A shorter proof of Lemma A.6

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Abstract

For the convenience of readers of the article *No-arbitrage pricing under systemic risk: accounting for cross-ownership* (Fischer, 2012), a full proof of Lemma A.5 and a shorter proof of Lemma A.6 of that paper are provided.

1 Lemma A.5

Lemma A.5 provides a method of splitting a positive number (or an interval) into a pre-specified number of ordered summands (or subintervals) where we demand that, while the order increases, the summands (or subintervals) have pre-specified positive sizes (or pre-specified lengths) for as long as possible. While this formulation sounds trivial (the 'algorithm' for solving this problem certainly is trivial), the resulting formula (1.1) is possibly not directly obvious at first sight, especially if the previous formulation is not given.

Lemma A.5. *For $x \in \mathbb{R}$, $m \in \{1, 2, \ldots\}$, and $y^1, \ldots, y^m \in \mathbb{R}_0^+$,*

$$(1.1)\quad x \;=\; \min\left\{y^1, x\right\} + \sum_{j=1}^{m-1} \min\left\{ y^{j+1}, \left(x - \sum_{i=1}^{j} y^i \right)^+ \right\} + \left(x - \sum_{i=1}^{m} y^i \right)^+ .$$

Proof. The case $x \leq y^1$ is clear. The case $x \geq \sum_{i=1}^m y^i$ is clear, because then

$$(1.2)\qquad x - \sum_{i=1}^{j} y^i \;\geq\; \sum_{i=j+1}^{m} y^i \;\geq\; y^{j+1} \quad (j \in \{0, \ldots, m-1\}).$$

Excluding the two cases above, one must have $m \geq 2$ and

$$(1.3)\qquad y^1 \;<\; x \;<\; \sum_{i=1}^{m} y^i.$$

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By the assumptions, there exists now an $i_0 \in \{1, \ldots, m-1\}$ with

$$\sum_{i=1}^{i_0} y^i \;<\; x \;\leq\; \sum_{i=1}^{i_0+1} y^i, \tag{1.4}$$

which is equivalent to

$$0 \;<\; x - \sum_{i=1}^{i_0} y^i \;\leq\; y^{i_0+1}. \tag{1.5}$$

From the left inequality in (1.4) one obtains

$$x - \sum_{i=1}^{j} y^i \;>\; \sum_{i=j+1}^{i_0} y^i \;\geq\; y^{j+1} \quad (j \in \{0, \ldots, i_0 - 1\}). \tag{1.6}$$

From the right inequality of (1.4), one obtains

$$x - \sum_{i=1}^{i_0+1} y^i \;\leq\; 0. \tag{1.7}$$

Applying (1.3), (1.6), (1.5) and (1.7) to the right hand side of (1.1), we obtain

$$y^1 + \ldots + y^{i_0} + \left(x - \sum_{i=1}^{i_0} y^i \right) \;=\; x. \tag{1.8}$$

□

2 Lemma A.6

The next lemma is a result which gives a condition under which the difference of two numbers which have been split into the same amount of summands (of which some can be zero) according to Lemma 5 can be expressed as the sum of the absolute values of the differences of their summands.

LEMMA 2.1. *Assume* $x_1, x_2 \in \mathbb{R}$ *where*

$$x_1 \;\geq\; x_2 \tag{2.1}$$

and $y_1^i, y_2^i \in \mathbb{R}_0^+$ *(*$i = 1, \ldots, m$*) where*

$$y_1^i \;\geq\; y_2^i \quad (i = 1, \ldots, m) \tag{2.2}$$

and

$$x_1 - x_2 \;\geq\; \sum_{i=1}^{m} \left(y_1^i - y_2^i \right). \tag{2.3}$$

Then, the following equation holds:

$$x_1 - x_2 = \left|\min\left\{y_1^1, x_1\right\} - \min\left\{y_2^1, x_2\right\}\right| + \sum_{j=1}^{m-1} \left|\min\left\{y_1^{j+1}, \left(x_1 - \sum_{i=1}^{j} y_1^i\right)^+\right\} - \min\left\{y_2^{j+1}, \left(x_2 - \sum_{i=1}^{j} y_2^i\right)^+\right\}\right| + \left|\left(x_1 - \sum_{i=1}^{m} y_1^i\right)^+ - \left(x_2 - \sum_{i=1}^{m} y_2^i\right)^+\right|. \tag{2.4}$$

Proof. (2.3) together with (2.2) implies

$$x_1 - x_2 \geq \sum_{i=1}^{j} \left(y_1^i - y_2^i\right) \quad (j = 1, \ldots, m), \tag{2.5}$$

and therefore

$$x_1 - \sum_{i=1}^{j} y_1^i \geq x_2 - \sum_{i=1}^{j} y_2^i \quad (j = 1, \ldots, m). \tag{2.6}$$

(2.1), (2.2) and (2.6) imply that all differences on the right hand side of (2.4) are non-negative. We can therefore apply (1.1) (with $x \mathrel{\widehat{=}} x_{1/2}$ and $y^i \mathrel{\widehat{=}} y^i_{1/2}$) and obtain the result. □

Obviously, $y_1^i = y_1^j = y_2^i = y_2^j \geq 0$ for all $i, j \in \{1, \ldots, m\}$ would satisfy the conditions of the lemma.

Lemma A.6 of Fischer (2012) follows now by noting that, without loss of generality, $y^1 \geq y^2$ can be assumed in the proof of that lemma. Therefore, Lemma 2.1 (with $x_j \mathrel{\widehat{=}} x + y^j$ and $y_j^i \mathrel{\widehat{=}} \psi^i(y^j)$ for $i = 1, \ldots, m$ and $j = 1, 2$) can be applied.

References

[1] Fischer, T. (2012): No-arbitrage pricing under systemic risk: accounting for cross-ownership. Mathematical Finance. doi: 10.1111/j.1467-9965.2012.00526.x