

Computing Quantiles in Regime-Switching Jump-Diffusions with Application to Optimal Risk Management: a Fourier Transform Approach

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Abstract

In this paper we consider the problem of calculating the quantiles of a risky position, the dynamic of which is described as a continuous time regime-switching jump-diffusion, by using Fourier Transform methods. Furthermore, we study a classical option-based portfolio strategy which minimizes the Value-at-Risk of the hedged position and show the impact of jumps and switching regimes on the optimal strategy in a numerical example. However, the analysis of this hedging strategy, as well as the computational technique for its implementation, is fairly general, i.e. it can be applied to any dynamical model for which Fourier transform methods are viable.

Key words: regime switching jump-diffusion models, Value at Risk, risk management, Fourier transform methods.

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1 Introduction

Quantiles computation of profit and losses of a given financial position is a basic task for measuring and managing portfolio market risks. The dynamic of the driving risk factors, as well as the chosen risk measure are the basic ingredient for the analysis of any risk management strategy. In this paper the model we consider for the risky position is of the form $S(t) = s_0 e^{X(t)}$, where $X(t)$ is specified on a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathcal{P})$ as a jump-diffusion whose

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parameters change over time, driven by a continuous time and stationary Markov Chain on the finite state space $\mathcal{S} = \{1, 2, \dots, M\}$, representing the unobserved *state of the world*. In fact, empirical studies on the behavior of financial markets show the ability of regime-switching models to capture some peculiarities in the observed data, as firstly highlighted in the seminal paper by Hamilton [13]. Since then there has been a growing effort in applying switching models to a wide class of financial and/or economic problems, such as time series analysis, portfolio theory, derivative pricing and risk management. On the other hand, the necessity of including jumps in the underlying models to provide better representation of their dynamical properties is widely recognized (see e.g. [9]). Regime-switching jump-diffusions turns out to be an appealing and flexible class of dynamic models.

Among many different risk measures proposed in the literature, Value-at-Risk, although sharply criticized for the lack of sub-additivity and its inability to quantify the severity of an exposure to rare events, has been adopted as a benchmark in the financial industry and for regulatory purposes. It plays a central role in banking regulation and internal risk management, mainly due to its simplicity. We therefore take the VaR as a starting point of our analysis of risk management strategies. The computation of VaR in Regime-Switching models has been considered by several authors mainly in discrete-time setting (see e.g. [6] or more recently [14], [23]). Here we consider this problem directly in the continuous time framework: as a matter of fact, the required computations can be very efficiently implemented with the help of Fourier Transform methods (see e.g. [20]). The use of this kind of technique for the analytical calculation of VaR has been considered in Duffie and Pan [12] who exploited the classical characterization of the distribution function in terms of the Fourier inversion of its characteristic function. The use of Generalized Fourier Transform and the FFT algorithm is more recent: see Le Courtois and Walter [17] who calculate the VaR for the Variance Gamma (VG) model and Kim et al. [15], Scherer et al. [22], who consider the class of tempered stable and infinitely divisible distributions.

As an application to risk management, we investigate the influence of jumps and switching regimes on the exposure to an underlying risky asset. More precisely we study a classical hedging policy based on options followed by an institutional manager whose aim is to minimize the VaR of a position. This type of analysis has been initiated by [1] a decade ago for a portfolio made by a risky asset following a log-normal random dynamic, and hence analytically solved in a Black-Scholes setting. More recently, it has been considered for a bond portfolio in [11], [3], [4].

By taking the VaR as the risk measure for potential losses L of a portfolio at a given level α (i.e. the value $\text{VaR}_\alpha(L)$ such that $\text{Prob}\{L > \text{VaR}_\alpha(L)\} = \alpha$), the strategy considered consists on minimizing the VaR of the option-hedged portfolio $L^{h,K}$ with respect to the strike price and the quantity of the put option written on the risky asset, subject to a budget constraint: in other words, we hedge the risky position by buying a fraction $h \in (0, 1]$ of a put option with maturity T and strike price K , but what K and h ? The optimal hedging strategy is therefore

given by the solution of the following *program*:

$$\begin{cases} \min_{K,h} \text{VaR}_\alpha(L^{h,K}) \\ h\Pi_0^P(K,T) = C, \quad h \in (0,1] \end{cases}$$

$\Pi_0^P(K,T)$ being the time 0 put price with strike K and maturity T . To implement the strategy, it is therefore needed *i*) the calculation of the VaR for the risky asset and *ii*) the corresponding value of a put option $\Pi_0^P(K,T)$. Under the classical no-arbitrage assumption, the price of the put option can be represented as the expected value of the payoff with respect to a risk-neutral probability (see e.g. [7]). Conversely, VaR is obtained under the objective or historical probability measure. Both steps can be efficiently faced with the Fourier transform technique.

The paper is organized as follows: we firstly derive the optimality conditions for the VaR minimizing strategy (Section 2) and then we specify the Fourier Transform technique for calculating quantiles and put/call option prices in a very general setting (Section 3). In Section 4 we introduce the regime-switching dynamic model, its generalized characteristic function and the main change-of-measure result for switching from the historical to the risk-neutral probability. Finally, in Section 5 some numerical experiments are reported to show the impact of jumps and regime-switching on the process quantiles and on the optimal hedging strategy. We use in particular a simple two-state model with gaussian jumps and quantify in such a case what is the effect of a wrong model choice.

2 VaR and optimal risk management

To measure the risk of a financial position the quantiles of its distribution function are commonly used. Given a confidence level $\alpha \in (0,1)$, the set of ϵ -quantiles of the random variable Y is the interval $[q_\alpha^-(Y), q_\alpha^+(Y)]$ where

$$q_\alpha^-(Y) = \inf\{q \in \mathbb{R} | P(Y \leq q) \geq \alpha\}, \quad q_\alpha^+(Y) = \inf\{q \in \mathbb{R} | P(Y \leq q) > \alpha\}.$$

For a random variable having continuous and strictly increasing distributions function $F_Y(y)$, $q_\alpha^-(Y) = q_\alpha^+(Y) \equiv q_\alpha(Y)$ and $q_\alpha(Y) = F_Y^{-1}(\alpha)$, i.e. it solves the equation

$$P(Y \leq q_\alpha(Y)) = \alpha.$$

Here we take the portfolio loss L to describe a financial position in a fixed time interval and, in order to simplify notations, we assume in this section that L has a continuous and strictly increasing distributions function. The Value-at-Risk at level α is defined as

$$\text{VaR}_\alpha(L) \equiv \inf\{q \in \mathbb{R} | P(L \leq q) \geq \alpha\}.$$

Let S_t be the value of the risky asset, $t \in [0, T]$ and r be the risk-free rate, that without loss of generality we consider fixed in the period: we define the loss at time 0 of such a position as

$$L^u = S_0 - e^{-rT} S_T.$$

Then

$$\text{VaR}_\alpha(L^u) = S_0 - e^{-rT} q_{1-\alpha}(S_T).$$

Let us now consider a classical hedging problem in which an institution has an exposure to a risky asset S_t and decide to hedge such an exposure in the interval $[0, T]$ by buying a fraction $h \in (0, 1]$ of an European put option on the asset with maturity T and strike price K . Analogously to the situation considered by Ahn et al. (1999), we take as the hedged position the portfolio composed by the risky asset and the put option: the loss of the hedged portfolio at time 0 is therefore

$$L^{h,K} = S_0 + h\Pi_0^P(K, T) - e^{-rT}(S_T + h(K - S_T)^+)$$

where $\Pi_t^P(K, T)$ is the price of the put option at time t . By defining the strictly increasing function

$$g(u) = u - h(u - \bar{K})^+ + h\Pi_0^P(K, T)$$

where $\bar{K} = S_0 - e^{-rT}K$, it is immediately seen that

$$L^{h,K} = g(L^u);$$

therefore

$$\begin{aligned} \text{VaR}_\alpha(L^{h,K}) &= g(\text{VaR}_\alpha(L^u)) = S_0 - e^{-rT} q_{1-\alpha}(S_T) + h\Pi_0^P(K, T) - e^{-rT} h(K - q_{1-\alpha}(S_T))^+ \\ &= \text{VaR}_\alpha(L^u) + h\Pi_0^P(K, T) - e^{-rT} h(K - q_{1-\alpha}(S_T))^+. \end{aligned}$$

Let us firstly notice that if $K \leq q_{1-\alpha}(S_T)$, then

$$\text{VaR}_\alpha(L^{h,K}) > \text{VaR}_\alpha(L^u)$$

since $\Pi_0^P(K, T) > 0$. Therefore the optimal hedging strategy is given by the following problem:

$$\begin{cases} \min_{K,h} \text{VaR}_\alpha(L^u) + h\Pi_0^P(K, T) - e^{-rT} h(K - q_{1-\alpha}(S_T)) \\ h\Pi_0^P(K, T) = C, \\ h \in (0, 1], \quad K > q_{1-\alpha}(S_T), \end{cases} \quad (1)$$

C being the budget constraint. Since $h = C/\Pi_t^P(K, T)$, the optimality first order condition for K is given by the following non-linear equation:

$$\Pi_0^P(K, T) = (K - q_{1-\alpha}(S_T)) \frac{\partial}{\partial K} \Pi_0^P(K, T). \quad (2)$$

Assuming that (2) has a solution $K^* > q_{1-\alpha}(S_T)$ and the twice differentiability of the price functional we can prove that this is actually a minimum since

$$\frac{\partial^2 \text{VaR}_\alpha(L^{h,K})}{\partial K^2} = \frac{e^{-rT} C (K^* - q_{1-\alpha}(S_T))}{\Pi^P(K^*, T)^2} \frac{\partial^2 \Pi^P}{\partial K^2}(K^*, T) > 0$$

by the convexity of the price functional w.r.t. the strike. Correspondingly, the optimal amount of the hedging put option is

$$h^* = \frac{C}{\Pi_0^P(K^*, T)}. \quad (3)$$

We now assume the following:

Assumption 2.1. *The price of the put option can be represented as the discounted expected value of the payoff at time T under a risk-neutral measure \mathcal{Q} :*

$$\Pi_t^P(K, T) = e^{-rT} \mathbb{E}^{\mathcal{Q}}[(K - S_T)^+].$$

Furthermore, let $F_S(s) = \mathcal{Q}(S_T \leq s)$ be the cumulative distribution function (cdf) of the random variable S_T under such a measure: hence

$$\begin{aligned} \Pi_t^P(K, T) &= e^{-rT} \int_{-\infty}^{+\infty} (K - s)^+ dF_S(s) = e^{-rT} \left(K \int_{-\infty}^K dF_S(s) - \int_{-\infty}^K s dF_S(s) \right) \\ &= e^{-rT} \left(K \mathcal{Q}(S_T \leq K) - \int_{-\infty}^K s dF_S(s) \right) \end{aligned}$$

and

$$\frac{\partial}{\partial K} \Pi_t^P(K, T) = e^{-rT} \mathcal{Q}(S_T \leq K).$$

We can finally prove the following property:

Proposition 2.1. *If $K^* > q_{1-\alpha}(S_T)$, then $\text{VaR}_\alpha(L_0^{h^*, K^*}) < \text{VaR}_\alpha(L^u)$.*

Proof. Since K^* and h^* are characterized through (2) and (3), we get

$$\text{VaR}_\alpha(L_0^{h^*, K^*}) = \text{VaR}_\alpha(L^u) + C - e^{-rT} h^* (K^* - q_{1-\alpha}(S_T))$$

$$\begin{aligned}
&= \text{VaR}_\alpha(L^u) + \frac{C}{\Pi_t^P(K^*, T)} (\Pi_t^P(K^*, T) - e^{-rT}(K^* - q_{1-\alpha}(S_T))) \\
&= \text{VaR}_\alpha(L^u) + \frac{C}{\Pi_t^P(K^*, T)} (K^* - q_{1-\alpha}(S_T)) \left(\frac{\partial}{\partial K} \Pi_t^P(K^*, T) - e^{-rT} \right).
\end{aligned}$$

From Assumption 2.1, we have

$$\frac{\partial}{\partial K} \Pi_t^P(K, T) - e^{-rT} = e^{-rT} (\mathbb{Q}(S_T \leq K) - 1) < 0.$$

Therefore

$$\text{VaR}_\alpha(L_0^{h^*, K^*}) = \text{VaR}_\alpha(L^u) + \frac{C}{\Pi_t^P(K^*, T)} (K^* - q_{1-\alpha}(S_T)) e^{-rT} (\mathbb{Q}(S_T \leq K^*) - 1) < \text{VaR}_\alpha(L^u).$$

□

Remark 2.1. Notice that the optimality condition (2) under Assumption 2.1 becomes

$$e^{-rT} \left(K \mathbb{Q}(S_T \leq K) - \int_{-\infty}^K s dF_S(s) \right) = (K - q_{1-\alpha}(S_T)) e^{-rT} \mathbb{Q}(S_T \leq K)$$

which simplifies to

$$\frac{1}{\mathbb{Q}(S_T \leq K)} \int_{-\infty}^K s dF_S^{\mathbb{Q}}(x) = q_{1-\alpha}(S_T) \quad (4)$$

and depends on both the objective and the risk neutral distributions \mathcal{P} and \mathcal{Q} . Furthermore, it easily seen that the l.h.s. is equal to the conditional expectation $\mathbb{E}^{\mathbb{Q}}[S_T | S_T \leq K]$ which is an increasing function of K bounded by $\mathbb{E}^{\mathbb{Q}}[S_T]$. Therefore, the eq. (4) has a unique solution if and only if $q_{1-\alpha}(S_T) < \mathbb{E}^{\mathbb{Q}}[S_T]$.

The minimum VaR as a function of the budget C is therefore

$$\text{VaR}^* = \text{VaR}_\epsilon(V) + C \frac{\Pi(K^*, T) e^{rT} - (K^* - q_X^+(\epsilon))}{\Pi(K^*, T)}, \quad (5)$$

which is a linear function with negative slope. In the plane budget-risk the previous equation describe an *efficient frontier* giving for each level of the budget C the minimum VaR.

Remark 2.2. It is easy to show that the problem of looking for the hedging strategy with the minimum cost $C(h, K) = h\Pi(K, T)$ for a target level v of VaR, results in the same first order optimality condition for K and that we get the same linear efficient frontier

$$C^* = h^* \Pi(K^*) = (v - \text{VaR}_\epsilon(V)) \frac{\Pi(K^*)}{\Pi(K^*) e^{rT} - (K^* - q_X^+(\epsilon))},$$

in the budget-risk plane.

3 The Fourier transform method

Fourier transform methods are efficient techniques emerged in recent years as one of the main methodology for the evaluation of derivatives. In fact, the no-arbitrage price of an european style contingent claim can be represented as the (conditional) expectation of the derivative payoff under a proper risk-neutral measure (see e.g. [7]). These methods essentially consist on the representation of such an expectation as a "convolution" of two Generalized Fourier Transforms. Since the value of most derivatives depend on a trigger parameter, two main variants have been developed depending on which variable of the payoff is transformed into the Fourier space. Here we consider the technique introduced in [8] which consider the generalized Fourier transform with respect to the trigger parameter.

More formally, let $\Pi(S, K)$ be the payoff at maturity of the derivative: for example, $\Pi(S, K) = (K - S)^+$ is the payoff of the put option. The no-arbitrage price is therefore given by

$$\Pi_0 = e^{-rT} \mathbb{E}^{\mathcal{Q}}[\Pi(S_T, K)].$$

Due to the exponential structure of typical underlying dynamics of the form $S_t = s_0 \exp(X_t)$, it is convenient to represent the payoff with respect to the new variables $X_T = \log(S_T) - \log(s_0)$ and $k = \log(K)$, in such a way $\Pi(S_T, K) = \Pi(s_0 \exp(X_T), \exp(k)) = \Pi(X_T + \log(s_0), \log(K))$.

Therefore, let us denote with $\Pi(x, k)$ an arbitrary payoff function and with $\hat{\Pi}_x(z)$ its generalized Fourier transform (GFT) w.r.t. k , that is

$$\hat{\Pi}_x(z) = \int_{\mathbb{R}} e^{izk} \Pi(x, k) dk, \quad z \in \mathbb{C};$$

under proper regularity conditions (see e.g. [18]), Fourier inversion gives

$$\Pi(x, k) = \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-izk} \hat{\Pi}_x(z) dz,$$

in some strip of \mathbb{C} , from which

$$\begin{aligned} B_T \Pi_0 &\equiv \mathbb{E}[\Pi(X(T) + \log(s_0), k)] = \int_{\mathbb{R}} \Pi(x, k) \mathcal{Q}(dx) \\ &= \int_{\mathbb{R}} \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-izk} \hat{\Pi}_x(z) dz \mathcal{Q}(dx) = \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-izk} \int_{\mathbb{R}} \hat{\Pi}_x(z) \mathcal{Q}(dx) dz \\ &= \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-izk} \mathbb{E}^{\mathcal{Q}}[\hat{\Pi}_{X(T)+\log(s_0)}(z)] dz. \end{aligned}$$

In order to implement our program, we need to evaluate

1. the VaR of the hedged position: this step require to solve w.r.t. v the equation $\mathcal{P}\{S_T < v\} = \alpha$;
2. the value of a put option $\Pi_0^P(k, T) = e^{-rT} \mathbb{E}^Q[(e^k - S_0 e^{X_T})^+]$.

Let us consider the following "payoff" functions,

$$\Phi_1(x, k) = (e^k - e^x)^+, \quad \text{and} \quad \Phi_2(x, k) = \mathbb{I}_{\{x \leq k\}},$$

in such a way $\Pi_0^P(k, T) = e^{-rT} \mathbb{E}^Q[\Phi_1(X_T + \log(S_0), k)]$ and $\mathcal{P}\{S_T < v\} = \mathcal{P}\{X_T < k\} = \mathbb{E}[\Phi_2(X_T, k)]$, with $k = \log(v/S_0)$. Their GFT w.r.t. the trigger parameter k are

$$\hat{\Phi}_1(x, z) = \frac{e^{x(1+iz)}}{iz - z^2}, \quad \nu > 1 \quad \text{and} \quad \hat{\Phi}_2(x, z) = \frac{i}{z} e^{ixz}, \quad \nu > 0$$

giving therefore the formulas

$$\begin{aligned} \Pi_0^P(k, T) &= e^{-rT} \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-ikz} \frac{e^{i \log(S_0)(z-i)} \phi_X(z-i)}{iz - z^2} dz, \quad \nu > 1 \\ &= e^{-rT} \frac{e^{\nu k} S_0^{1-\nu}}{\pi} \Re \left(\int_0^{+\infty} e^{-iu(k-\log(S_0))} \frac{\phi_X(u+i(\nu-1))}{\nu^2 - u^2 - \nu + iu(1-2\nu)} du \right), \end{aligned} \quad (6)$$

and

$$\begin{aligned} \mathcal{P}\{S_T < v\} &= \frac{i}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-iz \log(v/S_0)} \frac{\phi_X(z)}{z} dz, \quad \nu > 0 \\ &= \frac{(v/S_0)^\nu}{\pi} \Re \left(\int_0^{+\infty} e^{-iu \log(v/S_0)} \frac{\phi_X(u+i\nu)}{\nu - iu} du \right) \end{aligned} \quad (7)$$

$\phi_X(z)$ being the GFT of the process X_T under the appropriate measure. If this is a regular functions in a properly defined strip of \mathbb{C} , the transform method can be applied in both cases (see Lee (2004)).

Since under the Assumption (2.1) the optimality condition is

$$(K - q_{1-\alpha}(S_T)) = \frac{\Pi_0^P(K, T)}{\frac{\partial}{\partial K} \Pi_0^P(K, T)} \equiv \frac{\Pi_0^P(K, T)}{e^{-rT} \mathcal{Q}(S_T \leq K)}.$$

the optimal hedging strategy is then implemented by running

1. root search algorithm to find the value q^* solution of

$$\frac{(q^*/S_0)^\nu}{\pi} \Re \left(\int_0^{+\infty} e^{-iu \log(q^*/S_0)} \frac{\phi_X^P(u+i\nu)}{\nu - iu} du \right) = \alpha, \quad \nu > 1;$$

2. root search algorithm to find the value K^* solution of

$$(K^* - q^*) = \frac{S_0 \Re \left(\int_0^{+\infty} e^{-iu(\log(K^*/S_0))} \frac{\phi_X^{\mathcal{Q}}(u+i(\nu-1))}{\nu^2 - u^2 - \nu + iu(1-2\nu)} du \right)}{\Re \left(\int_0^{+\infty} e^{-iu(\log(K^*/S_0))} \frac{\phi_X^{\mathcal{Q}}(u+i(\bar{\nu}-1))}{\bar{\nu} - iu} du \right)}, \quad \nu > 1, \bar{\nu} > 0.$$

Numerical quadrature must be used for integral evaluation. Alternatively the FFT algorithm can be used to efficiently approximate integrals (see [18]) and then a standard root-finding routine will find the required solutions.

4 Regime-Switching Jump Diffusions and measure change

Let us consider on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ a stochastic process of the form $S_t = S_0 e^{X_t}$, $S_0 > 0$, modeling the value, or price, of a risky asset for $t \in [0, T]$. We consider a jump-diffusion setting in which the jump process is described as a marked point process (MPP), the parameters of which are driven by a finite state and continuous time Markov chain.

We briefly recall here (see e.g. Runggaldier (2003)) that a MPP can be characterized through the couple (T_n, Y_n) , where $\{T_n\}$ is an univariate point process on \mathbb{R}^+ and $\{Y_n\}$ is a sequence of random variables on a given measurable space (E, \mathcal{E}) , as a random measure $p(dy, dt)$ for which

$$\int_0^t \int_E \mathcal{H}(y, s) p(dy, ds) = \sum_{n=1}^{N_t} \mathcal{H}(Y_n, T_n)$$

N_t being the Poisson point process. The corresponding intensity for p is a measure-valued process $\lambda_t(dy)$ for which

$$\int_0^t \int_E \mathcal{H}(y, s) (p(dy, ds) - \lambda_s(dy) ds)$$

is a martingale for each predictable process \mathcal{H} and it characterizes the MPP. A common form of the intensity is $\lambda_t(dy) = \lambda_t m_t(dy)$, where λ_t represents the intensity of the Poisson counting process and $m_t(dy)$ is a probability measure on the mark space (E, \mathcal{E}) describing the jump component. Finally, the couple $(\lambda_t, m_t(dy))$ is called the $(\mathcal{P}, \mathcal{F}_t)$ -local characteristic of $p(ds, dy)$. This setting has been introduced in the financial literature by Bjork et al. (1997). Although jump diffusion models can be described in somewhat different ways, the approach based on MPP turns out to be particularly useful for managing absolutely continuous change of measures. As a matter of fact, for our application we have to specify the dynamic model under both the objective (or historical) measure \mathcal{P} and an equivalent risk-neutral (or pricing) measure \mathcal{Q} .

Let $\alpha(t)$ be a continuous time, homogeneous and stationary Markov Chain on the state space $\mathcal{S} = \{1, 2, \dots, M\}$ with a generator $H \in \mathbb{R}^{M \times M}$; furthermore, $\mu : \mathcal{S} \rightarrow \mathbb{R}$, $\sigma : \mathcal{S} \rightarrow \mathbb{R}$ and $\gamma : E \times \mathcal{S} \rightarrow \mathbb{R}$ are given functions, (E, \mathcal{E}) being the measurable mark space. Without loss of

generality, we can assume in the following $E \subseteq \mathbb{R}$. In a given interval $0 \leq t \leq T$, we consider therefore the dynamic

$$dX(t) = (\mu(\alpha(t)) - \frac{1}{2}\sigma^2(\alpha(t))dt + \sigma(\alpha(t))dW(t) + \int_E \gamma(y, \alpha(t^-))p^\alpha(dy, dt), \quad X(0) = 0, \quad (8)$$

where $W(t)$ is a standard brownian motion and $p^\alpha(dy, dt)$ is a MPP characterized by the intensity

$$\lambda_t^\alpha(dy) \equiv \lambda(\alpha)m(\alpha, dy).$$

Here $\lambda(\cdot)$ represents the (regime-switching) intensity of the Poisson process N_t , while $m(\cdot, dy)$ are a set of probability measures on E , one for each state (regime) $i \in \mathcal{S}$ of the chain. The function $\gamma(y, \alpha)$ represents the jump amplitude relative to the mark y in regime α . Throughout the paper we assume that the processes $\alpha(\cdot)$ and $W(\cdot)$ are independent and that $W(\cdot)$ and $p^\alpha(dy, dt)$ are conditionally independent given $\alpha(t)$. We denote $\mathcal{F}_t^\alpha = \sigma\{\alpha(s) : 0 \leq s \leq t\}$ the σ -algebra generated by the Markov chain. Furthermore, we assume that $\mathbb{E}[e^{\gamma(Y, \alpha)}] = \int_E e^{\gamma(y, \alpha)}m(\alpha, dy)$ is finite for each regime α . As usual, we also define the compensated point process $q^\alpha(dy, dt) = p^\alpha(dy, dt) - \lambda(\alpha(t^-))m(\alpha(t^-), dy)dt$ in such a way $\int_0^t \int_E H(y, \alpha(s^-))q^\alpha(dy, ds)$ is a martingale in t for each predictable process H satisfying appropriate integrability conditions.

An application of the generalized Ito's Formula gives the corresponding jump-diffusion SDE for the asset price

$$\begin{aligned} \frac{dS(t)}{S(t^-)} &= \mu(\alpha(t))dt + \sigma(\alpha(t))dW(t) + \int_E (e^{\gamma(y, \alpha(t^-))} - 1)p^\alpha(dy, dt) \\ &= [\mu(\alpha(t)) + \lambda(\alpha(t))\kappa(\alpha(t))]dt + \sigma(\alpha(t))dW(t) + \int_E (e^{\gamma(y, \alpha(t^-))} - 1)q^\alpha(dy, dt), \end{aligned} \quad (9)$$

with $S(0) = s_0$ and $\kappa(\alpha) = \mathbb{E}[e^{\gamma(Y, \alpha)} - 1]$.

Measure changes. An absolutely continuous transformation of measures in a jump-diffusion setting allows to change the intensities of the MPP and the Markov chain in addition to the translation of the Wiener process (see Runggaldier (2003)). In this context it results convenient to represent the underlying Markov chain itself as a MPP (see Landen (2001)) in such a way

$$d\alpha(t) = \int_S \delta(z)v(dt, dz) \quad (10)$$

where $v(dt, dz)$ is a marked point process with finite mark space $(S, \mathcal{P}(S))$, $S = \{z = (i, j) : i \neq j, i, j \in \{1, 2, \dots, N\}\}$ and $\mathcal{P}(S) = 2^S$ and compensator

$$\lambda_v(t, \alpha(t^-), dz) = \sum_{i \neq j} h_{ij} 1_{(\alpha(t^-)=i)} \epsilon_{(i,j)}(dz) \quad (11)$$

$\epsilon_{(i,j)}(dz)$ being the Dirac measure. In the previous formula the numbers h_{ij} are positive and such that $\sum_{j \neq i, j=1}^N h_{ij} = -h_{ii}$, for $i = 1, \dots, N$.

Consequently, let $\vartheta_t(\alpha)$ be a square integrable predictable processes, $h_t(\alpha, y)$ a non-negative function such that

$$\int_E h_t(\alpha, y) m(\alpha, dy) = 1, \forall t \in [0, T]$$

and let $\psi(\alpha)$ and $\Phi(z)$ be strictly positive functions defined on \mathcal{S} and \mathbb{S} , respectively. We can define a new measure \mathcal{Q} on the measurable space by setting

$$\begin{cases} dW_t^{\mathcal{Q}} = dW_t - \vartheta_t(\alpha(t)) dt \\ q^{\mathcal{Q}, \alpha}(dt, dy) = p(dt, dy) - \psi(\alpha(t)) \lambda(\alpha(t)) h_t(\alpha(t), y) m(\alpha(t), dy) dt \\ \lambda_v^{\mathcal{Q}}(t, \alpha, dz) = \Phi(z) \lambda_v(t, \alpha, dz). \end{cases} \quad (12)$$

Besides the translation of the Wiener process W_t , we perform a change in the intensity of the MPP giving the compensated process $q^{\mathcal{Q}, \alpha}(dt, dy)$ with $(\mathcal{Q}, \mathcal{F}_t)$ -local characteristic $(\psi(\alpha(t)) \lambda(\alpha(t)), h_t(\alpha(t), y) m(\alpha(t), dy))$ and a change of the intensity of the Markov chain which under \mathcal{Q} has generator $H^{\mathcal{Q}} = \{h_{ij}^{\mathcal{Q}}\}$ where

$$h_{ij}^{\mathcal{Q}} = \Phi(i, j) h_{ij}, \quad h_{ii}^{\mathcal{Q}} = - \sum_{k=1, k \neq i}^N \Phi(i, k) h_{ik}, \quad i = 1, \dots, N.$$

By taking the Radon-Nikodym derivative

$$\begin{aligned} L_t &= \exp \left(-\frac{1}{2} \int_0^t \vartheta_s(\alpha(s))^2 ds + \int_0^t \vartheta_s(\alpha(s)) dW_s^{\mathcal{Q}} \right. \\ &+ \int_0^t (1 - \psi(\alpha(s)) \lambda(\alpha(s))) ds + \int_0^t \int_E \log(\psi(\alpha(s)) h_t(\alpha(s), y)) p(ds, dy) \\ &\left. + \int_0^t (1 - \Phi(z)) \lambda_v(t, \alpha(t-), dz) ds + \int_0^t \int_{\mathbb{S}} \log(\Phi(z)) \nu(ds, dz) \right) \end{aligned} \quad (13)$$

and supposing that $\mathbb{E}^{\mathcal{P}}[L_t] = 1$ for all t , we have a probability measure \mathcal{Q} on \mathcal{F} equivalent to \mathcal{P} with $d\mathcal{Q} = L_T d\mathcal{P}$, under which

$$\begin{aligned} \frac{dS(t)}{S(t^-)} &= [\mu(\alpha(t)) + \sigma(\alpha(t)) \vartheta_t(\alpha(t)) + \psi(\alpha(t)) \lambda(\alpha(t)) \kappa^{\mathcal{Q}}(\alpha(t))] dt + \sigma(\alpha(t)) dW^{\mathcal{Q}}(t) \\ &+ \int_E (e^{\gamma(y, \alpha(t-))} - 1) q^{\mathcal{Q}, \alpha}(dy, dt), \end{aligned} \quad (14)$$

where $\kappa^{\mathcal{Q}}(\alpha) = \mathbb{E}^{\mathcal{Q}}[e^{\gamma(Y, \alpha)} - 1]$.

In order to price derivatives under the model (9) we need to specify a risk-neutral or martingale measure, that is a measure under which the discounted price process $e^{-rt}S_t$ is a martingale. This is done by taking

$$\vartheta(\alpha) \equiv \frac{r - \mu(\alpha) - \psi(\alpha)\lambda(\alpha)\kappa^{\mathcal{Q}}(\alpha)}{\sigma(\alpha)} \quad (15)$$

from which we finally get the risk-neutral dynamic for the underlying

$$\begin{aligned} \frac{dS(t)}{S(t^-)} &= rdt + \sigma(\alpha(t))dW^{\mathcal{Q}}(t) + \int_E (e^{\gamma(y, \alpha(t^-))} - 1)q^{\mathcal{Q}, \alpha}(dy, dt) \\ &= [r - \psi(\alpha(t))\lambda(\alpha(t))\kappa^{\mathcal{Q}}(\alpha(t))]dt + \sigma(\alpha(t))dW^{\mathcal{Q}}(t) + \int_E (e^{\gamma(y, \alpha(t^-))} - 1)p^{\mathcal{Q}, \alpha}(dy, dt). \end{aligned} \quad (16)$$

Consequently, from (15) and (9), the market price of risk is (see [21])

$$\begin{aligned} \rho(\alpha) &\equiv \mu(\alpha) - \lambda(\alpha)\kappa(\alpha) - r = \lambda(\alpha)[\kappa(\alpha) - \psi(\alpha)\kappa^{\mathcal{Q}}(\alpha)] - \sigma(\alpha)\theta(\alpha) \\ &= \lambda(\alpha) \left[\int_E (e^{\gamma(y, \alpha)} - 1)(1 - \psi(\alpha)h_t(\alpha, y))m(\alpha, dy) \right] - \sigma(\alpha)\theta(\alpha). \end{aligned} \quad (17)$$

Correspondingly, for the process $X(t)$ we have

$$\begin{aligned} dX(t) &= [r - \frac{1}{2}\sigma^2(\alpha(t))]dt + \sigma(\alpha(t))dW^{\mathcal{Q}}(t) + \int_E \gamma(y, \alpha(t^-))q^{\mathcal{Q}, \alpha}(dy, dt) \\ &= [r - \frac{1}{2}\sigma^2(\alpha(t)) - \psi(\alpha(t))\lambda(\alpha(t))\kappa^{\mathcal{Q}}(\alpha(t))]dt + \sigma(\alpha(t))dW^{\mathcal{Q}}(t) \\ &\quad + \int_E \gamma(y, \alpha(t^-))p^{\mathcal{Q}, \alpha}(dy, dt). \end{aligned} \quad (18)$$

The measure transformation defined by (12) through (13) preserves the probability structure of the stochastic process $X(t)$ under both \mathcal{P} and \mathcal{Q} . It worth noting that we can specify infinitely many equivalent measures \mathcal{Q} . In practice, the usual way to select one of the equivalent measures is to calibrate the model to a set of observed data.

GFT for regime-switching jump-diffusions. The formulas (6) and (7) depend from the dynamic model only through its generalized Fourier transform $\phi_X(z)$. In this section we report the GTF for the regime-switching jump-diffusion model. Since we have to consider the process X_T under two different measure, we derive its characteristic function for the following general dynamic

$$dX(t) = \xi(\alpha(t))dt + \sigma(\alpha(t))dW(t) + \int_E \gamma(y, \alpha(t^-))p(dy, dt)$$

where

$$\xi(\alpha) = \begin{cases} r - \frac{1}{2}\sigma^2(\alpha) - \psi(\alpha)\lambda(\alpha)\kappa^{\mathcal{Q}}(\alpha) & \text{under the measure } \mathcal{Q} \\ \mu(\alpha) - \frac{1}{2}\sigma^2(\alpha) & \text{under the measure } \mathcal{P} \end{cases} \quad (19)$$

the MPP $p(dy, dt)$ has intensity

$$\lambda(\alpha, dy) = \begin{cases} \psi(\alpha)\lambda(\alpha)h(\alpha, y)m(\alpha, dy) & \text{under the measure } \mathcal{Q} \\ \lambda(\alpha)m(\alpha, dy) & \text{under the measure } \mathcal{P} \end{cases}$$

and the Markov chain has generator $Q = \{q_{ij}\}_{i,j=1,\dots,N}$ where

$$q_{ij} = \begin{cases} e^{\Phi(i,j)}h_{ij} & \text{under the measure } \mathcal{Q} \\ h_{ij} & \text{under the measure } \mathcal{P} \end{cases} \quad i, j = 1, \dots, N, i \neq j, \text{ and } q_{ii} = - \sum_{k=1, k \neq i}^N q_{ik}. \quad (20)$$

In [20] (see also [10]) it was proved the following

Proposition 4.1. *Let $\phi_j(z) = \mathbb{E}[e^{iz\gamma(Y(j),j)}]$ be the generalized Fourier transform of the jump magnitude under the given measure. Then, by letting*

$$\vartheta_j(z) = z\xi(j) + \frac{1}{2}iz^2\sigma^2(j) - i\lambda(j)(\phi_j(z) - 1) \quad (21)$$

and $\tilde{\vartheta}_i(z) = \vartheta_j(z) - \vartheta_M(z)$, we have

$$\begin{aligned} \varphi_T(z) &= e^{i\vartheta_M(z)T} \left(\mathbf{1}' \cdot e^{(Q' + i \operatorname{diag}(\tilde{\vartheta}_1(z), \dots, \tilde{\vartheta}_{M-1}(z), 0))T} \cdot \mathbb{I}(0) \right) \\ &= \mathbf{1}' \cdot e^{(Q' + i \operatorname{diag}(\vartheta_1(z), \dots, \vartheta_M(z)))T} \cdot \mathbb{I}(0), \end{aligned} \quad (22)$$

where $\mathbf{1} = (1, \dots, 1)' \in \mathbb{R}^{M \times 1}$, $\mathbb{I}(0) = (\mathbb{I}_{\alpha(0)=1}, \dots, \mathbb{I}_{\alpha(0)=M})' \in \mathbb{R}^{M \times 1}$ and Q' is the transpose of Q .

Different models can be recovered with simple linear constraints on the full parameter set of our dynamics (9), (16). This follows by noticing that if $\xi(i) = \xi$, $\sigma(i) = \sigma$, $\lambda(i) = \lambda$ and $\phi_i(z) = \phi(z)$ we are implicitly assuming a unique regime so recovering the well-known characteristic function of the (single-regime) jump-diffusion dynamic $\varphi_T(z) = \exp(z\xi + \frac{1}{2}iz^2\sigma^2 - i\lambda(\phi(z) - 1))$ which includes the standard geometrical Brownian motion (GBM) ($\lambda = 0$) and the Merton jump-diffusion models (JDM). By letting $\lambda_i = 0$ in (22) we get the regime-switching version of GBM (RSGBM) and finally the regime-switching jump diffusion model (RSJDM) with the full set of parameters

$$\xi_i, \sigma_i, \lambda_i, h_{ij}, \quad i, j = 1, \dots, M.$$

The evaluation of the characteristic function requires to compute matrix exponentials for which efficient numerical techniques are available (see Higham (2009)); conversely, the case $M = 2$ can be considered explicitly. The following can be proved (see [20] and the references therein).

Proposition 4.2. Let $y_{1,2}$ be the solutions of the quadratic equation $y^2 + (q_1 + q_2 - i\theta)y - i\theta q_2 = 0$ and

$$\begin{aligned} q_1^T(\theta) &= \frac{1}{y_1 - y_2} (e^{y_1 T} (y_1 + q_1 + q_2) - e^{y_2 T} (y_2 + q_1 + q_2)) \\ q_2^T(\theta) &= \frac{1}{y_1 - y_2} (e^{y_1 T} (y_1 + q_1 + q_2 - i\theta) - e^{y_2 T} (y_2 + q_1 + q_2 - i\theta)). \end{aligned}$$

Then

$$\mathbb{E}_t[e^{i\theta T_1}] = \mathbb{I}_{\alpha(t)=1} q_1^T(\theta) + \mathbb{I}_{\alpha(t)=2} q_2^T(\theta)$$

and therefore

$$\varphi_X(z) = e^{i\theta_2(z)T} (\mathbb{I}_{\alpha(t)=1} q_1^T(\theta(z)) + \mathbb{I}_{\alpha(t)=2} q_2^T(\theta(z)))$$

□

5 Numerical results

In this section we report results about two kind of numerical experiments:

1. calculation of quantiles for RSJD. The objective is to study the behavior of such a quantities by varying diffusions and jumps parameters.
2. valuation of the optimal hedging strategy in the regime-switching jump-diffusion framework.

We consider a two-state regime switching version of the jump-diffusion model with gaussian jumps. This is defined by choosing $\gamma(y, \alpha) = y$ and two kinds of normal jumps, i.e. $Y(i) \sim \mathcal{N}(a_i, b_i)$ from which $\kappa(i) = \mathbb{E}[(e^{Y(i)} - 1)] = e^{a_i + b_i^2/2} - 1$, $i = 1, 2$. The two state Markov chain $\alpha(t) \in \mathcal{S} = \{1, 2\}$ has generator under the chosen measure $Q = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}$. Let $\sigma_i, \lambda_i > 0$ and $\mu_i, i = 1, 2$ be given parameters: the regime switching jump-diffusion Merton model is defined as

$$dX(t) = \xi(\alpha(t))dt + \sigma(\alpha(t))dW(t) + \int_0^t \int_E yp^\alpha(dy, ds) \quad (23)$$

where $\xi(\alpha(t))$ is given by (19), $\lambda(t, \alpha(t), dy) = \lambda(\alpha(t))\phi_{\alpha(t)}(y)dy$ is the intensity process of the Poisson jump component and $\lambda(\alpha(t)) \in \{\lambda_1, \lambda_2\}$, $\sigma(\alpha(t)) \in \{\sigma_1, \sigma_2\}$, $\mu(\alpha(t)) \in \{\mu_1, \mu_2\}$, $\phi_i(y)$ being the density of a normal distribution $\mathcal{N}(a_i, b_i)$, $i = 1, 2$.

All numerical procedures were implemented in the MatLab[©] framework.

5.1 Quantiles and VaR Calculation

Quantiles were computed by solving the equation

$$\frac{(q/S_0)^\nu}{\pi} \Re \left(\int_0^{+\infty} e^{-iu \log(q/S_0)} \frac{\phi_X^{\mathcal{P}}(u + i\nu)}{\nu - iu} du \right) = \alpha,$$

with respect to q (see Section 3). A standard root-search algorithm was used together with the Gauss-Lobatto quadrature for approximating the integral. Few milliseconds were needed to get the required quantile on an Intel[®] Core i5.

We calculate the quantiles of the RSJD models for different values of the parameters under the historical probability \mathcal{P} . We start by considering the simple regime-switching geometrical brownian motion ($\lambda_i = a_i = b_i = 0$): as expected, the effect of switching drift and volatility results in a sort of mixing behavior between the corresponding GBM models, see figures (1), (2), and such effect becomes more evident for growing time T . The frequency of Markov chain switching, driven by the generator Q , induces a quite different behavior, whose impact also depends on the value of T (figures (3), (4)).

It is well-known ([9]) that introducing (gaussian) jumps in the GBM dynamic has a great impact on the tails of the distribution, see figure (5). As before, the Markov switching generates a mixing effect: quantiles curves of the RSJD models are between the curves of the corresponding JD models without regime switching.

5.2 Optimal risk management

In the second set of numerical experiments we face two main issues: (a) Risk Reduction: how much reduction of risk is obtained by implementing the optimal hedging strategy in the RSJD framework and in particular what is the impact of jumps and switching regimes on the main quantities driving the strategy? (b) Misspecified Modeling: what is the effect of a wrong model specification which discards regime switchings and jumps, when they are indeed present in the market, and consider the simpler GBM model?

(a) Risk reduction and model impact. Inside each model - GBM, JD, RSGBM, RSJD - we show the behavior of the optimal hedging strategy obtained by changing the value of some relevant parameters: this corresponds to specify different levels of the market price of risk. In the figures (9), (10), (11), (12) and (13) the optimal strike K^* and the corresponding value of the VaR are depicted together with the risk reduction percentage

$$R = 1 - \frac{\text{VaR}_\alpha(L_0^{h^*, K^*})}{\text{VaR}_\alpha(L^u)}.$$

It is apparent how the the strategy can be effective in reducing risk. On the other hand, it can be noticed that a perturbation of a single parameter results in a change of the optimal strategy.

	Optimal strategy RSJD	Optimal strategy GBM ($\hat{\sigma}$)	β_{RSJD}
T=.5	64.7442, 0.7197, 37.0189	65.6191, 0.8433, 35.3880 (0.2905)	0.0132
T=1	55.5928, 0.6165, 47.1767	57.1579, 0.7644, 44.4718 (0.2689)	0.0148
T=3	41.6851, 0.5294, 62.3356	43.5664, 0.7382, 58.6379 (0.2254)	0.0157

Table 1: Optimal hedging strategy (K^*, h^*, VaR^*) under the simulated true model (first column), the fitted GBM model (second column - estimated volatility) and the corresponding value of β_{RSJD} . Here $\sigma_1 = .3$, $\sigma_2 = .05$, $\lambda_1 = 2$, $\lambda_2 = .8$, $a_1 = 0.0$, $a_2 = 0.0$, $b_1 = 0.08$, $b_2 = 0.15$, $q_1 = 1$, $q_2 = 0.2$; furthermore $r = 0.5\%$, $\alpha = 0.01$ and the budget constraint is $C = 0.1$.

	Optimal strategy RSJD	Optimal strategy GBM ($\hat{\sigma}$)	β_{RSJD}
T=.5	61.1841, 0.0581, 45.2341	63.3076, 0.0816, 41.3746 (0.3137)	0.0166
T=1	45.8347, 0.0833, 60.5069	52.7089, 0.0744, 52.6106 (0.3047)	0.0255
T=3	18.8056, 0.4015, 83.7630	32.7103, 0.0797, 72.5986 (0.2930)	0.0640

Table 2: Optimal hedging strategy (K^*, h^*, VaR^*) under the simulated true model (first column), the fitted GBM model (second column - estimated volatility) and the corresponding value of β_{RSJD} . Here $\sigma_1 = .3$, $\sigma_2 = .05$, $\lambda_1 = 2$, $\lambda_2 = .8$, $a_1 = 0.05$, $a_2 = -0.3$, $b_1 = 0.08$, $b_2 = 0.15$, $q_1 = 1$, $q_2 = 0.2$; furthermore $r = 0.5\%$, $\alpha = 0.01$ and the budget constraint is $C = 0.01$.

(b) Misspecified Modeling. In order to explore the model sensitivity of the optimal hedging strategy, we implemented the following exercise. We firstly fixed a RSJD model by choosing a complete set of parameters. Then we generated a set of call/put prices on which we calibrate the GBM model: hence we run the optimal hedging strategy obtaining K_{GBM}^*, h_{GBM}^* and correspondingly the minimal VaR, VaR_{GBM}^* . We finally calculated the probability

$$\beta_{RSJD} = \mathcal{P}(L^{K^*, h^*} \geq \text{VaR}_{GBM}^*)$$

under the RSJD model. Results are shown in Table (1) and (2). Notice that even when the optimal strategies are similar, the probability that the portfolio loss exceeds the (optimal) VaR is greater than the fixed level $\alpha = 0.01$. Of course this behavior depends on the choice of the parameters, but the underestimation produced by a wrong model choice can be quite severe: see Table (3), where parameters from a real data set were used (see [20]).

6 Conclusions

In this paper we considered the problem of computing the quantiles of a risky position described by a regime-switching jump-diffusion dynamic model. The knowledge of generalized characteristic function for this class of processes allowed us to use the Fourier Transform methods to design

	Optimal strategy RSJD	Optimal strategy GBM ($\hat{\sigma}$)	β_{RSJD}
T=.5	38.3721, 0.2497, 66.0564	60.0168, 0.0785, 44.8655 (0.3479)	0.1165
T=1	26.6034, 0.4103, 76.3270	49.4859, 0.0737, 55.8926 (0.3320)	0.1304
T=1.5	19.6884, 0.6506, 81.8069	41.9632, 0.0741, 63.4963 (0.3291)	0.1430

Table 3: Optimal hedging strategy (K^*, h^*, VaR^*) under the simulated true model (first column), the fitted GBM model (second column - estimated volatility) and the corresponding value of β_{RSJD} . Here $\sigma_1 = .27$, $\sigma_2 = .13$, $\lambda_1 = 6.8$, $\lambda_2 = .8$, $a_1 = -0.13$, $a_2 = -0.34$, $b_1 = 0.08$, $b_2 = 0.15$, $q_1 = 6.5$, $q_2 = 0.002$; furthermore $r = 0.5\%$, $\alpha = 0.01$ and the budget constraint is $C = 0.01$.

an efficient algorithm for the calculation of quantiles. With this same technique, we analyzed a static hedging policy based on the constrained minimization of the VaR of the option-hedged portfolio. Numerical examples showed the impact of jumps and switching regimes on the optimal strategy in a two-regime, gaussian jumps framework and moreover the risk of a wrong model choice.

Some final comments can be briefly outlined. Firstly, notice that different kind of jumps, as well as the number of regimes, can be readily considered in our computational framework, such as the double exponential Kou model ([16]). Furthermore, the analysis of the hedging strategy is fairly general, that is it can be applied to any dynamical model for which Fourier transform methods are viable, for example it can be extended to Variance-Gamma or Bates models. Finally, besides the choice of different dynamic models, it would be interesting to consider alternative risk measures, such as the Conditional Value at Risk (CVaR). This is certainly less commonly used in finance industry, but it is widely used in insurance industry being a coherent, convex and stable risk measure (see [5]).

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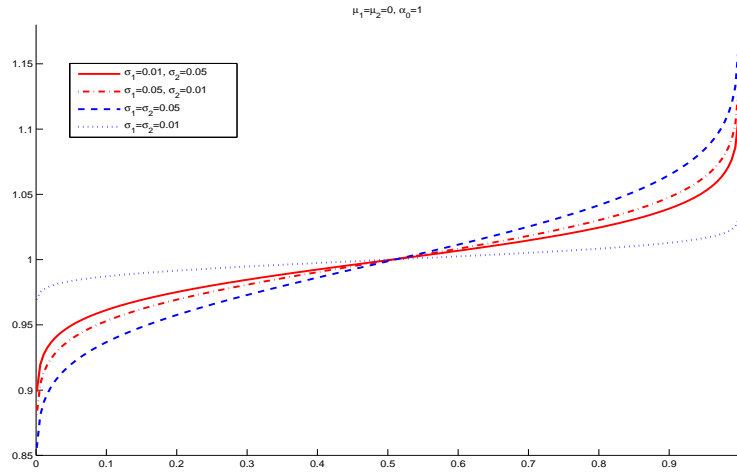


Figure 1: Regime-switching GBM, starting state $\alpha_0 = 1$, $q_1 = q_2 = 5$ and $T = 1$. Comparison with standard GBM. Analogous results for starting state $\alpha_0 = 2$.

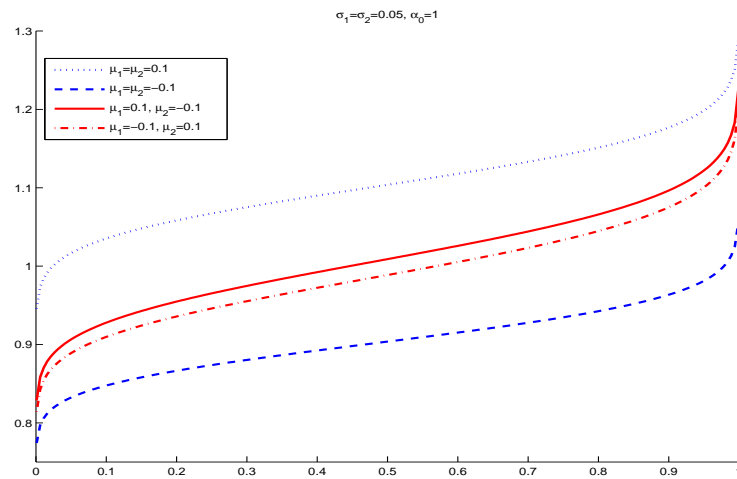


Figure 2: Regime-switching GBM, starting state $\alpha_0 = 1$, $q_1 = q_2 = 5$ and $T = 1$.

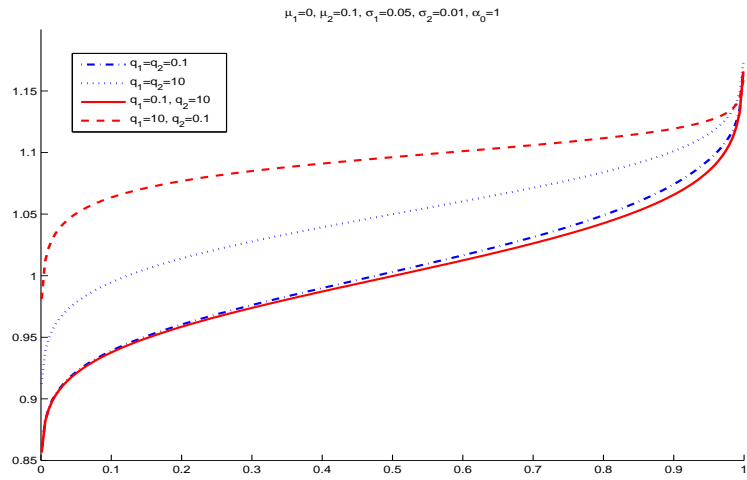


Figure 3: Regime-switching GBM for different values of q_i and $T = 1$.

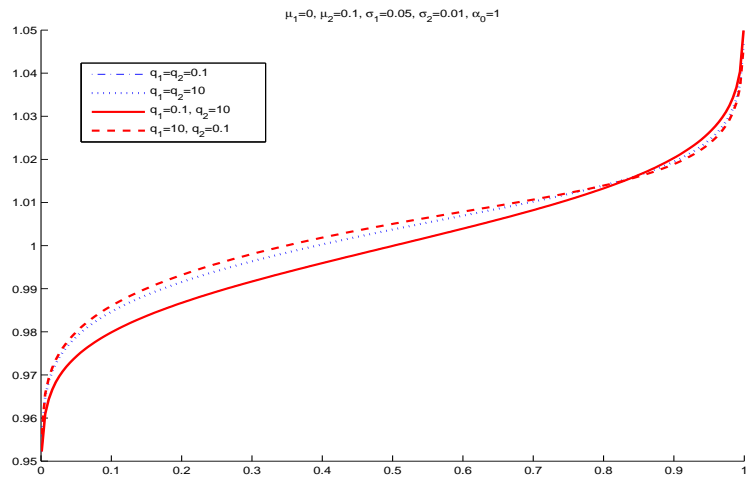


Figure 4: Regime-switching GBM for different values of q_i and $T = 10$.

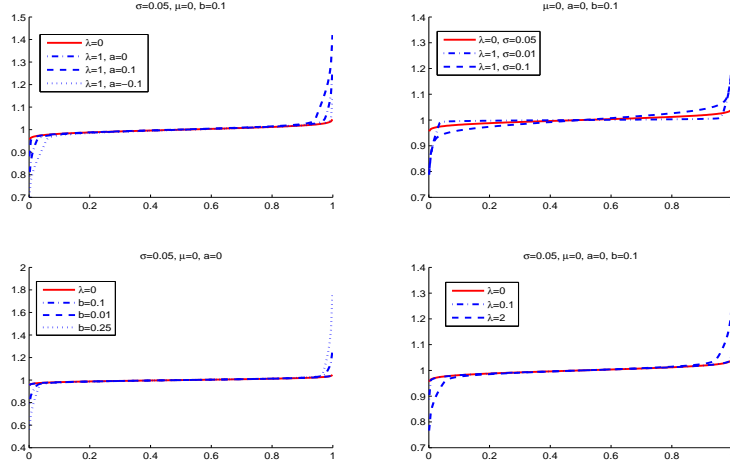


Figure 5: Jump diffusion model with gaussian jumps: red lines correspond to GBM. $T = 1$.

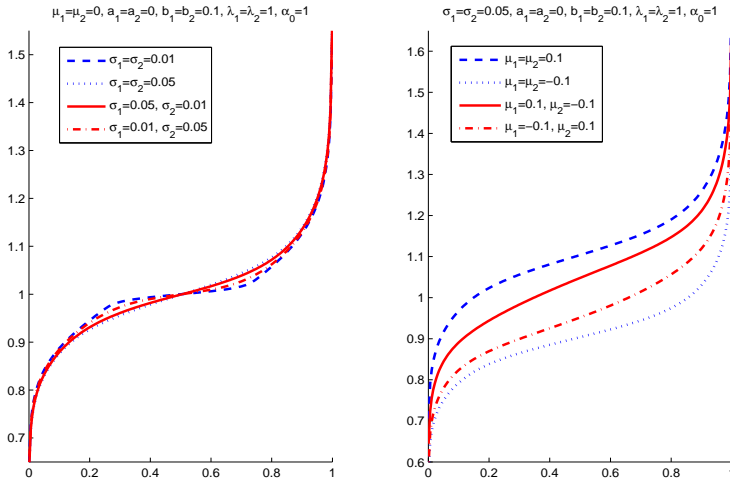


Figure 6: RSJD model with gaussian jumps: switching volatility (left picture) and drift (right picture). $T = 1$.

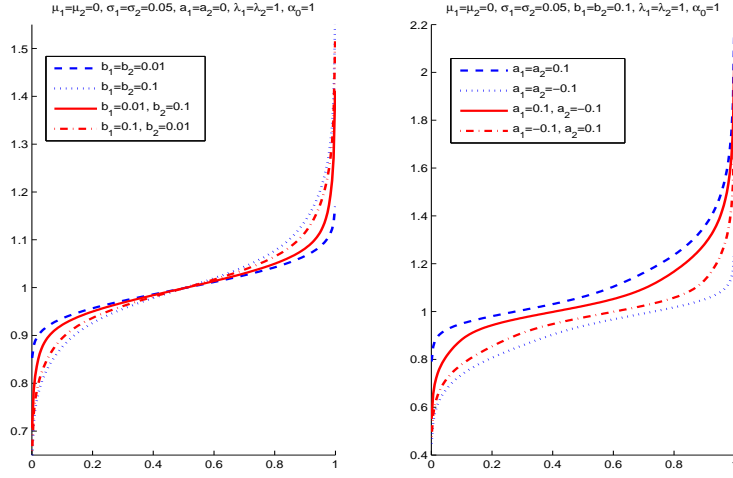


Figure 7: RSJD model: switching variance (left picture) and mean (right picture) of the gaussian jumps. $T = 1$.

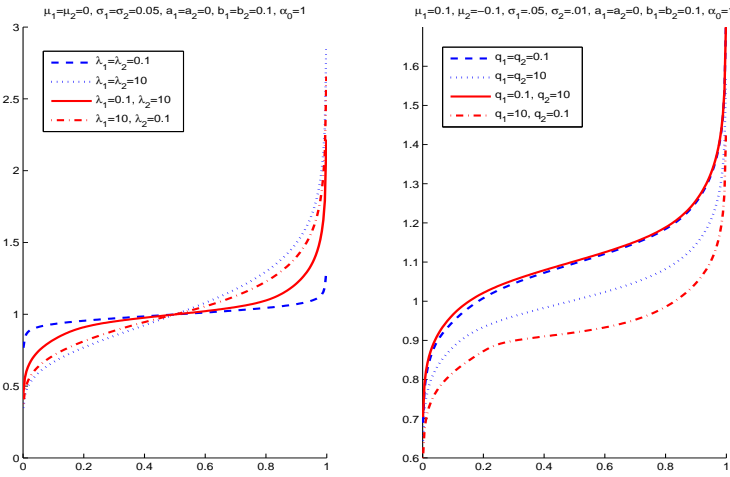


Figure 8: RSJD model: switching jump arrival times λ_i (left picture) and intensity of the Markov chain q_i (right picture). $T = 1$.

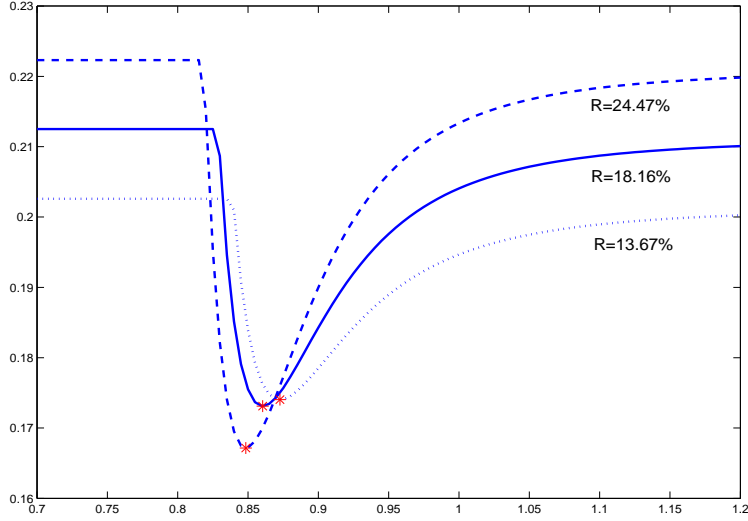


Figure 9: Risk reduction in GBM model: the continuous line corresponds to a model in which the risk is not priced, the dotted line to a variation of +25% of the drift and the dashed line to a variation of -25%. The other parameters are $\sigma = 0.1$, $r = 0.05$, $T = 1$ and $S_0 = 1$.

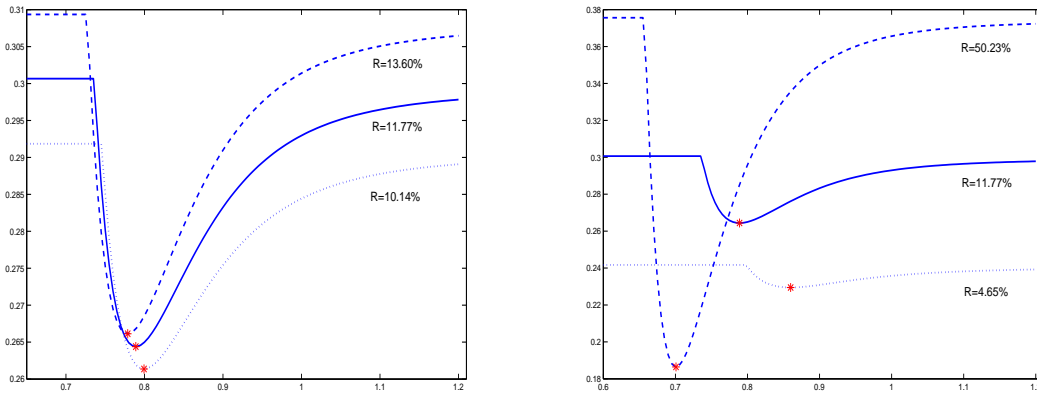


Figure 10: Risk reduction in JDM model: the continuous line corresponds to a model in which the risk is not priced, the dotted line to a variation of +25% of the drift (μ - left figure) and mean jump (a - right figure) and the dashed line to a variation of -25%. The other parameters are $\sigma = 0.1$, $r = 0.05$, $T = 1$ and $S_0 = 1$.

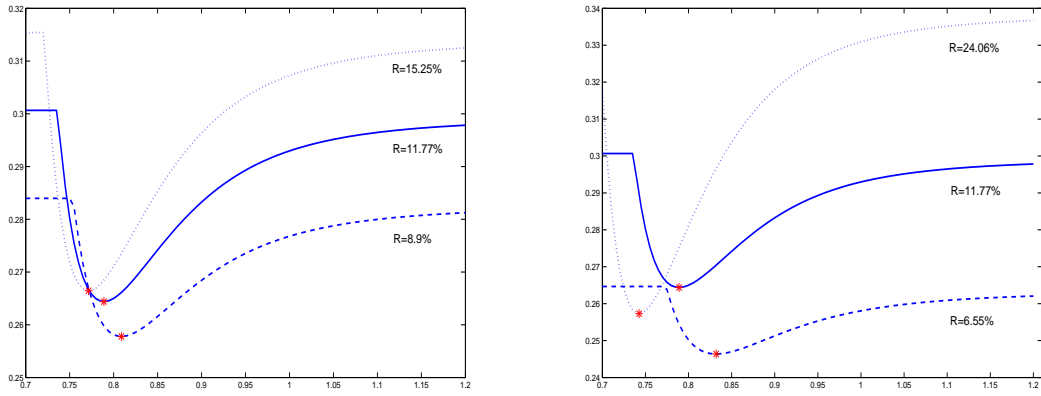


Figure 11: Risk reduction in JDM model: the continuous line corresponds to a model in which the risk is not priced, the dotted line to a variation of +25% of the jump intensity (λ - left figure) and jump std (b - right figure) and the dashed line to a variation of -25%. The other parameters are $\sigma = 0.1$, $r = 0.05$, $\lambda = 1$, $a = 0$, $T = 1$ and $S_0 = 1$.

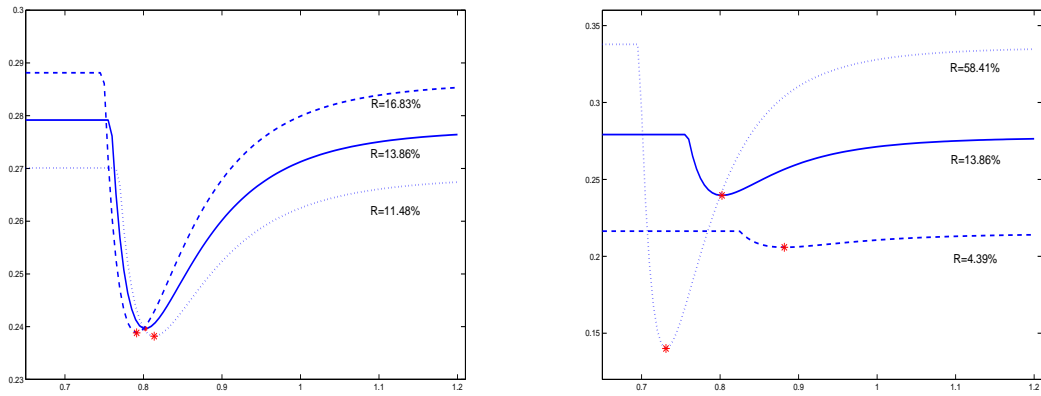


Figure 12: Risk reduction in RS model: the continuous line corresponds to a model in which the risk is not priced, the dotted line to a variation of +25% of the drift (μ_1, μ_2 - left figure) and volatility (σ_1, σ_2 - right figure) and the dashed line to a variation of -25%. The parameters are $\sigma_1 = 0.15$, $\sigma_2 = 0.05$, $r = 0.05$, $q_1 = q_2 = 1$, $T = 1$ and $S_0 = 1$. No substantial changes are observed for this kind of variation of the parameters q_i .

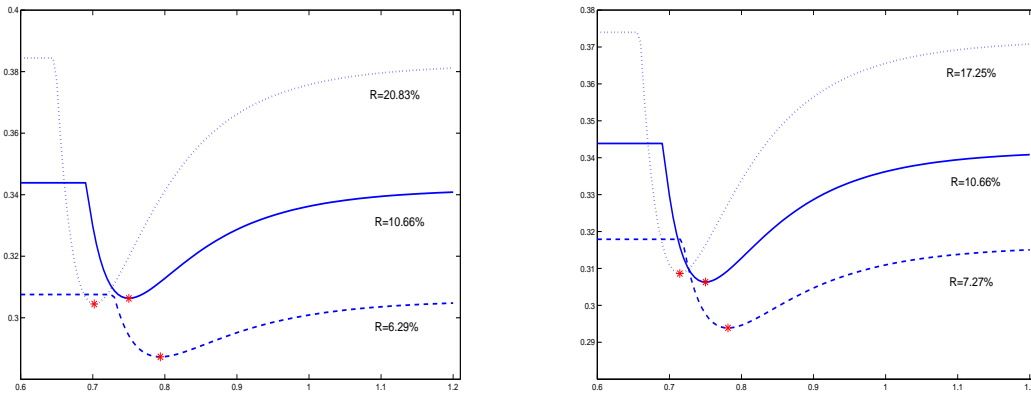


Figure 13: Risk reduction in RSJD model: the continuous line corresponds to a model in which the risk is not priced, the dotted line to a variation of +25% of the volatility (σ_1, σ_2 - left figure) and jump std (b_1, b_2 - right figure) and the dashed line to a variation of -25%. The parameters are $\sigma_1 = 0.15, \sigma_2 = 0.05, b_1 = 0.1, b_2 = 0.2, r = 0.05, q_1 = q_2 = 1, T = 1$ and $S_0 = 1$. No substantial changes are observed for this kind of variation of the other parameters $\lambda_1 = 1, \lambda_2 = 0.1, a_1 = 0, a_2 = -.1, q_i = 1$.