Finite quantum mechanical
model for the stock market

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ABSTRACT: The price of a given stock is exactly known only at the time of sale when the stock is between the traders. If we know the price (owner) then we have no information on the owner (price). A more general description including cases when we have partial information on both price and ownership is obtained by using the quantum mechanics methods. The relation price-ownership is similar to the relation position-momentum. Our approach is based on the mathematical formalism used in the case of quantum systems with finite-dimensional Hilbert space. The linear operator corresponding to the ownership is obtained from the linear operator corresponding to the price by using the finite Fourier transform. In our idealized model, the Schrödinger type equation describing the time evolution of the stock price is solved numerically.

1   INTRODUCTION

The mathematical modeling of price dynamics is a very complex problem. The information and market psychology [1] play important role in price dynamics, and therefore the methods of traditional financial dynamics are not sufficient for an adequate description. The extreme irregularities in the evolution of prices in financial markets can be better understood by using the mathematical formalism of quantum mechanics [2-7]. The time evolution of the price depends on many factors including the political environment, market information, economic policies and psychology of traders.

The presence of a minimal trading value of stocks results in the possibility to use the discrete quantum mechanics methods [8]. A given stock has not a definite price until it is traded. Its value is actualized as a consequence of the contextual interactions in the trading process [7, 9]. The usual description is in terms of probabilities, and the existence of probability interference suggests the necessity to use probability amplitudes, that is, wavefunctions.

2   STOCK PRICE AND STOCK OWNERSHIP

We consider a stock market with a large number N of traders $T_0, T_1, \ldots, T_{N-1}$, and investigate the price of a fixed stock. By choosing an adequate unit of cash, we can assume that the only possible values of the price of the considered stock are $0, 1, 2, \ldots, N-1$. Following the analogy with the description of quantum systems [10], we assume that the stock price at a fixed moment of time can be described by a function

$$\Psi : \{0, 1, 2, \ldots, N-1\} \longrightarrow \mathbb{C}$$

chosen such that $|\Psi(n)|^2$ represent the probability to have a price equal to n units of cash if a transaction takes place. In order to have a consistent probabilistic interpretation, the function Ψ must satisfy the relation

$$\sum_{n=0}^{N-1} |\Psi(n)|^2 = 1$$

that is, to be a normalized function. The finite Fourier transform [8, 11] allows us to associate to Ψ the function

$$F[\Psi] : \{0, 1, 2, \ldots, N-1\} \longrightarrow \mathbb{C}$$

defined by the relation

$$F[\Psi](k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N} nk} \, \Psi(n).$$

The function $F[\Psi]$ is normalized, and following the analogy with the description of quantum systems, we assume that $|F[\Psi](k)|^2$ represents the probability that the stock owner is T_k.

For example, in the case when the stock price is described by the function

$$\Psi_1 : \{0, 1, \ldots, 20\} \longrightarrow \mathbb{C}, \qquad \Psi_1(n) = \left\{ \begin{array}{ll} 1 & \text{for} \quad n = 7 \\ 0 & \text{for} \quad n \neq 7 \end{array} \right.$$

the corresponding Fourier transform is

$$\mathcal{F}[\Psi_1] : \{0, 1, ..., 20\} \longrightarrow \mathbb{C}, \qquad \mathcal{F}[\Psi_1](k) = \frac{1}{\sqrt{21}} e^{-\frac{2\pi i}{3} k}$$

and we have (see Figure 1)

$$|\Psi_1(n)|^2 = \begin{cases} 1 & \text{for} \quad n = 7 \\ 0 & \text{for} \quad n \neq 7 \end{cases} \qquad \text{and} \qquad |\mathcal{F}[\Psi_1](k)|^2 = \frac{1}{21}.$$

This means that the stock price is 7 (units of cash), and the probability for each trader to be the owner is the same, namely, $1/21$. We know with precision the price but we have no information on the stock owner.
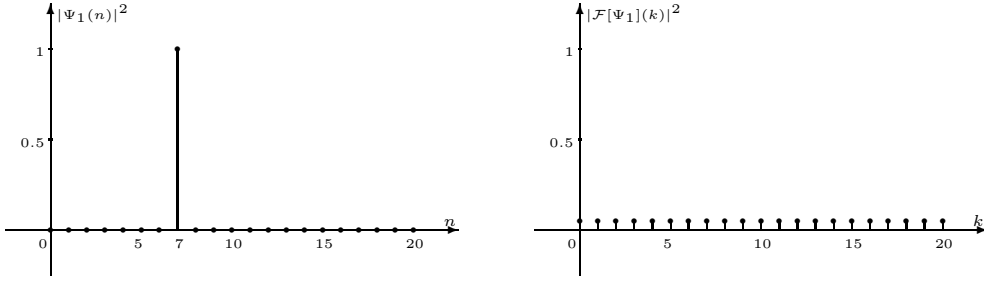

Figure 1: The distributions of probability $|\Psi_1(n)|^2$ and $|\mathcal{F}[\Psi_1](k)|^2$.

In the case when the stock price is described by the function

$$\Psi_2 : \{0, 1, ..., 20\} \longrightarrow \mathbb{C}, \qquad \Psi_2(n) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \quad n = 7 \\ \frac{1}{2} & \text{for} \quad n \in \{6, 8\} \\ 0 & \text{for} \quad n \notin \{6, 7, 8\} \end{cases}$$

we have only partial information on the price

$$|\Psi_2(n)|^2 = \begin{cases} \frac{1}{2} & \text{for} \quad n = 7 \\ \frac{1}{4} & \text{for} \quad n \in \{6, 8\} \\ 0 & \text{for} \quad n \notin \{6, 7, 8\} \end{cases}$$

but we have at the same time partial information on the owner. The stock owner is $T_0, T_1, ... , T_{N-1}$ with the probabilities $|\mathcal{F}[\Psi_2](0)|^2, |\mathcal{F}[\Psi_2](1)|^2, ... , |\mathcal{F}[\Psi_2](N-1)|^2$, respectively (see Figure 2). If a transaction takes place, the stock price is 6, 7 or 8 with the probabilities $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$, respectively.
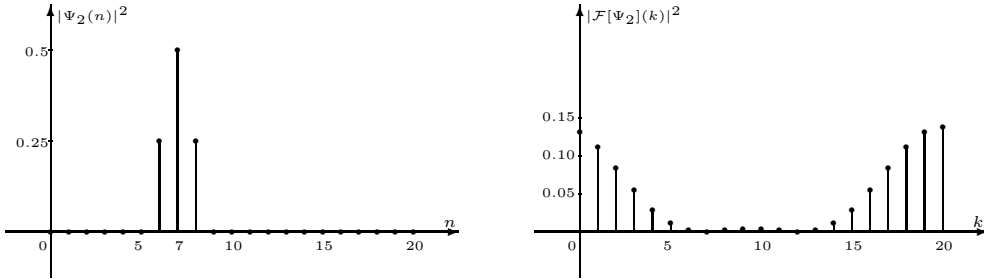

Figure 2: The distributions of probability $|\Psi_2(n)|^2$ and $|\mathcal{F}[\Psi_2](k)|^2$.

The scalar product of two functions

$$\Psi : \{0, 1, 2, ..., N-1\} \longrightarrow \mathbb{C} \qquad \text{and} \qquad \Phi : \{0, 1, 2, ..., N-1\} \longrightarrow \mathbb{C}$$

is defined by the formula

$$\langle \Psi, \Phi \rangle = \sum_{n=0}^{N-1} \overline{\Psi(n)} \, \Phi(n)$$

where $\overline{\Psi(n)}$ is the complex conjugate of $\Psi(n)$. The space $\mathcal{H}$ of all the functions

$$\Psi : \{0, 1, 2, ..., N-1\} \longrightarrow \mathbb{C}$$

is a N-dimensional Hilbert space.

3 PRICE AND OWNERSHIP OPERATORS

In quantum mechanics, in the case of a particle moving along an axis, the position is described by the linear operator [10]

$$\psi \mapsto \hat{x}\,\psi \qquad \text{where} \qquad (\hat{x}\,\psi)(x) = x\,\psi(x)$$

and the momentum by the linear operator ($\hbar$ is Planck's constant h divided by 2π)

$$\psi \mapsto \hat{p}\,\psi \qquad \text{where} \qquad \hat{p}\,\psi = -\mathrm{i}\hbar\frac{d\psi}{dx}.$$

In a state described by the normalized wavefunction ψ, the numbers

$$\langle \hat{x} \rangle = \langle \psi, \hat{x}\,\psi \rangle \qquad \text{and} \qquad \langle \hat{p} \rangle = \langle \psi, \hat{p}\,\psi \rangle$$

represent the mean value of coordinate and momentum, respectively. The position and momentum operators are related through the Fourier transform as

$$\hat{p} = F^{-1}\hat{x}F.$$

The relation price-ownership is, in a certain sense, similar to the relation position-momentum: if we know the stock owner then we have no information on the price, and conversely, if we know the price we have no information on the stock owner. Nevertheless, we have to recognize that the numerical description we use for the ownership is rather artificial.

By following the analogy with quantum mechanics we define the *price operator* as

$$\wp : \mathcal{H} \longrightarrow \mathcal{H} : \varphi \mapsto \wp\,\varphi, \qquad (\wp\,\varphi)(n) = n\,\varphi(n)$$

and the *ownership operator* as

$$\mathcal{O} : \mathcal{H} \longrightarrow \mathcal{H}, \qquad \mathcal{O} = \mathcal{F}^{-1}\wp\mathcal{F}.$$

If Φ is a normalized function then the number

$$\langle \wp \rangle = \langle \Phi, \wp\Phi \rangle = \sum_{n=0}^{N-1} n\,|\Phi(n)|^2$$

represents the *mean value* of the price.

4 TIME EVOLUTION OF THE STOCK PRICE

The main purpose of a mathematical model concerning the stock price is to anticipate the price. We can only predict probabilities. By following the analogy with the quantum mechanics, we assume that the time dependent function $\Phi(n, t)$ describing the stock price satisfies a Schrödinger type equation [10]

$$\mathrm{i}\frac{\partial}{\partial t}\Phi = \left(\frac{1}{2\mu}\mathcal{O}^2 + \mathcal{V}(\wp, t)\right)\Phi$$

where μ is a positive parameter and $\mathcal{V}(\wp, t)$ is a time dependent function describing the interactions between traders as well as the external economic conditions. The function $\Phi(n, t)$ is well-determined if its values $\Phi(n, t_0)$ at a fixed moment of time t_0 are known.

A large variety of cases can be easily investigated by using, for example, the computer program in MATHEMATICA presented in [12]. The particular case of the equation

$$\mathrm{i}\frac{\partial\Phi}{\partial t} = \left(\frac{1}{2\mu}\mathcal{O}^2 + \beta\,\cos\omega t\,\wp\right)\Phi$$

considered for $\alpha = 0.2$, $\mu = 1$, $\beta = 1/10$ and $\omega = 1/10000$, is ilustrated in Figure 3. At the initial moment of time, $t = 0$, the price of the considered stock is 7 units of cash, but we do not have any information on the owner. After 4 hours the most probable price is 8, but the price may also be 7 or 9 with certain probabilities. The most probable owner is T_{20}, but $T_0, T_1, T_2, T_{17}, T_{18}, T_{19}$ may also be the owners with smaller probabilities. After 8 hours the most probable price is 9 and the probability for T_{20} to be the owner is 0.43.
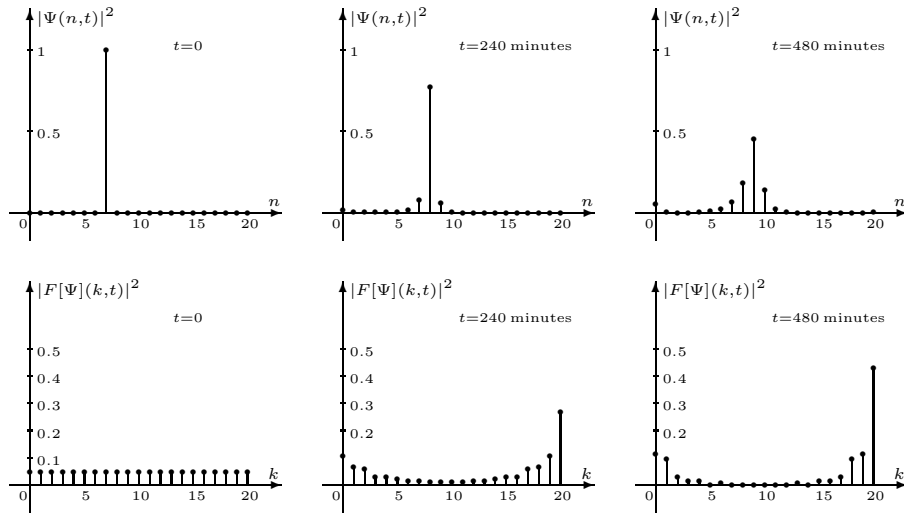
Figure 3: Time evolution of the distributions of probability describing the price and ownership.

5   CONCLUDING REMARKS

The relation price-ownership is similar to the relation position-momentum from quantum mechanics. Despite the rather artificial numerical description we use for the ownership, the proposed quantum mechanical model may help us to better understand the relation price-ownership playing a fundamental role in finance. The distribution probability concerning the price is related to the distribution probability describing the ownership. The price and the ownership cannot be simultaneously known with infinite precision. Generaly, the accuracy in the knowledge of the price cannot be improved without a corresponding loss in the accuracy in the knowledge of the ownership.

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