given vector of traded securities $\mathbf{C}$.

4.1. **How much Risk should an Agent Share?** In the situation where agents optimally share their endowments, we are asking whether it is preferable for an agent to report (and hence share) the exact random variable that represents her endowment or it is "*better*" to report a different random quantity for the calculation of the optimal sharing rule. The word *better* refers to the level of the utility at equilibrium, i.e. quantity $v_i^o(\mathbf{p}^*)$. Let's consider agent-1, which enters the risk sharing market, where the rest of agents have reported endowments $\mathcal{E}_2, \mathcal{E}_3, ..., \mathcal{E}_n$. All agents will agree to optimal sharing their endowments following the solution of problem (9) or problem (10). If agent-1 has the option to report any random quantity $B \in \mathcal{X}$ she may not report $\mathcal{E}_1$. Since her only criterion in this setting is the maximization of expected utility, it is reasonable to argue that she is going to report as endowment the random variable that maximizes her level of utility when the transaction is made, given that the rest of agents have reported endowments $\mathcal{E}_2, \mathcal{E}_3, ..., \mathcal{E}_n$. In other words, we are asking which is the *best response* of agent-1 given the responses of the rest of the agents?

Below, we introduce and solve this game theoretic problem for agents with quadratic utilities, where we are able to provide analytic and well-interpreted solutions. We adapt the risk sharing setting of problem (9) (or the equivalent problem (10)). If the reported endowments are $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n$, the optimal sharing rule is that agent-1 gets the payoff $C_1^* = \frac{\gamma}{\gamma_1}\mathcal{E} - \mathcal{E}_1$ and pays $\mathbf{a}_1^* \cdot \mathbf{p}_{\mathcal{E}}^*$, where $\mathbf{a}_1^*$ is given in (12) and $\mathbf{p}_{\mathcal{E}}^*$ in (22). Therefore, given that the rest agents have reported endowments $\mathcal{E}_2, \mathcal{E}_3, ..., \mathcal{E}_n$ and $\mathcal{E}_{-1}$ denotes their aggregate endowment, i.e. $\mathcal{E}_{-1} = \sum_{i=2}^n \mathcal{E}_i$ and if agent-1 reports as her random endowment some random variable $B \in \mathbb{L}^2(\mathcal{F})$, then according to optimal sharing rule she would get contract with payoff

$$C_1^*(B) = \frac{\gamma}{\gamma_1}\mathcal{E}_{-1} + \frac{\gamma - \gamma_1}{\gamma_1}B, \tag{26}$$

the accumulate cash transaction is

$$\mathbf{p}_1^*(B) = \left(\frac{\gamma - \gamma_1}{\gamma_1}, \frac{\gamma}{\gamma_1}, ..., \frac{\gamma}{\gamma_1}\right) \cdot \begin{pmatrix} \mathbb{E}[B] - 2\gamma \operatorname{Cov}(B + \mathcal{E}_{-1}, B) \\ \mathbb{E}[\mathcal{E}_2] - 2\gamma \operatorname{Cov}(B + \mathcal{E}_{-1}, \mathcal{E}_2) \\ \vdots \\ \mathbb{E}[\mathcal{E}_n] - 2\gamma \operatorname{Cov}(B + \mathcal{E}_{-1}, \mathcal{E}_n) \end{pmatrix} \tag{27}$$

and her utility after the transaction would be

$$G_1(B; \mathcal{E}_{-1}) = \mathbb{U}_1(\mathcal{E}_1 + C_1^*(B) - \mathbf{p}_1^*(B)) = \mathbb{E}[\mathcal{E}_1 + C_1^*(B)] - \gamma_1 \operatorname{\mathbb{V}ar}[\mathcal{E}_1 + C_1^*(B)] - \mathbf{p}_1^*(B). \tag{28}$$

Similarly, we define the utility level of agent-i when she reports B and the rest of the agents have reported $\mathcal{E}_1, ..., \mathcal{E}_{i-1}, \mathcal{E}_{i+1} ..., \mathcal{E}_n$ and denote it by $G_i(B; \mathcal{E}_{-i})$. Note that the dependence is restricted to the rest of the agents' aggregate endowment.

Therefore, the best response of agent-i is the solution of the following maximization problem

$$\max_{B \in \mathbb{L}^2(\mathcal{F})} \{G_i(B; \mathcal{E}_{-i})\} \tag{29}$$

Note that agent-i's best response should be a random variable in $\mathbb{L}^2(\mathcal{F})$, which implies that agent's reported endowment is measurable with respect to the filtration that is generated by the true endowments (there is no other security in the optimal risk sharing environment, see also the related discussion in the Introduction and Section 2). The solution of problem (29) is given in the following proposition.

Proposition 4.1. *Let all agents have quadratic utility functionals. The best endowment response of agent-i when the rest of the agents have reported endowments $\mathcal{E}_1, ..., \mathcal{E}_{i-1}, \mathcal{E}_{i+1}, ..., \mathcal{E}_n$ is given by*

$$B_i^* = \frac{\gamma_i}{\gamma_i + \gamma} \mathcal{E}_i + \frac{\gamma^2}{\gamma_i^2 - \gamma^2} \mathcal{E}_{-i} \tag{30}$$

for each $i \in \{1, 2, ..., n\}$, where $\mathcal{E}_{-i} = \sum_{j \neq i}^n \mathcal{E}_j$. Also, B_i^* is unique up to constant additions.

Proof. See Appendix A. □

If agent-i chooses to share her best endowment response, the optimal risk sharing rule implies that she is going to get the contract with payoff

$$C_i^*(B_i^*) = \frac{\gamma}{\gamma_i + \gamma} \mathcal{E}_{-i} - \frac{\gamma_i - \gamma}{\gamma_i + \gamma} \mathcal{E}_i$$

instead of

$$C_i^* = \frac{\gamma}{\gamma_i} \mathcal{E}_{-i} - \frac{\gamma_i - \gamma}{\gamma_i} \mathcal{E}_i.$$

This gives that

$$C_i^*(B_i^*) = \frac{\gamma_i}{\gamma_i + \gamma} C_i^*,$$

hence the best optimal contract of agent-i is in fact a percentage of the contract that the optimal risk sharing procedure induces.

Remark 4.1. *At this point, a number of observations worth some attention. First, the best response B_i^* does not depend on any of the moments of the endowments and hence it is model free in the sense of probability formulation of the random endowments. Given that agent-i has learned the endowments of the rest of the agents, she should report as her endowment a certain percentage of her true endowment and a part of the other agents' aggregate endowment. Note that for risk averse agents ($\gamma_i > 1$), the percentage of the true endowment that maximizes the expected utility after the* *optimal transaction is higher. Furthermore, the factor* $\frac{\gamma^2}{\gamma_i^2-\gamma^2}$ *in front of the aggregate endowment of the rest of the agents is a decreasing function of the risk aversion coefficient* γ_i*. In fact, it is straightforward to check that the limit of* B_i^* *as* γ_i *goes to infinity is equal to* $\mathcal{E}_i$*. These observations imply that the strategic behavior induced by problem (29) is more intense for risk seekers than for risk averse agents and in a sense this strategy incorporates a speculating attitude.*

Remark 4.2. *At the best response, one can calculate that the utility level of agent-i is*

$$G_i(B_i^*;\mathcal{E}_{-i}) = \frac{\gamma_i^3}{\gamma_i^2-\gamma^2}\,\mathbb{V}\mathrm{ar}\,[C_i^*] + \mathbb{U}_i(\mathcal{E}_i)$$

which means that the increase of agent-i utility from following this strategic behavior is equal to $\gamma_i\,\mathbb{V}\mathrm{ar}[C_i^*]\frac{\gamma^2}{\gamma_i^2-\gamma^2}$ *or in other words the percentage increase of utility is* $\frac{\gamma^2}{\gamma_i^2-\gamma^2}$*. This increase is higher when the agent is less risk averse and when the other agents is more risk averse.*

4.2. **Best percentage of the true endowment.** However, the situation of problem (29) can be characterized as an extreme one, since the cases where an agent can report *any* random variable as her random income is rare. In the majority of the OTC risk sharing settings, agents which are about to trade their risk could not hide the direction of their risk exposure (for example in insurance industry, big competitors know more or less the kind of risk exposure each of them are facing). In favor of this argument is that B_i^* includes part of the endowments of the rest of agents. However, the nature of agents' incomes may be of a special nature and if agent-i argue that she is exposed to it, she gives a signal that it is more a risk seeker (speculator) than a hedger.

Alternatively, in the case where B_i^* can not be reported, agent-i may have the option to report not any random variable as her endowment but a percentage $b_i \geq 0$ of her true endowment $\mathcal{E}_i$. To emphasize the difference with problem (29), we hereafter use the notation $g_i(b;\mathcal{E}_{-i})$ instead of $G_i(b\mathcal{E}_i;\mathcal{E}_{-i})$, where the domain of the function g_i is the set $[0,\infty)$. In these situations, problem (29) is written as

$$\max_{b\in[0,\infty)}\{g_i(b;\mathcal{E}_{-i})\} \tag{31}$$

We deal with this restriction in the following proposition.

Proposition 4.2. *Let all agents have quadratic utility functionals. Suppose that* $\mathcal{E}_i$ *is not constant and the rest of the agents have reported endowments* $\mathcal{E}_1,...,\mathcal{E}_{i-1},\mathcal{E}_{i+1},...,\mathcal{E}_n$*. The best percentage response of agent-i is*

$$b_i^* = 0 \vee \left(\frac{\gamma_i}{\gamma_i+\gamma} + \frac{\gamma^2}{\gamma_i^2-\gamma^2}\frac{\mathrm{Cov}(\mathcal{E}_i,\mathcal{E}_{-i})}{\mathbb{V}\mathrm{ar}[\mathcal{E}_i]}\right) \tag{32}$$

for each $i\in\{1,2,...,n\}$*.*

Proof. Following the lines of the proof of Proposition 4.1, we need to minimize the quantity

$$(\gamma_i - \gamma)(1-b_i)^2\,\mathbb{V}\mathrm{ar}[\mathcal{E}_i] + (1-b_i)2\gamma\,\mathrm{Cov}(\mathcal{E},\mathcal{E}_i) + 2b_i^2\gamma\,\mathbb{V}\mathrm{ar}[\mathcal{E}_i] + 2b_i\gamma\left(\frac{\gamma_i-2\gamma}{\gamma_i-\gamma}\right)\mathrm{Cov}(\mathcal{E}_i,\mathcal{E}-\mathcal{E}_i).$$

Equation (32) follows by straightforward calculations. □

Remark 4.3. *From* (32) *we make the following observations:*

(i) *As agent-i becomes more and more risk averse, the best percentage approaches one, i.e., $\lim_{\gamma_i\to\infty} b_i^* = 1$, no matter the correlations and the endowments or their variances. This is consistent with the fact that the speculation attitude that drives the strategic behavior vanishes as the risk aversion increases. On the other hand, this attitude is more intense when γ_i goes to zero. In limit we have:*

$$\lim_{\gamma_i\to 0} b_i^* = \begin{cases} 0, & \rho\left(\mathcal{E}_i,\mathcal{E}_{-i}\right) < 0; \\ \frac{1}{2}, & \rho\left(\mathcal{E}_i,\mathcal{E}_{-i}\right) = 0; \\ +\infty, & \rho\left(\mathcal{E}_i,\mathcal{E}_{-i}\right) > 0. \end{cases}$$

(ii) *The best percentage response of agent-i is a linear (increasing) function of $\rho\left(\mathcal{E}_{-i},\mathcal{E}_i\right)$. b_i^* can well be zero if this correlation is sufficiently small. Note also that in the special case of zero correlation the best percentage report of agent-i is only a percentage of her endowment.*

4.3. **The Case of Constrained Risk Sharing.** We may consider a similar agent's behavior in the case where the risks are shared through a given vector of securities $\mathbf{C}$ (the equilibrium problem defined in Definition 3.1). In these situations each agent reports her demand function $Z_i(\mathbf{p})$ and the sharing rule is the allocation of $\mathbf{C}$ at the price vector that makes the sum of demand functions equal to zero. If agent-i knows or could extract the demand functions of the rest of the agents, she could report an adjusted demand function that makes the market clears out and at the same time maximizes her expected utility after the equilibrium transaction. In particular, we suppose that the demand functions of all agents except agent-i are given and we first ask: *which is the best equilibrium price for agent-i?* In the spirit of previous subsection, the word "*best*" refers to utility maximization. Hence, this problem is written as

$$\max_{\mathbf{p}\in\mathbb{R}^k}\{\phi_i(\mathbf{p})\} \tag{33}$$

where

$$\phi_i(\mathbf{p}) = \mathbb{U}_i\left(\mathcal{E}_i - \sum_{j\neq i} Z_j(\mathbf{p})\cdot\mathbf{C}\right) + \sum_{j\neq i} Z_j(\mathbf{p})\cdot\mathbf{p} \tag{34}$$

The maximizer of ϕ gives the equilibrium price that is preferable for agent-i, given the demand function of the rest of the agents. In this oligopoly market, she can make the market equilibrate at this price by bid/ask the corresponding quantities of $\mathbf{C}$, i.e., by reporting the appropriate demand function.

In case of quadratic utility functionals the calculation of this maximizer is straightforward thanks to the linearity of demand functions. The result is stated below.

Proposition 4.3. *Let all agents have quadratic utility functionals and assume that* $\mathbf{C} \in (\mathbb{L}^2(\mathcal{F}))^k$, *with* $\mathbb{V}\mathrm{ar}[\mathbf{C}]$ *invertible. The price that maximizes utility of agent-i given the demand functions* $(Z_j(\mathbf{p}))_{j\neq i}$ *is given by*

$$\mathbf{p}_i^* = \mathbb{E}[\mathbf{C}] - 2\gamma \operatorname{Cov}\left(\mathbf{C}, \frac{\gamma_i^2}{\gamma_i^2 - \gamma^2}\mathcal{E}_{-i} + \frac{\gamma_i}{\gamma_i + \gamma}\mathcal{E}_i\right). \tag{35}$$

Remark 4.4. *It follows by simple calculations that given the demand functions* $(Z_j(\mathbf{p}))_{j\neq i}$, *the price* (35) *is the equilibrium price of* $\mathbf{C}$ *if and only if the endowment* B_i *that is induced by agent-i's reported demand function that satisfies*

$$\operatorname{Cov}(\mathbf{C}, B_i) = \operatorname{Cov}(\mathbf{C}, B_i^*)$$

where, B_i^* *is given in* (30). *That is, the demand that gives price* $\mathbf{p}_i^*$ *as the equilibrium price is*

$$Z_i^*(\mathbf{p}) = \left(\frac{\mathbb{E}[\mathbf{C}] - \mathbf{p}}{2\gamma_i} - \operatorname{Cov}(\mathbf{C}, B_i^*)\right)\mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}].$$

This means that the best endowment response of agent-i is always B_i^* *for unconstrained and constrained risk sharing market.*

4.4. **Nash Equilibrium in Risk Sharing.** In this subsection, we examine how a risk sharing market functions if all the participated agents follow a similar strategic behavior regarding their endowments and their demand functions. As it is shown below, this behavior results in a significant reduction of the shared risk, i.e., a significant risk sharing inefficiency, as the latter is defined in (17).

We first consider the case of optimal OTC risk sharing market where the optimal sharing rules are the solutions of problem (9) or problem (10). However, we assume that each of the agents follows the strategic behavior induced by problems (29). If each agent knows that the rest of agents are going to follow the above strategies and is able to negotiate her reported endowments, a meaningful question is whether this risk sharing game has an equilibrium, (*a Nash-type equilibrium* in particular), the definition of which is given below.

Definition 4.1. *We call a vector of random variables* $(B_1^*, B_2^*, ..., B_n^*) \in \mathcal{X}^n$ *Nash equilibrium of risk sharing game if for each* $i \in \{1, 2, ..., n\}$

$$G_i(B_i^*; (B_j^*)_{j\neq i}) \geq G_i(B_i; (B_j^*)_{j\neq i}) \tag{36}$$

for all $B_i \in \mathcal{X}$.

Under the assumption of quadratic utilities, the optimal sharing is done through equations (11), (12) and the pricing rule (22), where the endowments in (11) and (22) are now the reported ones[4].

In the following proposition we solve this problem under the quadratic utilities.

Proposition 4.4. *Let all agents have quadratic utility functionals and impose the optimal risk sharing rules* (11)*,* (12) *and* (22)*. If each of the agents follow the strategy induced by problem* (29)*, there exists a unique (up to constant additions) Nash equilibrium of the risk sharing game and for each* $i \in \{1, 2, ..., n\}$ *the following statements hold:*

(i) *The best endowment response of agent-i at Nash equilibrium is given by*

$$B_i^* = \frac{c_i}{1+\theta_i}\mathcal{E}_i + \frac{\theta_i}{1+\theta_i}\mathcal{B}^* \tag{37}$$

where

$$c_i = \frac{\gamma_i}{\gamma_i + \gamma} \quad \text{and} \quad \theta_i = \frac{\gamma^2}{\gamma_i^2 - \gamma^2}$$

and $\mathcal{B}^*$ *is the aggregate shared risk,* $\mathcal{B}^* = \sum_{i=1}^n B_i^*$*, given by*

$$\mathcal{B}^* = \frac{\mathcal{E} - \gamma \sum_{i=1}^n \frac{\mathcal{E}_i}{\gamma_i}}{1 - \sum_{i=1}^n \frac{\gamma^2}{\gamma_i^2}} \tag{38}$$

(ii) *The contract that agent-i gets at the optimal sharing is*

$$C_i(B_i^*) = \frac{\gamma_i - \gamma}{\gamma_i}\left(-\mathcal{E}_i + \frac{\gamma}{\gamma_i}\mathcal{B}^*\right) \tag{39}$$

(iii) *The inefficiency caused by agents' strategic behavior is*

$$\sum_{i=1}^n \mathbb{U}_i(\mathcal{E}_i + C_i^*) - \sum_{i=1}^n \mathbb{U}_i(\mathcal{E}_i + C_i(B_i^*)) = \sum_{i=1}^n \gamma_i \operatorname{\mathbb{V}ar}[\mathcal{E}_i - B_i^*] - \gamma \operatorname{\mathbb{V}ar}[\mathcal{E} - \mathcal{B}^*]. \tag{40}$$

Proof. See Appendix A □

Remark 4.5. *As agent-i becomes more risk averse, her reported equilibrium endowment approaches her true endowment. On the other hand, when agent become less risk averse she puts more weight in the speculation part of the risk sharing scheme, that is, she reports a large part of endowment she does not possess and she shares only part of her endowment. For all the non-trivial cases the inefficiency is always positive, since it equals zero if and only if* $\gamma_i \mathcal{E}_i = \gamma_j \mathcal{E}_j$ *for each* $i, j \in \{1, 2, ..., n\}$*, which results in no transaction among the agents (*C_i^* *is constant for all i).*

[4]It should be mentioned at this point that the Nash equilibrium is a reasonable proposed solution only if each player (agent) knows (or is quiet certain) that a Nash-type game is going to be played by each of the agents. In practical situations, an OTC transaction among agents can be thought as a Nash game if the participants renegotiate the risks that are going to share.

Corollary 4.1. *The aggregate sum of endowments is $\mathcal{E}$, i.e. $\mathcal{B}^* = \mathcal{E}$ (all risk are shared, although not optimally) if and only if agents' risk aversion coefficients are equal.*

The situation of agent-1 at Nash equilibrium when $n = 2$ is summarized and compared to the Pareto equilibrium in the following table.

	Pareto Sharing	Nash Sharing
Aggregate shared endowment	$\mathcal{E}$	$\frac{\gamma_1\mathcal{E}_1+\gamma_2\mathcal{E}_2}{2\gamma}$
Reported endowment	$\mathcal{E}_1$	$\frac{2\gamma_1+\gamma_2}{2(\gamma_1+\gamma_2)}\mathcal{E}_1 + \frac{\gamma_2^2}{2\gamma_1(\gamma_1+\gamma_2)}\mathcal{E}_2$
Purchased contract	$\frac{\gamma_2\mathcal{E}_2-\gamma_1\mathcal{E}_1}{\gamma_1+\gamma_2}$	$\frac{1}{2}\frac{\gamma_2\mathcal{E}_2-\gamma_1\mathcal{E}_1}{\gamma_1+\gamma_2}$
Gain of utility from sharing	$\gamma_1 \mathbb{V}\text{ar}\left[\frac{\gamma_1\mathcal{E}_1-\gamma_2\mathcal{E}_2}{\gamma_1+\gamma_2}\right]$	$\frac{\gamma_1+2\gamma_2}{4}\mathbb{V}\text{ar}\left[\frac{\gamma_1\mathcal{E}_1-\gamma_2\mathcal{E}_2}{\gamma_1+\gamma_2}\right]$
Inefficiency	0	$\frac{1}{\gamma_1+\gamma_2}\mathbb{V}\text{ar}\left[\frac{\gamma_1\mathcal{E}_1-\gamma_2\mathcal{E}_2}{2}\right]$

Remark 4.6. *Although the aggregate utility is lower in the case of Nash risk sharing equilibrium, there exist situations where some agents enjoy higher utility when a Nash-type of game is played. The factor that determines whether this happens is the agents' risk aversion coefficients. It is clear from the table above that utility of agent-1 in Nash equilibrium is higher than the one in the Pareto equilibrium if her risk aversion is sufficiently lower that the other agent's risk aversion (when in particular $\gamma_1 < \frac{2}{3}\gamma_2$). We conclude that a Nash-type of game in the risk sharing setting encourages agents to become less risk averse.*

The special case where agents have the same risk aversion level is summarized in the following corollary. In this case, the risk sharing inefficiency is shared equally among the agents.

Corollary 4.2. *Impose the conditions of Proposition 4.4 and assume in addition that agents have common risk aversion coefficient. Then, for each $i \in \{1, 2, ..., n\}$ the following statements hold:*

(i) *The best endowment response of agent-i at Nash equilibrium is given by*

$$B_i^* = \frac{1}{n^2}\sum_{i\neq j}^{n}\mathcal{E}_j + \frac{n(n-1)+1}{n^2}\mathcal{E}_i \tag{41}$$

(ii) *The contract that agent-i gets at the optimal sharing is*

$$C_i(B_i^*) = \frac{n-1}{n^2}\sum_{i\neq j}^{n}\mathcal{E}_j - \frac{(n-1)^2}{n^2}\mathcal{E}_i = \frac{n-1}{n}C_i^* \tag{42}$$

(iii) *The risk sharing inefficiency caused by the strategic behavior of agents is*

$$\sum_{i=1}^{n}\mathbb{U}_i(\mathcal{E}_i + C_i^*) - \sum_{i=1}^{n}\mathbb{U}_i(\mathcal{E}_i + C_i(B_i^*)) = \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathbb{V}\text{ar}[\mathcal{E}_i] - \frac{\mathbb{V}\text{ar}\left[\mathcal{E}\right]}{n}\right). \tag{43}$$

This decrease is equally shared to the agents and equals zero if and only if all endowments $\mathcal{E}_i$'s are constant.

4.5. **Nash equilibrium on reported percentages of true endowments.** The arguments that lead to an equilibrium when agents follow a strategic behavior in risk sharing market can also be implied in the case where the best response of each agent is the percentage of their endowment that is going to be shared (problem (31)). First, we give a similar definition as Definition 4.1 where the equilibrium refers to the reported percentage of the agents' true endowments.

Definition 4.2. *We call a vector* $(b_1^*, b_2^*, ..., b_n^*) \in [0, \infty)^n$ *Nash equilibrium of risk sharing percentage game if for every agent-i*

$$g_i\left(b_i^*; (b_j^* \mathcal{E}_j)_{j \neq i}\right) \geq g_i\left(b_i; (b_j^* \mathcal{E}_j)_{j \neq i}\right) \tag{44}$$

for every $b_i \geq 0$.

As in the previous subsection, under quadratic utility functionals we are able to get the existence and the uniqueness of the Nash equilibrium. We state this result in the following proposition, which proof follows by Glicksburg-Fan-Debreu Theorem (see among others Chapter 1 of [22]) and the linearity of the best response function (see (32)).

Proposition 4.5. *Let all agents have quadratic utility functionals and impose the optimal risk sharing rules of* (11), (12) *and* (22). *If there is an upper bound* $\kappa > 0$ *such that* $b_i \leq \kappa$ *for all* $i \in \{1, 2, ..., n\}$, *there exists a unique Nash equilibrium of the risk sharing percentage game.*

In the simplified case when $n = 2$ the solution of the associated equations is given by

$$b_i^* = 0 \vee \left(\frac{\gamma_i}{\gamma_i + \gamma} + \frac{\gamma^2 b_{-i}^*}{\gamma_i^2 - \gamma^2} \sqrt{\frac{\mathbb{V}\mathrm{ar}[\mathcal{E}_{-i}]}{\mathbb{V}\mathrm{ar}[\mathcal{E}_i]}} \rho(\mathcal{E}_{-i}, \mathcal{E}_i) \right).$$

The cases under common risk aversion coefficients assumption, for three different choices of the fraction $\frac{\mathbb{V}\mathrm{ar}[\mathcal{E}_2]}{\mathbb{V}\mathrm{ar}[\mathcal{E}_1]}$ are illustrated in Figures 1 and 2.

We observe the following:

- (i) The aggregate shared endowment is equal to $\mathcal{E}$ only in the trivial case where $\mathbb{V}\mathrm{ar}[\mathcal{E}_1] = \mathbb{V}\mathrm{ar}[\mathcal{E}_2]$ and $\rho(\mathcal{E}_1, \mathcal{E}_2) = 1$. In such a case however, the optimal risk sharing contract has constant payoff.
- (ii) The agent with the riskier endowment (agent-2) reported lower percentage when $\rho(\mathcal{E}_1, \mathcal{E}_2)$ is positive and higher when $\rho(\mathcal{E}_1, \mathcal{E}_2)$ is negative. This is because the large variance of endowment means lower equilibrium price of $\mathcal{E}_2$ and hence lower price of the contract that the agent-2 is going to short at equilibrium. This price is even less when endowments are positively correlated. However, when the correlation is negative, the price of $\mathcal{E}_2$ is higher and this together with the need for hedging induced by the high variance of $\mathcal{E}_2$ imply an increased b_2^*. This fact is more intense when the difference in $\mathbb{V}\mathrm{ar}[\mathcal{E}_2] - \mathbb{V}\mathrm{ar}[\mathcal{E}_1]$ is larger (figure 2).

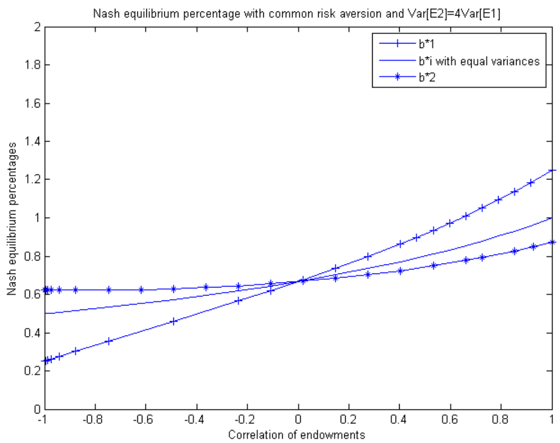

FIGURE 1. Low risk of $\mathcal{E}_2$

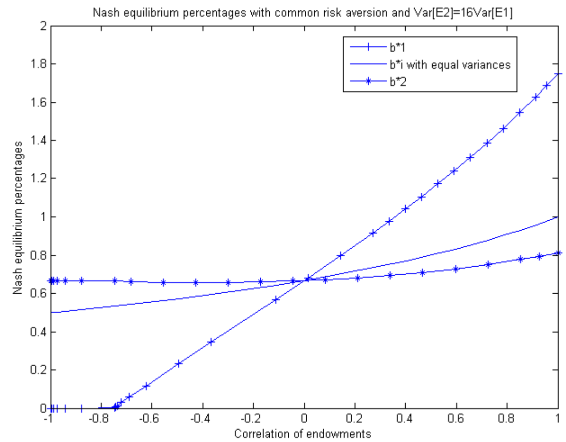

FIGURE 2. High risk of $\mathcal{E}_2$

In Figure 3 below, we illustrate the relation of the inefficiency of risk sharing with respect to $\rho(\mathcal{E}_1, \mathcal{E}_2)$. For $\mathbb{V}\text{ar}[\mathcal{E}_2] \neq \mathbb{V}\text{ar}[\mathcal{E}_1]$, this is a second order polynomial function, which is higher as the difference of variances increases. We also compare this inefficiency with the corresponding of the Nash equilibrium game of Definition 4.1. As we can see their difference is an increasing function of the difference $\mathbb{V}\text{ar}[\mathcal{E}_1] - \mathbb{V}\text{ar}[\mathcal{E}_2]$.

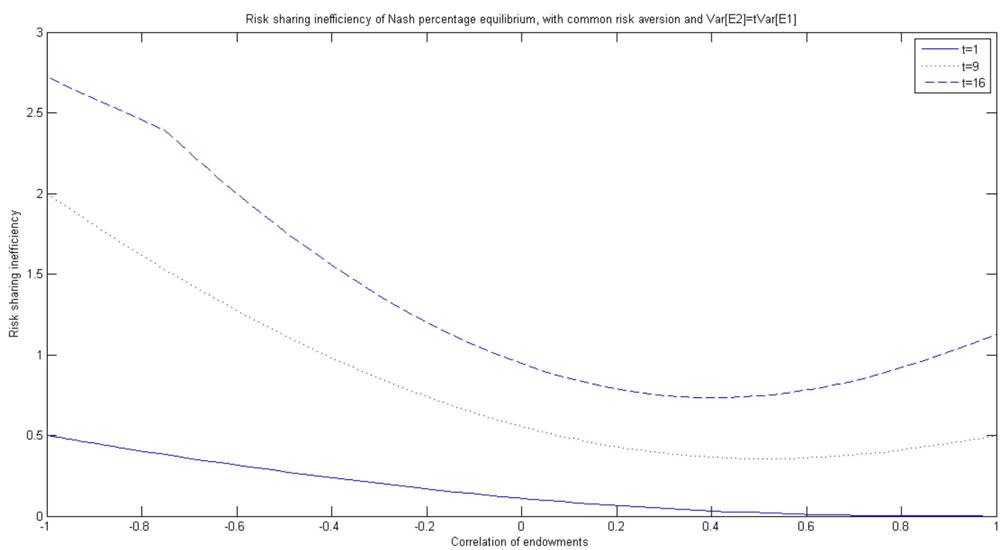

FIGURE 3. Inefficiency w.r.t. endowments' correlation

4.6. **Nash equilibrium in the constrained risk sharing market.** The arguments of Nash equilibrium risk sharing can be applied even in the case of constrained risk sharing setting. In such a market there is a vector of tradable securities $\mathbf{C} \in \mathcal{X}^k$ and each agent reports a demand function that leads to the equilibrium when the sum of all of them is zero. We then impose the strategic behavior discussed in subsection 4.3 for all the agents. In particular, each agent negotiates her demand function on $\mathbf{C}$ according to problem (35) and the (Nash) equilibrium price is the one at which all agents agree. The exact definition is given below.

Definition 4.3. *We call a vector* $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, ..., \hat{p}_k) \in \mathbb{R}^k$ *Nash equilibrium price of a vector of securities* $\mathbf{C} \in \mathcal{X}^k$ *if for every agent-i*

$$\phi_i(\hat{\mathbf{p}}) \geq \phi_i(\mathbf{p}) \tag{45}$$

for every $\mathbf{p} \in \mathbb{R}^k$*, where* ϕ_i *is given in* (34)*.*

Under quadratic utility the Nash equilibrium price can be easily calculated.

Proposition 4.6. *Let all agents have quadratic utility functionals. Then, the Nash equilibrium price of a vector of securities* $\mathbf{C} \in (\mathbb{L}^2(\mathcal{F}))^k$ *is*

$$\hat{\mathbf{p}} = \mathbb{E}[\mathbf{C}] - 2\gamma \operatorname{Cov}(\mathbf{C}, \mathcal{B}^*) \tag{46}$$

where $\mathcal{B}^*$ *is given in* (38)*. In particular the Nash equilibrium price coincides with the equilibrium price defined in Definition 3.1 if and only if* $\gamma_i = \gamma_j$ *for every* $i, j \in \{1, 2, ..., n\}$*. Furthermore:*

(i) *The allocation of* $\mathbf{C}$ *at Nash equilibrium price is*

$$\hat{\mathbf{a}}_i = \operatorname{Cov}(\mathbf{C}, C_i^*(B_i^*)) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \tag{47}$$

where $C_i^*(B_i^*)$ *is given by* (39)*.*

(ii) *For* $n = 2$ *the aggregate utility when the transaction is done at price* $\hat{\mathbf{p}}$ *is*

$$\sum_{i=1}^{2} \mathbb{U}_i(\mathcal{E}_i + \hat{\mathbf{a}}_i \cdot \mathbf{C} - \hat{\mathbf{a}}_i \cdot \hat{\mathbf{p}}) = \sum_{i=1}^{2} \mathbb{U}_i(\mathcal{E}_i) + \frac{3}{4} \operatorname{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \operatorname{Cov}(\mathbf{C}, C_i^*) \tag{48}$$

Hence, the aggregate gain of utility in Nash equilibrium is 75% the aggregate gain in the Pareto equilibrium.

Proof. Taking Proposition 4.3 and Remark 4.4 into account, we conclude that the market equilibrates when the covariances $\operatorname{Cov}(\mathbf{C}, B_i^*)$ equilibrate. As we have seen in Proposition 4.4, this may happen for endowments B_i^* given in (37), which gives price (46).

The equivalence of Nash equilibrium price and the Pareto equilibrium price when agents have common risk aversion is then induced by Corollary 4.2. Furthermore, the item (i) follows by Proposition 3.1 and item (ii) is delivered after simple calculations. □

Remark 4.7. *Although the Nash equilibrium price is equal to Pareto equilibrium price when agents have the same risk aversion, the volume of the transaction differs. Assuming the all γ's are equal then* $\hat{\mathbf{a}}_i = \frac{n-1}{n}\operatorname{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]$*, where the term* $\operatorname{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]$ *is the allocation at Pareto equilibrium (see equation* (20)*). We may follow the same lines as the ones in proof of Proposition 3.2 and get that the utility level of agent-i at this type of Nash equilibrium is* $\frac{n^2-1}{n^2}\gamma_i \operatorname{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \operatorname{Cov}(\mathbf{C}, C_i^*) + \mathbb{U}_i(\mathcal{E}_i)$*. This means that the gain in terms of utility of trading securities* $\mathbf{C}$ *is decreased by the factor* $\frac{n^2-1}{n^2}$ *for each agent, when all of them adapt the strategic behavior induced by* (33)*. Furthermore, the inefficiency of risk sharing is increased by* $\frac{1}{n^2}\sum_{i=1}^{n}\gamma_i \operatorname{Cov}(\mathbf{C}, C_i^*) \cdot \mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}] \cdot \operatorname{Cov}(\mathbf{C}, C_i^*)$.

5. The Agents's Participation as an Endogenous Problem

So far we have assumed that the participation of agents in the risk sharing market is given. However, agents have always the option to share their risks not with all the members of the market but only with a subgroup of them. According to our utility criterion, each agent is going to look for the subgroup of agents in which the risk sharing maximizes her utility function. A possible result of this strategic behavior is the formation of more than one risk sharing subgroups, something that reduces the efficiency of the risk sharing.

In this section, we introduce this subgroup formulation problem in the financial risk sharing setting in the following sense: First, we ask when the enlarged participation in an OTC market would result a higher utility of the existed members of the market and second, we give an introductory example of risk sharing market formation where 3 agents choose with whom they will share their endowments.

The formation of risk sharing markets has been studied in economics literature namely for the case of rural markets (see [10] and [27]). However, there is no formulation of this problem in the case of the OTC markets, where the structured finance transactions take place.

5.1. **When the participation of an extra agent is beneficial?** Consider the case where agents share their risks by trading a given vector of securities $\mathbf{C} \in \mathcal{X}^k$ (a special case is when $\mathbf{C}$ consists of agents' random endowments). In the previous sections we have assumed that the number of agents is fixed and each agent enters to equilibrium risk sharing market if and only if the induced equilibrium price is different than her reservation price. In this section, the following question is addressed:

"*when the participation of an extra agent is going to be beneficial for the rest of the agents?*"

In other words, we want to compare equilibrium induced by n agents with the equilibrium with $(n+1)$ agents. The comparison is based on the level of utility of the first n agents at the equilibrium. More precisely, the n-agents equilibrium implies a level of utility equal to $v_i(\mathbf{p}_n^*)$ (see (7)) for each $i \in \{1, 2, ..., n\}$, where $\mathbf{p}_n^*$ is the associated equilibrium price. If however another agent (throughout called the agent-$(n+1)$) participates in the transaction, the equilibrium allocation of the traded securities and the associated equilibrium price change. We denote the equilibrium price of the $(n+1)$-agents equilibrium by $\mathbf{p}_{n+1}^*$ and hence we call the participation of the agent-$(n+1)$ *beneficial* for all n first agents if

$$v_i(\mathbf{p}_n^*) \leq v_i(\mathbf{p}_{n+1}^*), \quad \text{for each } i \in \{1, 2, ..., n\}. \tag{49}$$

When the condition (49) is satisfied, the agents are willing to encourage the participation of the agent-$(n+1)$ in the market something that increases the aggregate gain of utility and hence the efficiency of the risk sharing transaction. As we have seen in the previous section, even if agents follow strategic behavior in these transactions the enlargement of the market implies the reduction of the inefficiency. In the structure finance markets this questions have an increased interest.

Below we are looking for the exact conditions on the characteristics of the agent-$(n+1)$ (that is, her risky endowment and risk aversion coefficient) so that (49) holds. A primary (at first glance surprising) result is that in the special case of one security ($k = 1$), a necessary condition for this is that agent-$(n+1)$ is the unique buyer or the unique seller at the $(n+1)$ agents equilibrium.

Proposition 5.1. *Let $n+1$ agents have cash invariant utility functionals and $C \in \mathcal{X}$ a given non-constant security. A necessary condition for the participation of the agent-$(n+1)$ in the equilibrium trading of C to be beneficial for all n agents is that agent-$(n+1)$ is the only buyer or the only seller of the equilibrium of $n+1$ agents.*

Proof. First consider the n agents equilibrium. The utility level at equilibrium price p_n^* is given by $v_i(p_n^*)$. Note that function $v_i(p)$ is continuous, convex and its minimizer is the agent-i reservation price, p_{i0} (hence it is strictly decreasing for prices in $(-\infty, p_{i0})$ and strictly increasing otherwise). It also holds that

$$\min_{1 \leq i \leq n} p_{i0} \leq p_n^* \leq \max_{1 \leq i \leq n} p_{i0}.$$

This is because, if we assume otherwise (say $p_n^* > \max\limits_{1 \leq i \leq n} p_{i0}$), it holds that $Z_i(p_n^*) < 0$ (all agents are willing to take the short position on C), which implies that p_n^* can not be the equilibrium price of C. In fact, for buyers at equilibrium it holds that $p_{i0} < p_n^*$ and for the sellers $p_{i0} > p_n^*$ (if $p_{i0} = p_n^*$ for some agent, then her demand is zero). Now assume that the participation of agent-$(n+1)$ is beneficial for all n agents and suppose without loss of generality that $p_{n+1}^* < p_n^*$. This means that the buyers at equilibrium of n agents remain buyers and also their level of utility increases when price is p_{n+1}^*. However, the sellers' level of utility increase only when they become buyers too, i.e. a necessary condition is the $p^*_{n+1} \leq \min_{1\leq i\leq n} p_{i0}$. Similarly, if $p^*_{n+1} > p^*_n$ it should also hold that $p^*_{n+1} \geq \max_{1\leq i\leq n} p_{i0}$. □

Following the arguments of the above proof, it is clear (see also Figure 4) that the sufficient condition for the participation of agent-$(n+1)$ to be beneficial for all n agents is that

$$p^*_{n+1} \leq \min\{v^{-1}_{(1)}\left(v_{(1)}(p^*_n)\right)\} \quad \text{or} \quad p^*_{n+1} \geq \max\{v^{-1}_{(n)}\left(v_{(n)}(p^*_n)\right)\} \tag{50}$$

where agent-(1) (agent-(n)) is the one with the lowest (highest) reservation price and $v^{-1}_{(i)}(\cdot)$ is the set of the inverse images of function $v_{(i)}(\cdot)$. The following graph describes the situation in the simplified case of $n = 2$.

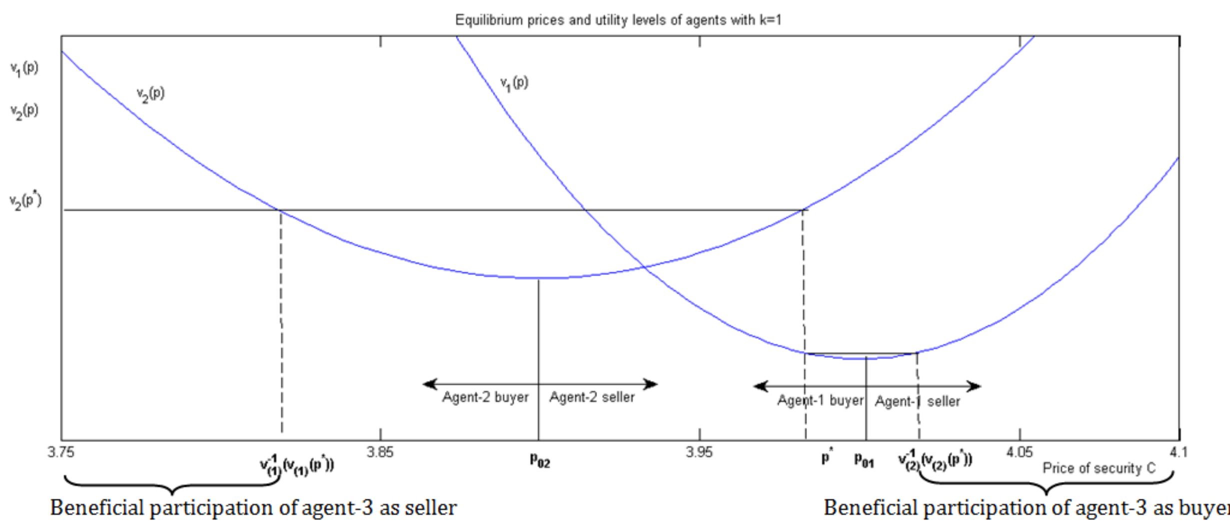

FIGURE 4. Participation of agent-3 when $k = 1$ (this graph is for agents with quadratic utility and $\gamma_1 = \mathbb{V}\text{ar}[C] = \mathbb{U}_1(\mathcal{E}_1) = 1$, $\text{Cov}(C, \mathcal{E}_1) = -2$ and $\gamma_2 = 2$, $\text{Cov}(C, \mathcal{E}_2) = -0.975$ and $\mathbb{U}_2(\mathcal{E}_2) = 1.001$).

However, Proposition 5.1 does not hold in the cases where number of securities are more than 1. Here is a counter example.

Example 5.1. Suppose that $n = 2$ and $\mathbf{C} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) \in (\mathbb{L}^2(\mathcal{F}))^3$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are the random endowments of the two agents (where both have quadratic utility with $\gamma_1 = \gamma_2 = 1$). Since the available for transaction securities include the agents' random endowments, the equilibrium risk sharing is given by (10) and the equilibrium allocation is

$$\mathbf{a}^*_1 = \left(-\tfrac{1}{2}, \tfrac{1}{2}, 0\right) \text{ and } \mathbf{a}^*_2 = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0\right),$$

that is agent-1 (agent-2) is the seller (buyer) of $\mathcal{E}_1$ and the buyer (seller) of $\mathcal{E}_2$ and there is no transaction on $\mathcal{E}_3$. Also, at equilibrium of two agents: $v_i(\mathbf{p}_2^*) = \frac{1}{4}\,\mathbb{V}\mathrm{ar}[\mathcal{E}_1 - \mathcal{E}_2] + \mathbb{U}_i(\mathcal{E}_i)$, for $i = 1, 2$. Suppose now that agent-3 has random endowment $\mathcal{E}_3$ and quadratic utility with $\gamma_3 = 1$. Again by (10)

$$\mathbf{a}_1^* = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right),\ \mathbf{a}_2^* = \left(\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}\right) \text{ and } \mathbf{a}_3^* = \left(\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{1}{3}\right),$$

that is agent-3 is not the only buyer of contracts with payoff $\mathcal{E}_1$ and $\mathcal{E}_2$. In order to complete the counterexample, we need to see cases where agent-1 and agent-2 benefit for the participation of agent-3 in equilibrium. Note that at equilibrium of three agents $v_1(\mathbf{p}_3^*) = \frac{1}{9}\,\mathbb{V}\mathrm{ar}[\mathcal{E}_2 + \mathcal{E}_3 - 2\mathcal{E}_1] + \mathbb{U}_1(\mathcal{E}_1)$, and $v_2(\mathbf{p}_3^*) = \frac{1}{9}\,\mathbb{V}\mathrm{ar}[\mathcal{E}_1 + \mathcal{E}_3 - 2\mathcal{E}_2] + \mathbb{U}_2(\mathcal{E}_2)$. It is left to consider random variables $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ for which

$$\frac{1}{9}\min\{\mathbb{V}\mathrm{ar}[\mathcal{E}_2 + \mathcal{E}_3 - 2\mathcal{E}_1], \mathbb{V}\mathrm{ar}[\mathcal{E}_1 + \mathcal{E}_3 - 2\mathcal{E}_2]\} \geq \frac{1}{4}\,\mathbb{V}\mathrm{ar}[\mathcal{E}_1 - \mathcal{E}_2]. \tag{51}$$

The above inequality is equivalent with the pair of inequalities

$$\begin{aligned}
\mathbb{V}\mathrm{ar}[\mathcal{E}_3] + 2\,\mathrm{Cov}(\mathcal{E}_2, \mathcal{E}_3) - 4\,\mathrm{Cov}(\mathcal{E}_1, \mathcal{E}_3) &\geq \frac{5}{4}\,\mathbb{V}\mathrm{ar}[\mathcal{E}_2] - \frac{7}{4}\,\mathbb{V}\mathrm{ar}[\mathcal{E}_1] - \frac{1}{2}\,\mathrm{Cov}(\mathcal{E}_1, \mathcal{E}_2)\\
\mathbb{V}\mathrm{ar}[\mathcal{E}_3] + 2\,\mathrm{Cov}(\mathcal{E}_1, \mathcal{E}_3) - 4\,\mathrm{Cov}(\mathcal{E}_2, \mathcal{E}_3) &\geq \frac{5}{4}\,\mathbb{V}\mathrm{ar}[\mathcal{E}_1] - \frac{7}{4}\,\mathbb{V}\mathrm{ar}[\mathcal{E}_2] - \frac{1}{2}\,\mathrm{Cov}(\mathcal{E}_1, \mathcal{E}_2).
\end{aligned}$$

For instance, for $\mathbb{V}\mathrm{ar}[\mathcal{E}_1] = \mathbb{V}\mathrm{ar}[\mathcal{E}_2] = 1$ and $\mathrm{Cov}(\mathcal{E}_1, \mathcal{E}_2) = 2$,

$$\begin{aligned}
2\,\mathrm{Cov}(\mathcal{E}_2, \mathcal{E}_3) - 4\,\mathrm{Cov}(\mathcal{E}_1, \mathcal{E}_3) &\geq -\frac{3}{2} - \mathbb{V}\mathrm{ar}[\mathcal{E}_3]\\
2\,\mathrm{Cov}(\mathcal{E}_1, \mathcal{E}_3) - 4\,\mathrm{Cov}(\mathcal{E}_2, \mathcal{E}_3) &\geq -\frac{3}{2} - \mathbb{V}\mathrm{ar}[\mathcal{E}_3]
\end{aligned}$$

where the first inequality refers to the agent-1 and the second to agent-2. In words, both agents are more likely to accept agent-3 when her random endowment is risky and its covariances with $\mathcal{E}_1$ and $\mathcal{E}_2$ is negatively enough. Notice however, that each agent puts more weight in her endowment's correlation with $\mathcal{E}_3$, than to what happens with other endowments. This makes the solution of the above inequalities possible. The following graph visualizes the areas of covariances that will make agents 1 and 2 welcome the entrance of agent-3 in the risk sharing scheme.

In the case where the utility function is quadratic for every agent, we are able to give a specific form of the necessary and sufficient condition under which the participation of the agent-$(n+1)$ is beneficial for the other n agents. The proof follows directly from equation (21).

Proposition 5.2. *Let all $n+1$ agents have quadratic utility functionals. The participation of agent-$(n+1)$ is beneficial for agent-i if and only if*

$$\mathrm{Cov}(\mathbf{C}, C_i^*(n))\,\mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]\,\mathrm{Cov}(\mathbf{C}, C_i^*(n)) \leq \mathrm{Cov}(\mathbf{C}, C_i^*(n+1))\,\mathbb{V}\mathrm{ar}^{-1}[\mathbf{C}]\,\mathrm{Cov}(\mathbf{C}, C_i^*(n+1)) \tag{52}$$

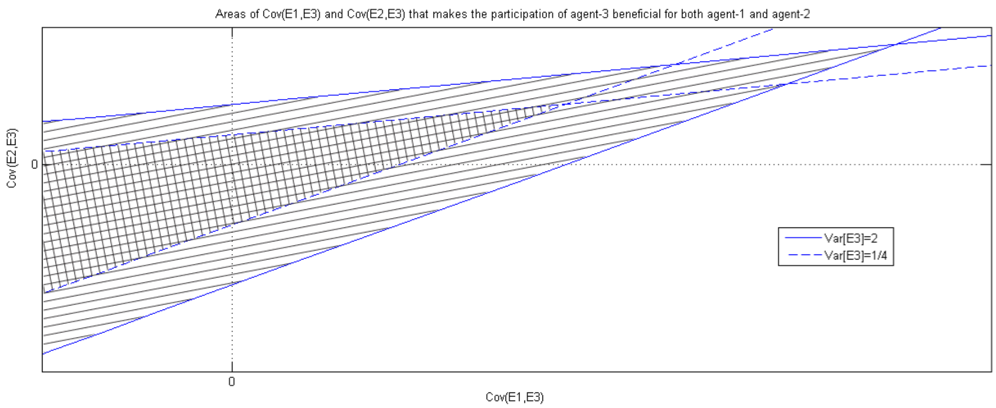

FIGURE 5. Areas for beneficial participation of agent-3.

for all $i \in \{1, 2, ..., n\}$*, where* $C_i^*(n)$ *and* $C_i^*(n+1)$ *are the optimal contracts of agent-i in risk sharing of* n *and* $n+1$ *agents, which are given in Proposition 2.1.*

The interpretation of condition (52) is consistent with the discussion of Section 3. Namely, the participation of agent-$(n+1)$ is beneficial for all the existed agents if and only if the market of $n+1$ agents results in higher scaled covariance between the traded securities and the optimal contract of agent-i, for every $i \in \{1, 2, ..., n\}$. If we further assume normal distribution for $\mathbf{C}$ and $\mathbf{E}$, we may rewrite (52) as

$$\mathbb{V}\text{ar}[\mathbb{E}[C_i^*(n)|\mathbf{C}]] \leq \mathbb{V}\text{ar}[\mathbb{E}[C_i^*(n+1)|\mathbf{C}]], \quad \text{for every } i \in \{1, 2, ..., n\}. \tag{53}$$

5.2. **An Example of Risk Sharing Market Formation.** Example 5.1 gives raise to an important question for OTC risk sharing environments that has not captured the analogous attention in the financial literature. It is often the case where the risk is not shared in the optimal way because some agents find more preferable to share their risks in a sub-market rather than with all the available agents. As we will see in the following case study, this situation implies a non-optimal risk sharing for the market as a whole. The problem of group formation in risk sharing agreements has been recently studied by a number of economists mainly in the problems of rural risk sharing (see [10] and [27]).

In the present work, we initiate the problem of risk sharing group formation in a financial market, where the sharing rules are based on the problems (9) or (10).

We adapt a similar situation as the one in Example 5.1, with three agents with common risk aversion coefficients. Agent-1 has three risk sharing options: sharing her endowment with agent-2, sharing her endowment with agent-3 or sharing her endowment with agent-2 and agent-3. If the sharing rules are given as solutions of problem (9), her criterion is the maximization of her utility level after the transactions. If we further impose quadratic quadratic utility, these options refer to the following utilities:

- For optimal sharing with agent-2: $\mathbb{V}\mathrm{ar}\left[\frac{\mathcal{E}_1-\mathcal{E}_2}{2}\right]+\mathbb{U}_1(\mathcal{E}_1)$
- For optimal sharing with agent-3: $\mathbb{V}\mathrm{ar}\left[\frac{\mathcal{E}_1-\mathcal{E}_3}{2}\right]+\mathbb{U}_1(\mathcal{E}_1)$
- For optimal sharing with agent-2 and agent-3: $\mathbb{V}\mathrm{ar}\left[\frac{\mathcal{E}_1-\mathcal{E}_2}{3}+\frac{\mathcal{E}_1-\mathcal{E}_3}{3}\right]+\mathbb{U}_1(\mathcal{E}_1)$.

The other agents face similar choices. Without loss of generality we may assume that

$$\mathbb{V}\mathrm{ar}[\mathcal{E}_2-\mathcal{E}_3]\leq\mathbb{V}\mathrm{ar}[\mathcal{E}_1-\mathcal{E}_3]\leq\mathbb{V}\mathrm{ar}[\mathcal{E}_1-\mathcal{E}_2] \tag{54}$$

Given (54), only two possible cases are left. Namely, agent-1 and agent-2 are going to reject the risk sharing only with agent-3 and hence the only possible market formations are (agent-1, agent-2) and (agent-1, agent-2, agent-3). In particular, agent-3 is going to be out of the risk sharing market if and only if

$$\mathbb{V}\mathrm{ar}\left[\frac{2\mathcal{E}_1-(\mathcal{E}_2+\mathcal{E}_3)}{3}\right]\leq\mathbb{V}\mathrm{ar}\left[\frac{\mathcal{E}_1-\mathcal{E}_2}{2}\right] \tag{55}$$

since this inequality means that agent-1 does not prefer the three agents market. Even if agent-2 wants agent-3 into the sharing (that is if $\mathbb{V}\mathrm{ar}\left[\frac{2\mathcal{E}_2-(\mathcal{E}_1+\mathcal{E}_3)}{3}\right]>\mathbb{V}\mathrm{ar}\left[\frac{\mathcal{E}_1-\mathcal{E}_2}{2}\right]$), agent-1 is going to refuse which makes agent-2 choose the sharing with agent-1. If inequality (55) holds, agent-1 and agent-2 are going to share their endowments using the optimal risk sharing rules (11) and (12) and agent-3 is left out of the transaction. The result is the risk sharing inefficiency which (according to equation (23)) equals to

$$\frac{1}{2}\mathbb{V}\mathrm{ar}[\mathcal{E}_1+\mathcal{E}_2]+\mathbb{V}\mathrm{ar}[\mathcal{E}_3]-\frac{1}{3}\mathbb{V}\mathrm{ar}[\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3]=\frac{1}{6}\mathbb{V}\mathrm{ar}[\mathcal{E}_1+\mathcal{E}_2-2\mathcal{E}_3].$$

Appendix A.

In this Section give the proof of Theorem 3.1. We begin with a Lemma the proof of which can be found in [20].

Lemma A.1. *Under the assumptions of Theorem 3.1 the following holds for each* $i\in\{1,2,...,n\}$

(1) $\mathbb{U}_i$ *admits the following robust representation:*

$$\mathbb{U}_i(X)=\inf_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\{\mathbb{E}_{\mathbb{Q}}[X]-V_i(\mathbb{Q})\},\quad \textit{for every } X\in\mathbb{L}^\infty(\mathcal{F},\mathbb{P}) \textit{ and}$$

(2)

$$\lim_{m\to+\infty}\frac{f_i(m\mathbf{b}\cdot\mathbf{C})}{m}=\inf_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\mathbb{E}_{\mathbb{Q}}[\mathbf{b}\cdot\mathbf{C}],\quad \textit{for every } \mathbf{b}\in\mathbb{R}^k.$$

where $\mathcal{P}(\mathbb{P})$ *is the set of all probability measures in* $(\Omega,\mathcal{F})$ *absolutely continuous with respect to* $\mathbb{P}$, $V_i:\mathcal{P}(\mathbb{P})\to[0,\infty]$ *is the so-called penalty function and* $\mathbb{E}_{\mathbb{Q}}[\cdot]$ *denotes the expectation under the probability measure* $\mathbb{Q}$.

Proof of Theorem 3.1. First, thanks to Lebesque property and the strict concavity assumption, function f_i is continuously differential in $\mathbb{R}^k$. This means that the function $f : \mathbb{A}_k \to \mathbb{R}$ defined by

$$f(A) = \sum_{i=1}^{n} f_i(\mathbf{a}_i),$$

is also strictly concave and differentiable (where $\mathbf{a}_i$ stands for the i-th row of matrix A). The problem (16) is then written as $\sup_{A\in\mathbb{A}_k} f(A)$. That is, if there exists an allocation A^* such that

$$\nabla f(A^*) = \mathbf{0}$$

or equivalently

$$\nabla f_i(\mathbf{a}_i^*) = \nabla f_n\left(-\sum_{i=1}^{n-1}\mathbf{a}_i^*\right), \quad \text{for every } i = 1, 2, ..., n-1, \tag{56}$$

A^* solves problem (16). In fact, A^* will be the unique solution, thanks to strict concavity of f. If we also denote by $\mathbf{p}^*$ the common for each $i = 1, 2, ..., n-1$ value $\nabla f_i(\mathbf{a}_i^*)$, we get that the pair $(\mathbf{p}^*, A^*)$ is the (unique) price-allocation equilibrium. Hence, it is left to show that ∇f has a root in $\mathbb{A}_k$, or in other words there exist vectors $\mathbf{a}_1^*, \mathbf{a}_2^*, ..., \mathbf{a}_{n-1}^* \in \mathbb{R}^k$ that solve equation (56).

We assume for contradiction that $\nabla f(A) \neq \mathbf{0}$ for all $A \in \mathbb{A}_k$. Since f is strictly concave, this means that for each $m \in \mathbb{N}$ there exists a sequence $A(m) \in \mathbb{R}^{(n-1)\times k}$ such that $||A(m)||_1 = m$ and $f(A(m)) \geq f(B)$ for every $B \in \mathbb{R}^{(n-1)\times k}$, with $||B||_1 \leq m$ (where $||B||_1 = \sum_{i=1}^{n-1}\sum_{j=1}^{k}|B_{i,j}|$). Taking into account Proposition 1.1.5 of [12], we will have a contradiction if $\lim_{m\to+\infty}\frac{f(A(m))}{m} < 0$.

Let $\mathbf{a}_i(m)$ be the i-th row of matrix $A(m)$, $i = 1, 2, ..., n-1$. Due to boundness of $\left|\left|\frac{\mathbf{a}_i(m)}{m}\right|\right|_1$, there exists a subsequence of $\frac{a_i(m)}{m}$ that converges in some vector $\mathbf{a}_i^{(0)} \in \mathbb{R}^k$, for each $i \in \{1, 2, ..., n-1\}$. Now, Lemma 4.3 of [21] implies that

$$\left|\frac{f_i(\mathbf{a}_i(m))}{m} - \frac{f_i(m\mathbf{a}_i^{(0)})}{m}\right| \leq \left|\left|\frac{\mathbf{a}_i(m)}{m} - \mathbf{a}_i^{(0)}\right|\right| \, ||\mathbf{C}|| \longrightarrow 0.$$

Taking the second item of Lemma A.1 into account we conclude that

$$\frac{f_i(\mathbf{a}_i(m))}{m} \longrightarrow \inf_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\mathbb{E}_{\mathbb{Q}}[\mathbf{a}_i^{(0)}\cdot\mathbf{C}].$$

for each $i \in \{1, 2, ..., n-1\}$. Similarly, since $\frac{\sum_{i=1}^{n-1}\mathbf{a}_i(m)}{m}$ converges to $\sum_{i=1}^{n-1}\mathbf{a}_i^{(0)}$ we get that

$$\frac{f_n\left(-\sum_{i=1}^{n-1}\mathbf{a}_i(m)\right)}{m} \longrightarrow \inf_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\mathbb{E}_{\mathbb{Q}}\left[-\sum_{i=1}^{n-1}\mathbf{a}_i^{(0)}\cdot\mathbf{C}\right].$$

Therefore,

$$
\begin{aligned}
\lim_{m\to+\infty}\frac{f(A(m))}{m} &= \sum_{i=1}^{n-1}\inf_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\mathbb{E}_{\mathbb{Q}}[\mathbf{a}_i^{(0)}\cdot\mathbf{C}] + \inf_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\mathbb{E}_{\mathbb{Q}}\left[-\sum_{i=1}^{n-1}\mathbf{a}_i^{(0)}\cdot\mathbf{C}\right]\\
&\geq \inf_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{n-1}\mathbf{a}_i^{(0)}\cdot\mathbf{C}\right] + \inf_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\mathbb{E}_{\mathbb{Q}}\left[-\sum_{i=1}^{n-1}\mathbf{a}_i^{(0)}\cdot\mathbf{C}\right]\\
&= \inf_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{n-1}\mathbf{a}_i^{(0)}\cdot\mathbf{C}\right] - \sup_{\mathbb{Q}\in\mathcal{P}(\mathbb{P})}\mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{n-1}\mathbf{a}_i^{(0)}\cdot\mathbf{C}\right] < 0
\end{aligned}
$$

where the last strict inequality follows from Assumption 2.1.

Proof of Proposition 4.1. We fix $\mathcal{E}_{-i} = (\mathcal{E}_1, ..., \mathcal{E}_{i-1}, \mathcal{E}_{i+1}, ..., \mathcal{E}_n) \in (\mathbb{L}^2(\mathcal{F}))^{n-1}$ and from (28) (with a slight abuse of notation) we get that

$$
\begin{aligned}
G_i(B) &= \mathbb{E}[\mathcal{E}_i] - \frac{\gamma^2}{\gamma_i}\mathbb{V}\mathrm{ar}[\mathcal{E}] - \frac{(\gamma_i-\gamma)^2}{\gamma_i}\mathbb{V}\mathrm{ar}[\mathcal{E}_i - B] - \frac{2\gamma(\gamma_i-\gamma)}{\gamma_i}\left(\mathrm{Cov}(\mathcal{E}, \mathcal{E}_i - B) + \mathbb{V}\mathrm{ar}[B]\right)\\
&+ \frac{2\gamma(2\gamma-\gamma_i)}{\gamma_i}\mathrm{Cov}(B, \mathcal{E}-\mathcal{E}_i). \qquad (57)
\end{aligned}
$$

Hence, in order to solve problem (29), it is enough to find $B \in \mathbb{L}^2(\mathcal{F})$ that minimizes

$$
\frac{\gamma_i-\gamma}{2\gamma}\mathbb{V}\mathrm{ar}[\mathcal{E}_i - B] + \mathrm{Cov}(\mathcal{E}, \mathcal{E}_i - B) + \mathbb{V}\mathrm{ar}[B] + \mathrm{Cov}(B, \mathcal{E}-\mathcal{E}_i)\left(1 - \frac{\gamma}{\gamma_i-\gamma}\right).
$$

Since the above quantity is constant invariant, we may consider for now that $\mathbb{E}[B] = 0$. Thus, we want to minimize function $W_i : \mathbb{L}^2(\mathcal{F}) \to \mathbb{R}$ defined by

$$
W_i(B) = \mathbb{E}\left[\frac{\gamma_i-\gamma}{2\gamma}(\mathcal{E}_i - B)^2 + \mathcal{E}(\mathcal{E}_i - B) + B^2 + \left(1 - \frac{\gamma}{\gamma_i-\gamma}\right)B(\mathcal{E}-\mathcal{E}_i)\right].
$$

The minimum of $W_i(\cdot)$ can be obtained through its *Fréchet derivative*. For this, we first show that the Fréchet derivative of $W_i(\cdot)$ at $B \in \mathbb{L}^2(\mathcal{F})$ (denoted by $D_{W_i(B)}$) is given by

$$
D_{W_i(B)}[X] = \mathbb{E}\left[\left(\frac{\gamma_i+\gamma}{\gamma}B - \frac{\gamma_i-\gamma}{\gamma}\mathcal{E}_i - \mathcal{E} + \left(1 - \frac{\gamma}{\gamma_i-\gamma}\right)(\mathcal{E}-\mathcal{E}_i)\right)X\right],
$$

for every $X \in \mathbb{L}^2(\mathcal{F})$. In order to prove this statement, it is enough to get the following limit

$$
\frac{|W_i(B+X) - W_i(B) - D_{W_i(B)}[X]|}{||X||_{\mathbb{L}^2(\mathcal{F})}} \longrightarrow 0 \qquad (58)
$$

as $||X||_{\mathbb{L}^2(\mathcal{F})} \to 0$. Indeed,

$$
\frac{|W_i(B+X) - W_i(B) - D_{W_i(B)}[X]|}{||X||_{\mathbb{L}^2}} = \left(\frac{\gamma_i+\gamma}{2\gamma}\right)\frac{|\mathbb{E}\left[X^2\right]|}{||X||_{\mathbb{L}^2}} = \left(\frac{\gamma_i+\gamma}{2\gamma}\right)||X||_{\mathbb{L}^2} \longrightarrow 0.
$$

Note that function $W_i(\cdot)$ is strictly convex in $\mathbb{L}^2(\mathcal{F})$ and hence, if there exists $B^* \in \mathbb{L}^2(\mathcal{F})$ such that $D_{W_i(B^*)}[X] = 0$ for each $X \in \mathbb{L}^2(\mathcal{F})$, then B^* is the unique (in the class of $\mathbb{L}^2(\mathcal{F})$ with expectation equal to zero) minimizer of $W_i(\cdot)$. Clearly, for B_i^* given in (30), $D_{W_i(B_i^*)}[X] = 0$ for every $X \in \mathbb{L}^2(\mathcal{F})$ which makes B_i^* the best response of agent-i.

Proof of Proposition 4.4. From Proposition 4.1 and the FOCs for the Nash equilibrium, we get that Nash equilibria are the solutions $(B_1^*, B_2^*, ..., B_n^*) \in (\mathbb{L}^2(\mathcal{F}))^n$ of the following system of n linear equations

$$\begin{aligned} B_1^* &= \frac{\gamma_1}{\gamma_1+\gamma}\mathcal{E}_1 + \frac{\gamma^2}{\gamma_1^2-\gamma^2}\sum_{j\neq 1}^{n} B_j^* \\ B_2^* &= \frac{\gamma_2}{\gamma_2+\gamma}\mathcal{E}_2 + \frac{\gamma^2}{\gamma_2^2-\gamma^2}\sum_{j\neq 2}^{n} B_j^* \\ &\vdots \\ B_n^* &= \frac{\gamma_n}{\gamma_n+\gamma}\mathcal{E}_n + \frac{\gamma^2}{\gamma_n^2-\gamma^2}\sum_{j=1}^{n-1} B_j^* \end{aligned} \tag{59}$$

We fix $\omega \in \Omega$ and we are looking for $\mathbf{B} = (B_1, B_2, ..., B_n)$ which solves the above linear system. For each $i \in \{1, 2, ..., n\}$, we have that $B_i^* = \frac{\gamma_i}{\gamma_i+\gamma}\mathcal{E}_i + \frac{\gamma^2}{\gamma_i^2-\gamma^2}\sum_{j=1}^n B_j^* - \frac{\gamma^2}{\gamma_i^2-\gamma^2}B_i^*$, which given (37). But then,

$$\mathcal{B}^* = \frac{\sum_{i=1}^n \frac{c_i}{1+\theta_i}\mathcal{E}_i}{1-\sum_{i=1}^n \frac{\theta_i}{1+\theta_i}},$$

which is equivalent to (38). Since $B_i^*(\omega)$ is a linear combination of $\mathcal{E}_1(\omega), \mathcal{E}_2(\omega), ..., \mathcal{E}_n(\omega)$, $B_i^* : \Omega \longrightarrow \mathbb{R}$ is a $\mathcal{F}$-measurable and in particular it belongs in $\mathbb{L}^2(\mathcal{F})$.

Finally, it is left to observe that if $\mathbf{B}^* = (B_1^*, B_2^*, ..., B_n^*)$ is a Nash equilibrium, every vector of the form $\mathbf{B}^* + \mathbf{c}$, for $\mathbf{c} \in \mathbb{R}^n$ is also a Nash equilibrium, since it satisfies (36). The fact that this form of Nash equilibria is unique follows from the strict convexity of $G_i(\cdot)$. Equations (39) and (40) follow by straightforward calculations.

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