

# Dynamic Debt Runs

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This article analyzes the dynamic coordination problem among creditors of a firm with a time-varying fundamental and a staggered debt structure. In deciding whether to roll over his debt, each maturing creditor is concerned about the rollover decisions of other creditors whose debt matures during his next contract period. We derive a unique threshold equilibrium and characterize the roles of fundamental volatility, credit lines, and debt maturity in driving runs. In particular, we show that when fundamental volatility is sufficiently high, commonly used measures such as temporarily keeping the firm alive under runs and increasing debt maturity can exacerbate rather than mitigate runs. (*JEL* G01, G20)

Runs by creditors on non-bank financial institutions, such as investment banks, special investment vehicles, conduits, and hedge funds, are widely regarded as one of the direct causes of the credit crisis of 2007–2009.<sup>1</sup> The freeze of the U.S. asset-backed commercial paper (ABCP) markets in 2007 provided a vivid illustration of runs on the financial institutions. Prompted by concerns about the mounting delinquencies of subprime mortgages, outstanding ABCP fell by a staggering \$400 billion (one-third of the existing amount) during the second half of 2007 (e.g., [Covitz, Liang, and Suarez 2009](#)).

While there is a large literature analyzing bank runs, most of the existing theories (e.g., [Diamond and Dybvig 1983](#); [Rochet and Vives 2004](#); [Goldstein and Pauzner 2005](#)) focus on static settings.<sup>2</sup> This article targets three questions regarding runs on financial firms that are related to fluctuations of firm

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<sup>1</sup> See comments of various regulators and researchers (e.g., [Bernanke 2008](#); [Cox 2008](#); [Gorton 2008](#); [Brunnermeier 2009](#); [Krishnamurthy 2010](#); [Shin 2009](#)).

<sup>2</sup> The classic bank-run model of [Diamond and Dybvig \(1983\)](#) features a setting in which bank depositors simultaneously decide whether to withdraw their demand deposits from a solvent but illiquid bank. There exist

fundamentals and thus motivate a dynamic setting. First, how does the price volatility of a firm's asset holdings affect its debt run risk? As volatility tends to spike up during financial crises, this question is especially relevant for understanding the stability of financial firms during crises. Second, different from banks, financial firms are mostly financed by short-term debt contracts, such as commercial paper and repo transactions. Would debt maturity choice compound the potential volatility effect on a firm's debt run risk? Third, these firms also hold (explicit and implicit) credit lines from other firms and the government, which can temporarily sustain them under runs. Do credit lines always mitigate debt run risk? In this article, we develop a dynamic model in continuous time to address these questions.

Specifically, our model focuses on a firm with a time-varying fundamental and a staggered debt structure. The firm finances its long-term asset holding by rolling over short-term debt with a continuum of small creditors. The firm's debt expirations are uniformly spread out across time, which implies that creditors decide whether to roll over their debt contracts with the firm at different times. This staggered structure is realistic,<sup>3</sup> and is distinct from the synchronous structure assumed by the extant static bank-run models. As a result, each maturing creditor does not need to worry much about the rollover decisions of other maturing creditors at the same time. However, he faces the risk that the firm could fail during his next contract period if future maturing creditors choose not to roll over their debt contracts. Because of this so-called rollover risk, he needs to coordinate his rollover decision with future maturing creditors.

We also make two assumptions on the asset side. First, the firm asset is illiquid. When some maturing creditors choose to run and the firm fails to raise new funds to repay them, it has to prematurely liquidate the asset at a fire-sale price equal to a fraction of its fundamental value. Second, the firm's asset fundamental is time-varying and every creditor observes the same public information about its current value. This assumption is realistic, as assets held

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multiple equilibria. In the self-fulfilling bank-run equilibrium, all depositors choose to withdraw and cause the bank to fail. This model highlights the key externality of depositors' withdrawals, although the existence of multiple equilibria makes it difficult to analyze timing and determinants of runs. More recently, Rochet and Vives (2004) and Goldstein and Pauzner (2005) adopt the insight of the global-games literature (e.g., Carlsson and van Damme 1993; Morris and Shin 2003) to derive a unique bank-run equilibrium in the Diamond-Dybvig setting. The key idea is to let depositors possess noisy private signals about bank fundamentals. The noise in their private signals introduces strategic uncertainty about others' actions and thus prevents the emergence of multiple equilibria. This global-games framework has proven useful in analyzing various issues related to banking regulation on liquidity ratios, central bank interventions as the lender of last resort, and banks' optimal demand-deposit contracts.

<sup>3</sup> This staggered debt structure is widely used in practice. For example, on February 10, 2009, the data from Bloomberg show that Morgan Stanley, one of the major U.S. investment banks, had short-term debt (with maturities less than 1.5 years) expiring on almost every day throughout February and March 2009. If we sum up the total value of Morgan Stanley's expiring short-term debt in each week, the values for the following five weeks are \$62 million, \$324 million, \$339 million, \$239 million, and \$457 million, respectively. The Federal Reserve Release also shows that the commercial paper issued by financial firms in aggregate has maturities well spread out over time.

by financial firms are mostly financial securities whose values change over time and are largely observable by the public. Furthermore, we also allow the firm to have some imperfect credit lines, which sustain the firm under runs until it breaks at a random Poisson time.

We derive in closed form a unique threshold equilibrium, in which each maturing creditor chooses to run on the firm if its fundamental falls below a certain endogenously determined threshold. To protect himself against the firm's future rollover risk caused by other creditors, each maturing creditor will choose to roll over his debt if and only if the current fundamental provides a sufficient safety margin. Each creditor's optimal threshold choice depends on that of others—if a creditor anticipates that the creditors maturing during his next contract period are more likely to run (i.e., using a higher rollover threshold), he has a greater incentive to run now (i.e., using an even higher threshold). In this way, creditors engage in a preemptive “rat race,” which leads each creditor to choose a rollover threshold substantially higher than he would in the absence of the coordination problem.

The uniqueness of the debt-run equilibrium is reminiscent of the global-games models (e.g., Carlsson and van Damme 1993; Morris and Shin 2003), although the underlying mechanism is different. Instead of relying on creditors' noisy private information, the unique threshold equilibrium in our model builds on the firm's time-varying fundamental and creditors' asynchronous rollover decisions. A time-varying fundamental introduces strategic uncertainty about other creditors' rollover decisions in the future and thus prevents the emergence of multiple equilibria. This equilibrium selection insight builds on Frankel and Pauzner (2000), who show that in dynamic coordination games, fundamental shocks can act as a coordination device for agents who choose actions at different times.<sup>4</sup>

It is also worth mentioning that we cannot directly apply the method of iterated deletions of dominated strategies, which is used by Frankel and Pauzner (2000), to derive the uniqueness equilibrium. This is because strategic complementarity in our setting arises only in creditors' continuation values rather than in flow payoffs. Instead, we have invoked a guess-and-verify approach. Thus, our model also provides a useful lesson for analyzing dynamic coordination problems in other realistic situations when strategic complementarity is not available in the standard form required by the standard models.

Like the static global-games models of Rochet and Vives (2004) and Goldstein and Pauzner (2005), our model integrates two distinct and long-standing views about runs: one based on fundamental concerns and the other based on

<sup>4</sup> Guimaraes (2006) and Plantin and Shin (2008) also build on the same equilibrium selection insight to study coordinated currency attacks and speculative dynamics in carry trades. In both models, time-varying fundamentals, together with frictions that prevent investors from instantaneously changing their investment positions in a currency, allow investors to coordinate around a unique equilibrium. Guimaraes highlights that small frictions can cause a long delay in investors' attacks on an overvalued currency, while Plantin and Shin focus on funding externalities created by carry trades and the resulting large negative movements in exchange rate dynamics.

unwarranted panics.<sup>5</sup> On the one hand, concerns about weak firm fundamentals prompt creditors to run; on the other, the externality of their runs exacerbates the concerns and leads them to run even when the firm is still solvent.

Our model allows us to characterize the timing and determinants of debt runs. In addition to the standard result that creditors tend to run on firms with weaker fundamentals and with greater asset illiquidity, we also derive several implications that are directly related to time-varying firm fundamentals and that are beyond the static bank-run models.

First, higher fundamental volatility tends to exacerbate creditors' incentives to run. This result aggregates two separate effects of higher fundamental volatility: it causes both larger insolvency risk and greater strategic uncertainty about other creditors' rollover decisions. Together, these effects motivate each creditor to use a higher rollover threshold. As asset price volatility tends to spike during crises, this implication highlights rising volatility as an important source of instability in financial firms.

The greater strategic uncertainty created by higher fundamental volatility differentiates the volatility effect in our model from the effect of fundamental uncertainty in the static global-games models. In these models with noisy private signals, each agent assesses a bank's insolvency risk based on his posterior belief about the bank fundamental (fundamental uncertainty), but strategic uncertainty about other agents' actions is determined by noise in their private signals.<sup>6</sup> While it is possible to incorporate time-varying fundamentals in dynamic global-games models, learning in the presence of private information can substantially complicate such a task.<sup>7</sup> The rich information structure embedded in the global-games models is useful for addressing important questions related to effects of public and private information,

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<sup>5</sup> The first view, advocated by [Friedman and Schwartz \(1963\)](#) and [Kindleberger \(1978\)](#), attributes many historical banking crises to unwarranted panics by arguing that the banks that were forced to liquidate in such episodes were illiquid rather than insolvent. The alternative view, proposed by [Mitchell \(1941\)](#) and others, suggests that runs occur when depositors have fundamental concerns about the health of banks. Each of these views has motivated a body of theoretical models of bank runs. [Diamond and Dybvig \(1983\)](#), [Postlewaite and Vives \(1987\)](#), [Peck and Shell \(2003\)](#), and [Caballero and Krishnamurthy \(2008\)](#) offer models of panic-driven runs, while [Bryant \(1980\)](#), [Gorton \(1988\)](#), [Chari and Jagannathan \(1988\)](#), [Jacklin and Bhattacharya \(1988\)](#), and [Allen and Gale \(1998\)](#) focus on the fundamental risk of bank loans and the depositors' signal extraction problem in driving runs. See [Gorton and Winton \(2003\)](#) and [Allen and Gale \(2007\)](#) for two recent reviews of the history of financial crises and different theories of runs.

<sup>6</sup> [Morris and Shin \(2003, Section 3\)](#) synthesize a unified static global-games framework. This framework features a continuum of agents who simultaneously make binary choices (i.e., to run or not to run) and who possess both private and public information about an unobservable fundamental. The agents' payoffs increase with the fundamental and exhibit strategic complementarity (i.e., each agent's payoff increases with the fraction of agents taking the same action as him). A unique equilibrium exists only if agents' private signals are sufficiently precise relative to their public information. When this condition holds true, the model comparative statics also indicate that variance of agents' prior beliefs may increase or decrease their equilibrium threshold strategy depending on whether the threshold is above or below their public information. [Angeletos, Hellwig, and Pavan \(2007\)](#) further confirm the important role played by information structure in a dynamic global-games model.

<sup>7</sup> See [Abreu and Brunnermeier \(2003\)](#), [Chamley \(2003\)](#), and [Angeletos, Hellwig, and Pavan \(2007\)](#) for examples of dynamic coordination models with private information. These models all feature constant fundamentals and, instead, focus on agents' dynamic learning regarding other agents' private information and the learning effect on coordination among agents.

but it is not essential for analyzing dynamic runs. In contrast, our model provides a convenient framework based on publicly observable time-varying fundamentals. The next two implications build on the interactions of the firm's time-varying fundamentals with its credit lines and debt maturity.

Second, and perhaps the most novel implication of all, credit lines can exacerbate creditors' incentives to run when fundamental volatility is sufficiently high. To be precise, when fundamental volatility is low, credit lines can mitigate creditors' incentives to run by sustaining the firm under the runs for a certain period during which a creditor's contract may mature. However, when fundamental volatility is sufficiently high, the opposite holds true because volatility can cause the firm's fundamentals to severely deteriorate during the period that it survives using its credit lines. This volatility effect prompts each creditor to run earlier if the firm can survive longer. This intriguing result suggests that the effort made by governments to temporarily bail out financial firms during crises (which acts like imperfect credit lines) can be counterproductive in deterring runs.

Third, longer debt maturities can have opposite effects on mitigating runs, again depending on the firm's fundamental volatility. A longer maturity has two offsetting effects from the perspective of an individual creditor. On the one hand, it reduces the firm's rollover frequency with other creditors and thus makes it less likely to fail under the runs of other creditors. On the other hand, each creditor faces a longer lock-in period, during which the firm's fundamentals could drop below other creditors' rollover threshold. The trade-off between these two effects implies that a longer maturity reduces the creditors' equilibrium rollover threshold during normal periods when volatility is modest, but increases the threshold when volatility is sufficiently high. Given the pervasive high volatility during financial crises, this result cautions the widely advocated policy of requiring financial firms to use long-term debt.

Our article complements several recent studies on firms' rollover risk. [Acharya, Gale, and Yorulmazer \(2011\)](#) show that fast rollover frequency can lead to diminishing debt capacity. [Brunnermeier and Oehmke \(2012\)](#) study the conflict between long-term and short-term creditors and show that this conflict can motivate all creditors to demand short-term debt. [He and Xiong \(2012\)](#) analyze the role played by market illiquidity in exacerbating the conflict between debt and equity holders. [Morris and Shin \(2009\)](#) build a global-games model to analyze the illiquidity component of financial institutions' credit risk. In contrast, our model focuses on preemptive runs caused by creditors' fear of a firm's future rollover risk.

The article is organized as follows. Section 1 describes the model setup. We derive a unique debt-run equilibrium in Section 2 and analyze the determinants of the equilibrium rollover threshold in Section 3. We provide some further discussion in Section 4 and conclude in Section 5. All technical proofs are given in the Appendix.

## 1. Model

We consider a continuous-time model with an infinite horizon. A firm invests in a long-term asset by rolling over short-term debt. One can interpret this firm as any firm, either financial or non-financial. Our model is perhaps more appealing for financial firms because they tend to have higher leverage and more short-term debt. To make debt runs a relevant concern, we assume that the capital markets are imperfect in the following sense: the firm cannot find a single creditor with “deep pockets” to finance all of its debt; instead, it has to rely on a continuum of small creditors. The firm spreads its debt expirations uniformly across time. Then, if some of the maturing creditors choose not to roll over their debt and the firm fails to raise new funds from its imperfect credit lines to pay them off, the firm is bankrupt and has to liquidate its asset in an illiquid secondary market at a discount.

### 1.1 Asset

We normalize the firm’s asset holding to be one unit. The firm borrows \$1 at time 0 to acquire its asset. Once the asset is in place, it generates a constant stream of cash flow, i.e.,  $r dt$  over the time interval  $[t, t + dt]$ . At a random time  $\tau_\phi$ , which arrives according to a Poisson process with intensity  $\phi > 0$ , the asset matures with a final payoff. An important advantage of assuming a random asset maturity with a Poisson process is that at any point before the maturity, the expected remaining time-to-maturity is always  $1/\phi$ .

The asset’s final payoff is equal to the time- $\tau_\phi$  value of a stochastic process  $y_t$ , which follows a geometric Brownian motion with constant drift  $\mu$  and volatility  $\sigma > 0$ :

$$\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t,$$

where  $\{Z_t\}$  is a standard Brownian motion. We assume that the value of the fundamental process is publicly observable at any time.

Taken together, the firm’s asset generates a constant cash flow of  $r dt$  before  $\tau_\phi$  and a final value of  $y_{\tau_\phi}$  at  $\tau_\phi$ . Then, by assuming that agents in this economy (including the firm’s creditors) are risk-neutral and have a discount rate of  $\rho > 0$ , we can compute the fundamental value of the firm’s asset as its expected discounted future cash flows:

$$F(y_t) = E_t \left[ \int_t^{\tau_\phi} e^{-\rho(s-t)} r ds + e^{-\rho(\tau_\phi-t)} y_{\tau_\phi} \right] = \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y_t, \tag{1}$$

where the two components,  $\frac{r}{\rho + \phi}$  and  $\frac{\phi}{\rho + \phi - \mu} y_t$ , correspond to the present values of the asset’s constant cash flow and final payoff, respectively. Since the asset’s fundamental value increases linearly with  $y_t$ , we will conveniently refer to  $y_t$  as the firm fundamental.

The assumption that the firm's fundamental is time-varying is natural. It is somewhat strong to assume that the fundamental is publicly observable. This assumption mainly serves to insulate our model from further complications caused by agents' private information about the firm's fundamental. In fact, our model would stay intact if we assume that the fundamental is unobservable and instead all agents only observe the same noisy public signals.

## 1.2 Debt financing

The firm finances its asset holding by issuing short-term debt. A key contributing factor to the recent credit crisis was the excessive use of short-term debt, such as commercial paper and repos, by financial institutions in the preceding period (e.g., Gorton 2008; Brunnermeier 2009; Krishnamurthy 2010; Shin 2009). Why do firms use short-term debt? Short-term debt is a natural response of outside creditors to a variety of agency problems inside the firm (e.g., Calomiris and Kahn 1991; Diamond and Rajan 2009). By choosing short-term financing, creditors keep the option to pull out if they discover that firm managers are pursuing value-destroying projects.<sup>8</sup> The short commitment period also makes short-term debt less information sensitive and thus less exposed to adverse-selection problems (e.g., Gorton and Pennacchi 1990). As a result, short-term debt also has a lower financing cost. To maintain the simplicity of our model, we take a realistic debt structure as given and focus on the coordination problem generated by short-term debt.<sup>9</sup>

We emphasize an important feature of real-life firms' debt structure: firms tend to spread out their debt expirations over time to reduce liquidity risk (see evidence given in Footnote 4). In this way, they avoid having to roll over a large fraction of their debt on a single day. Specifically, we assume that the firm finances its asset holding by issuing one unit of debt divided uniformly among a continuum of small creditors with measure 1. The promised interest rate is  $r$  so that the cash flow from the asset exactly pays off the interest payment until the asset matures or until the firm is forced to liquidate the asset prematurely.<sup>10</sup> Following the staggered-pricing model of Calvo (1983) and the credit-risk model of Leland (1998), we assume that each debt contract lasts for a random period, which ends upon the arrival of an independent Poisson shock with intensity  $\delta > 0$ . In other words, the duration of each debt contract

<sup>8</sup> See Kashyap, Rajan, and Stein (2008) for a recent review of this agency literature and capital regulation issues related to the recent financial crisis.

<sup>9</sup> In Section 4, we also discuss an extension of our model by Cheng and Milbradt (2012), who analyze a firm's optimal debt maturity choice based on a trade-off between incentive provision and debt run risk.

<sup>10</sup> To focus on the coordination problem between creditors, we also take the interest payment of the firm debt as given. One might argue that when facing rollover difficulties, the firm can attract the maturing creditors by promising higher interest rates. However, doing so dilutes the stakes of other creditors in the firm and would motivate earlier maturing creditors to demand higher interest rates preemptively, similar to the preemptive runs highlighted in our model. In other words, promising higher interest rates could become a self-enforcing tightening mechanism on the firm, instead of a way to bail it out. We will leave a more elaborate analysis of this effect for future research.



has an exponential distribution. Once the contract expires, the creditor chooses whether to roll over the debt or to run. The maturity shocks are independent across creditors so that each creditor expects some other creditors' contracts to mature before his. He is thus exposed to the firm's rollover risk.

In aggregate, the firm has a fixed fraction  $\delta dt$  of its debt maturing over  $[t, t + dt]$ , where the parameter  $\delta$  represents the firm's rollover frequency. The random maturity assumption simplifies the complication of keeping track of the remaining maturities of individual contracts, because at any time before the maturity the expected remaining maturity of each contract is always  $1/\delta$ . By matching  $1/\delta$  with the fixed maturity of a real-life debt contract, this assumption captures the first-order effect of debt maturity when a creditor makes his rollover decision.<sup>11</sup>

### 1.3 Runs and liquidation

When the maturing creditors choose to run, they expose the firm to bankruptcy risk if it cannot raise new funds to repay the running creditors. The firm would be extremely frail if a single creditor's run would cause it to fail. To prevent this, we allow the firm to draw on pre-committed credit lines from other institutions or the government. However, the credit lines are imperfect, as the issuing institutions may experience their own financial distresses and the government may face political pressures against supporting distressed firms. As a result, persistent runs will eventually cause the firm to fail.

More specifically, over a short time interval  $[t, t + dt]$ ,  $\delta dt$  fraction of the firm's debt contracts mature. If these creditors choose to run, the firm will draw on its credit lines to raise new funds to pay off the running creditors. We assume that with probability  $\theta \delta dt$ , the issuer of the firm's credit lines fails to provide liquidity and the firm is thus forced into liquidation. The parameter  $\theta > 0$  measures the unreliability of the firm's credit lines. The higher the value of  $\theta$ , the less reliable the firm's credit lines, and therefore the more likely the firm will be forced into liquidation given the same creditor outflow rate. With probability  $1 - \theta \delta dt$ , the firm is able to raise new funds through the credit lines to pay off the running creditors. For simplicity, we assume that the new funds raised from the credit lines have the same debt contract as the existing ones. Taken together, if every maturing creditor chooses to run, the firm fails with Poisson intensity  $\theta \delta$ , i.e., it survives on average for a period of  $\frac{1}{\theta \delta}$ .<sup>12</sup>

<sup>11</sup> This assumption also generates an artificial second-order effect: if the debt contracts have a fixed maturity, a creditor, after rolling over his contract, will go to the end of the maturity queue. The random maturity assumption makes it possible for the creditor to be released early and therefore to run before other creditors when the asset fundamental deteriorates. This possibility makes the creditor less worried about the firm's rollover risk than he would be if the debt contract had a fixed maturity. This in turn makes him more likely to roll over his debt. We have verified this outcome by numerically analyzing a variation of our model with fixed debt maturity. Thus, by assuming random debt maturity, our model underestimates the firm's rollover risk.

<sup>12</sup> The imperfect credit lines are realistic as credit lines were frequently withdrawn by issuers during the recent credit crisis, either because they also faced funding problems or because they were concerned about future funding problems and thus chose to hoard liquidity. Regarding the runs in the ABCP market in 2007,



Once the firm fails to raise new funds to pay off the running creditors, it falls into bankruptcy and has to liquidate its asset in an illiquid secondary market. We assume that the firm can only recover a fraction  $\alpha \in (0, 1)$  of its fundamental value. That is, the firm obtains a fire-sale price of

$$\tilde{L}(y_t) = \alpha F(y_t) = L + ly_t, \quad (2)$$

where

$$L = \frac{\alpha r}{\rho + \phi} \quad \text{and} \quad l = \frac{\alpha \phi}{\rho + \phi - \mu}. \quad (3)$$

For simplicity, we rule out partial liquidations in this model.

The liquidation value will then be used to pay off all creditors on an equal basis. In other words, both the running creditors and the other creditors who are locked in by their current contracts get the same payoff  $\min(\tilde{L}(y), 1)$ . Ex ante, each creditor's expected payoff from choosing to run is still 1 because the probability of the firm failure  $\theta \delta dt$  is in a higher  $dt$  order.<sup>13</sup>

Due to the staggered debt structure in our continuous-time setting, the fraction of maturing creditors over a small time interval (i.e.,  $\delta dt$ ) is small. This implies that a creditor's running decision is not affected by the concurrent decisions of other maturing creditors. This feature insulates our model from the [Diamond and Dybvig \(1983\)](#) type of static coordination problem, in which agents make simultaneous decisions, and instead allows us to focus on the coordination problem between creditors whose contracts mature at different times.

Our model implicitly assumes that once in distress, the firm cannot raise more capital by issuing new equity. This assumption is consistent with the existence of information asymmetry between the firm and outside equity holders, and with the existence of conflict of interest between debt and equity holders.<sup>14</sup> We also assume that it is impossible for the firm to renegotiate on the contracts of other creditors to make the maturing creditors more willing to roll over their debt. This assumption is realistic due to the complexity and high cost of renegotiating with a large number of creditors with different seniorities.

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[Covitz, Liang, and Suarez \(2009\)](#) find that across different ABCP programs, the reliability of their credit lines is also an important determinant of the likelihood of runs. One could also interpret  $\theta$  as inversely related to the firm's cash reserve. If the firm has more cash reserves, it can survive the creditors' runs for a longer period. Since outside creditors usually cannot directly observe the balance of a firm's cash reserve, from their perspective the failure of the firm under creditors' runs will occur at a random time.

<sup>13</sup> This observation implies that in our model the sharing rule in the event of bankruptcy is inconsequential. We can also assume that during bankruptcy those maturing creditors who have chosen to run get a full payoff 1, while the remaining creditors who are locked in by their current contracts get  $\min(\tilde{L}(y), 1)$ . This alternative assumption gives a greater incentive for maturing creditors to run. However, since the probability of the firm failure is  $\theta \delta dt$ , the difference in incentive is negligible.

<sup>14</sup> When a firm faces liquidity problems in the debt market, equity holders could find it optimal not to inject more equity. By injecting equity they bear all of the financial burden of keeping the firm from bankruptcy, but the benefit is shared by both debt and equity holders. See [He and Xiong \(2012\)](#) for a formal analysis of this distortion in short-term debt crises.

### 1.4 Parameter restrictions

To make our analysis meaningful, we impose several parameter restrictions. First, we bound the interest payment by

$$\rho < r < \rho + \phi. \tag{4}$$

The first part  $r > \rho$  makes the interest payment attractive to the creditors, who have a discount rate of  $\rho$ . The second part  $r < \rho + \phi$  rules out the scenario where the interest payment is so attractive that rollover becomes the dominant strategy even when the firm fundamental  $y_t$  is close to zero. Essentially, this condition ensures the existence of the lower dominance region in which each creditor's dominant strategy is to run if the firm's fundamental  $y_t$  is sufficiently low.

Second, we limit the growth rate of the firm fundamental by

$$\mu < \rho + \phi. \tag{5}$$

Otherwise, the firm's fundamental value in Equation (1) would explode.

Third, we also limit the premature liquidation recovery rate of the firm asset:

$$\alpha < \frac{1}{\frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu}}, \tag{6}$$

so that  $L + l < 1$  (see Equation (3)). Under this condition, the asset liquidation value is not enough to pay off all the creditors when  $y_t = 1$ . This condition ensures that the firm's future rollover risk significantly concerns each creditor when the firm fundamental  $y_t$  is in an intermediate region.

Finally, we assume that the parameter  $\theta$  is sufficiently high:

$$\theta > \frac{\phi}{\delta(1 - L - l)}, \tag{7}$$

so that the firm faces a serious bankruptcy probability when some creditors choose to run.

## 2. The Debt-run Equilibrium

Given the firm's asset and financing structures described in the previous section, we now analyze the debt-run equilibrium. We limit our attention to monotone equilibria, equilibria in which each creditor's rollover strategy is monotonic with respect to the firm fundamental  $y_t$  (i.e., to roll over if and only if the firm's fundamental is above a threshold). In making his rollover decision, a creditor rationally anticipates that once he rolls over the debt, he faces the firm's rollover risk during his contract period. This is because volatility could cause the firm's fundamental to fall below the other creditors' rollover thresholds. As a result, the creditor's optimal rollover threshold depends on the other creditors' threshold choices.

In this section, we first set up a creditor’s optimization problem in choosing his optimal threshold. We then construct a unique monotone equilibrium in closed form. Finally, we discuss the rat race in determining the equilibrium rollover threshold.

### 2.1 An individual creditor’s problem

We first analyze the optimal rollover decision of a creditor by taking as given that all other creditors use a monotone strategy with a rollover threshold  $y_*$  (i.e., other creditors will roll over their debt if and only if the firm’s fundamental is above  $y_*$  when their debt contracts mature). During the creditor’s contract period, his value function depends directly on the firm fundamental  $y_t$ , and indirectly on the other creditors’ rollover threshold  $y_*$ . Since the creditor’s future payoff is proportional to the unit of debt he holds, we denote  $V(y_t; y_*)$  as the creditor’s value function normalized by the debt unit.

For each unit of debt, the creditor receives a stream of interest payments  $r$  until

$$\tau = \min (\tau_\phi, \tau_\delta, \tau_\theta),$$

which is the earliest of the following three events, illustrated in Figure 1 at the end of three different fundamental paths. On the top path, the firm stays alive until its asset matures at  $\tau_\phi$ . At this time, the creditor gets a final payoff of  $\min(1, y_{\tau_\phi})$ , i.e., the face value 1 if the asset’s maturity payoff  $y_{\tau_\phi}$  is

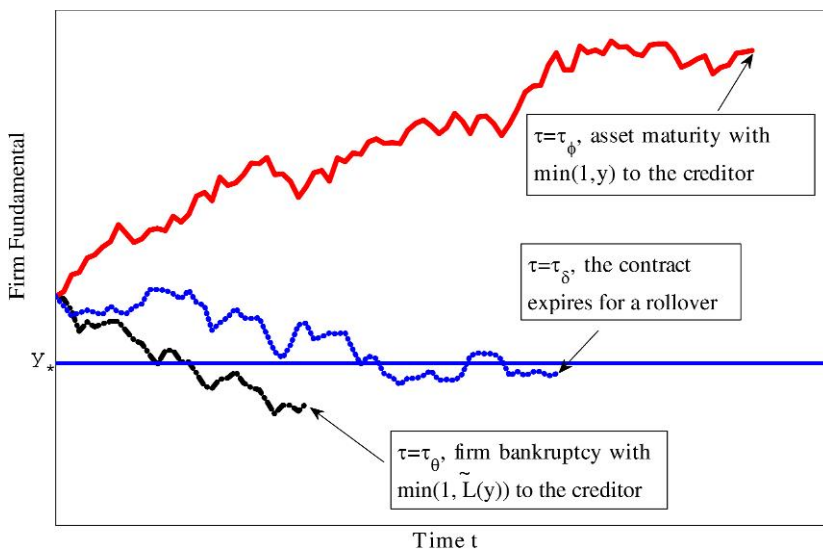


Figure 1  
Three possible outcomes for a creditor

sufficient to pay all the debt, and  $y_{\tau_\phi}$  otherwise. The possibility that the asset's maturity value may be insufficient to pay off the debt represents the firm's insolvency risk. On the bottom path, the firm's fundamental drops below the other creditors' rollover threshold and the firm is eventually forced to liquidate its asset prematurely at  $\tau_\theta$ . At this time, the creditor gets  $\min(1, L + ly_{\tau_\theta})$ . This outcome represents the firm's rollover risk. On the middle path, the firm stays alive (although its fundamental dips below the other creditors' rollover threshold) until  $\tau_\delta$ , when the creditor's contract expires. At this time, the creditor has an option, i.e., he can choose whether to roll over depending on whether the continuation value  $V(y_{\tau_\delta}; y_*)$  is higher than getting the one dollar back.

Due to risk neutrality, the creditor's value function is given by

$$\begin{aligned}
 V(y_t; y_*) = E_t \left\{ \int_t^\tau e^{-\rho(s-t)} r ds + e^{-\rho(\tau-t)} \left[ \min(1, y_\tau) \mathbf{1}_{\{\tau=\tau_\phi\}} \right. \right. \\
 + \min(1, L + ly_\tau) \mathbf{1}_{\{\tau=\tau_\theta\}} \\
 \left. \left. + \max_{\text{rollover or run}} \{V(y_\tau; y_*), 1\} \mathbf{1}_{\{\tau=\tau_\delta\}} \right] \right\}, \tag{8}
 \end{aligned}$$

where  $\mathbf{1}_{\{\cdot\}}$  is an indicator function that takes a value of 1 if the statement in the bracket is true or zero otherwise. The creditor's future payoff during his contract period depends on other creditors' rollover choices because other creditors' runs might force the firm to liquidate its asset prematurely, as illustrated by the bottom path of Figure 1. This dependence gives rise to strategic complementarities in the creditors' rollover decisions, and thus creates a coordination problem between creditors whose contracts mature at different times.

By considering the change in the creditor's continuation value over a small time interval  $[t, t + dt]$ , we can derive his Hamilton-Jacobi-Bellman (HJB) Equation:

$$\begin{aligned}
 \rho V(y_t; y_*) = \mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy} + r + \phi [\min(1, y_t) - V(y_t; y_*)] \\
 + \theta \delta \mathbf{1}_{\{y_t < y_*\}} [\min(L + ly_t, 1) - V(y_t; y_*)] \\
 + \delta \max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}. \tag{9}
 \end{aligned}$$

The left-hand-side term  $\rho V(y_t; y_*)$  represents the creditor's required return. This term should be equal to the expected increment in his continuation value, as summarized by the terms on the right-hand side. The first two terms  $\mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy}$  capture the expected change in the continuation value caused by the fluctuation in the firm fundamental  $y_t$ . The third term  $r$  is the interest payment

per unit of time. The next three terms capture the three events illustrated in Figure 1: the fourth term  $\phi [\min (1, y_t) - V (y_t; y_*)]$  captures the possibility that the asset matures during the time interval, which occurs with probability  $\phi dt$  and generates an impact of  $\min (1, y_t) - V (y_t; y_*)$  on the creditor's continuation value. The fifth term  $\theta \delta \mathbf{1}_{\{y_t < y_*\}} [\min (L + ly_t, 1) - V (y_t; y_*)]$  represents the expected effect of premature liquidation from other creditors' runs, which occurs with probability  $\theta \delta \mathbf{1}_{\{y_t < y_*\}} dt$  (other maturing creditors will run only if  $y_t < y_*$ ) and generates an impact of  $\min (L + ly_t, 1) - V (y_t; y_*)$  on the creditor's continuation value. The last term  $\delta \max_{\text{rollover or run}} \{0, 1 - V (y_t; y_*)\}$  captures the expected effect from the creditor's own contract expiration, which arrives with probability  $\delta dt$ . Upon its arrival, the creditor chooses whether to roll over or to run:  $\max_{\text{rollover or run}} \{0, 1 - V (y_t; y_*)\}$ .

It is obvious that a maturing creditor will choose to roll over his contract if and only if  $V (y_t; y_*) > 1$ , and to run otherwise. This implies that if the value function  $V$  only crosses 1 at a single point  $y'$ , i.e.,  $V (y'; y_*) = 1$ , then  $y'$  is the creditor's optimal threshold.

**2.1.1 Externality on future maturing creditors.** The rollover decision of current-period maturing creditors affects not only their own payoffs, but also future maturing creditors. In particular, their decision to run adds to the firm's bankruptcy probability and thus imposes an implicit cost on future maturing creditors. Since they do not internalize the cost on others, this externality is the ultimate source of debt runs in our model. To see this point precisely, we summarize the payoff (or continuation value) of the current-period maturing creditors and future maturing creditors depending on the choice of the current-period maturing creditors in Table 1. For simplicity, we treat all the current-period maturing creditors as one identity in this illustration.

The maturing creditors will choose run if  $1 \cdot (1 - \theta \delta dt) + \tilde{L} (y) \cdot \theta \delta dt > V (y)$ , which is  $V (y) < 1$  after ignoring the higher-order  $dt$  term. Their runs reduce the remaining creditors' continuation value by

$$V (y) - [V (y) \cdot (1 - \theta \delta dt) + \tilde{L} (y) \cdot \theta \delta dt] = [V (y) - \tilde{L} (y)] \theta \delta dt.$$

While this effect is of the  $dt$  order, a creditor needs to bear the accumulative externality effect of all maturing creditors before him, which, in expectation, could be significant.<sup>15</sup>

## 2.2 The unique monotone equilibrium

We first focus our attention on symmetric monotone equilibria, and then show that there cannot be any asymmetric monotone equilibrium. In a symmetric

<sup>15</sup> The current-period maturing creditors' runs also impose externalities on each other. But this effect is one time and of the  $dt$  order, and thus can be ignored.

**Table 1**  
**Externality on future maturing creditors**

Choice of current-period maturing creditors	Run		Rollover
	Failed	Survived	Survived
Possible firm outcomes			
Probability	$\theta\delta dt$	$1 - \theta\delta dt$	1
Payoff of current-period maturing creditors	$\tilde{L}(y)$	1	$V(y)$
Payoff of future maturing creditors	$\tilde{L}(y)$	$V(y)$	$V(y)$

monotone equilibrium, each creditor’s optimal threshold choice  $y'$  must be equal to the other creditors’ threshold  $y_*$ . Thus, we obtain the condition for determining the equilibrium threshold:

$$V(y_*; y_*) = 1.$$

We employ a guess-and-verify approach to derive a unique monotone equilibrium in four steps.<sup>16</sup> First, we derive a creditor’s value function  $V(y_t; y_*)$  from the HJB Equation in (9) by assuming that every creditor (including the creditor under consideration) uses the same monotone strategy with a rollover threshold  $y_*$ . Due to the terms  $\min(1, y_t)$  and  $\min(L + l_{y_t}, 1)$  in (9), the value function depends on the value of  $y_*$  in three cases:

1. If  $y_* < 1$ ,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + A_1 y_t^{\eta_1} & \text{when } 0 < y_t \leq y_* \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_t + A_2 y_t^{-\gamma_2} + A_3 y_t^{\eta_2} & \text{when } y_* < y_t \leq 1; \\ \frac{r+\phi}{\rho+\phi} + A_4 y_t^{-\gamma_2} & \text{when } y_t > 1. \end{cases}$$

2. If  $1 \leq y_* < \frac{1-L}{l}$ ,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + B_1 y_t^{\eta_1} & \text{when } 0 < y_t \leq 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + B_2 y_t^{-\gamma_1} + B_3 y_t^{\eta_1} & \text{when } 1 < y_t \leq y_*; \\ \frac{r+\phi}{\rho+\phi} + B_4 y_t^{-\gamma_2} & \text{when } y_t > y_*. \end{cases}$$

<sup>16</sup> Our model is substantially different from the standard dynamic coordination game frameworks. In our model, each creditor’s flow payoff from the debt contract (interest payment  $r$  and possible asset maturity payoff  $\min(y, 1)$ ) does not exhibit any strategic complementarity. Instead, strategic complementarities emerge from the implicit dependence of a creditor’s continuation value function on other creditors’ rollover decisions (Equation (8)). The standard game frameworks (e.g., Frankel and Pauzner 2000) typically specify strategic complementarity in agents’ flow payoffs, i.e., an agent’s payoff in a given period is higher if his current-period strategy overlaps with that of a greater fraction of the population. This important difference in model framework prevents us from readily applying the method of iterated deletion of dominated strategies used by Frankel and Pauzner (2000). Instead, we derive the equilibrium by invoking a guess-and-verify approach.

3. If  $y_* \geq \frac{1-L}{I}$ ,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + C_1 y_t^{\eta_1} & \text{when } 0 < y_t \leq 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + C_2 y_t^{-\gamma_1} + C_3 y_t^{\eta_1} & \text{when } 1 < y_t \leq \frac{1-L}{I} \\ \frac{r+\phi+\theta\delta+\delta}{\rho+\phi+(1+\theta)\delta} + C_4 y_t^{-\gamma_1} + C_5 y_t^{\eta_1} & \text{when } \frac{1-L}{I} < y_t \leq y_* \\ \frac{r+\phi}{\rho+\phi} + C_6 y_t^{-\gamma_2} & \text{when } y_t > y_* \end{cases}$$

The coefficients  $\eta_1, \eta_2, \gamma_1, \gamma_2, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4, C_5,$  and  $C_6$  are given in Section A.1 of the Appendix and are expressions of the model parameters and  $y_*$ .

Second, based on the derived value function, we show that there exists a unique fixed point  $y_*$  such that  $V(y_*, y_*) = 1$ . Third, we prove the optimality of the threshold  $y_*$  for any creditor, i.e.,  $V(y; y_*) > 1$  for  $y > y_*$  and  $V(y; y_*) < 1$  for  $y < y_*$ . Finally, we show that there cannot be any asymmetric monotone equilibrium.

We summarize the main results in Theorem 1.

**Theorem 1.** There exists a unique monotone equilibrium, in which each maturing creditor chooses to roll over his debt if  $y_t$  is above the threshold  $y_*$  and to run otherwise. The equilibrium threshold  $y_*$  is uniquely determined by the condition that  $V(y_*, y_*) = 1$ .

The Diamond-Dybvig model features multiple self-fulfilling equilibria. What leads to the unique equilibrium in our model? To understand this issue, first note the existence of lower and upper dominance regions. When the firm fundamental  $y_t$  is sufficiently low (i.e., close to zero), a creditor's dominant strategy is run (lower dominance region). This is because even if all other creditors choose to roll over in the future, the expected asset payoff at the maturity plus the interest payments before the asset maturity are not as attractive as getting one dollar back now. On the other hand, when the firm fundamental  $y_t$  is sufficiently high (i.e., close to infinity), the creditor's dominant strategy is rollover (upper dominance region). Even if all other creditors choose to run in the future, the firm's liquidation value is sufficient to pay off the debt in the event of a forced liquidation.

When the firm's fundamental is in the intermediate region between the two dominance regions, self-fulfilling multiple equilibria could arise if creditors make synchronous rollover decisions or if the firm's fundamental stays constant over time. In an earlier version of this article, which is listed as NBER



Working Paper 15482, we derive several variations of our model. In one of the variations, the firm still uses a staggered debt structure, but its fundamental stays constant over time. In another variation, all the debt contracts mature at the same time and the creditors simultaneously decide whether to roll over into new perpetual contracts, which last until the firm asset matures. In both variations, there exists an intermediate region in which self-fulfilling multiple equilibria emerge. We briefly discuss the second variation here as it provides a synchronous-rollover benchmark for evaluating the equilibrium rollover threshold in the asynchronous-rollover setting.

**2.2.1 The synchronous-rollover benchmark.** Suppose that the firm's debt contracts all expire at time 0, and the current firm's fundamental is  $y_0$ . At this time, each creditor decides whether to run or to roll over into a perpetual debt contract lasting until the firm asset matures at  $\tau_\phi$ . We also assume that if all creditors choose to run, the firm might fail with a probability of  $\theta_s \in (0, 1)$ . Because all creditors simultaneously choose their rollover decisions at time 0 and the firm does not face any future rollover risk, this setting closely resembles that in the Diamond-Dybvig model.

**Proposition 1.** Assume the aforementioned setting. Then, if  $y_0 > y_h \equiv \frac{1-L}{T}$  (the upper dominance region), a creditor's dominant strategy is to roll over; if  $y_0 < y_l$  (the lower dominance region), where the endogenous threshold  $y_l$  is less than  $y_h$ , the creditor's dominant strategy is to run; if  $y_0 \in [y_l, y_h]$  (the intermediate region), the creditor's optimal choice depends on the others, i.e., it is optimal to run if the others choose to run and it is optimal to roll over if the others choose to roll over.

Proposition 1 shows that in the synchronous-rollover setting, multiple self-fulfilling equilibria emerge if the firm's fundamental is in an intermediate region. In particular, creditors choose to run only if the fundamental is below a critical level  $\frac{1-L}{T}$ . This level serves as a useful benchmark to evaluate equilibrium rollover threshold in the asynchronous-rollover setting.

The emergence of self-fulfilling multiple equilibria in the two variations discussed above suggests that the unique equilibrium derived in Theorem 1 is a joint effect of the staggered debt structure and the time-varying fundamental. The underlying mechanism is analogous although different from that in the global-games models developed by Carlsson and van Damme (1993) and Morris and Shin (2003). In the global-games models, agents possess noisy private signals about an unobservable fundamental variable. Noise in their private signals introduces strategic uncertainty about other agents' actions and thus prevents the emergence of self-fulfilling multiple equilibria even when the fundamental variable lies inside the intermediate region. In our model, a time-varying fundamental and asynchronous rollover jointly imply that different creditors face different fundamentals when making their rollover decisions. As

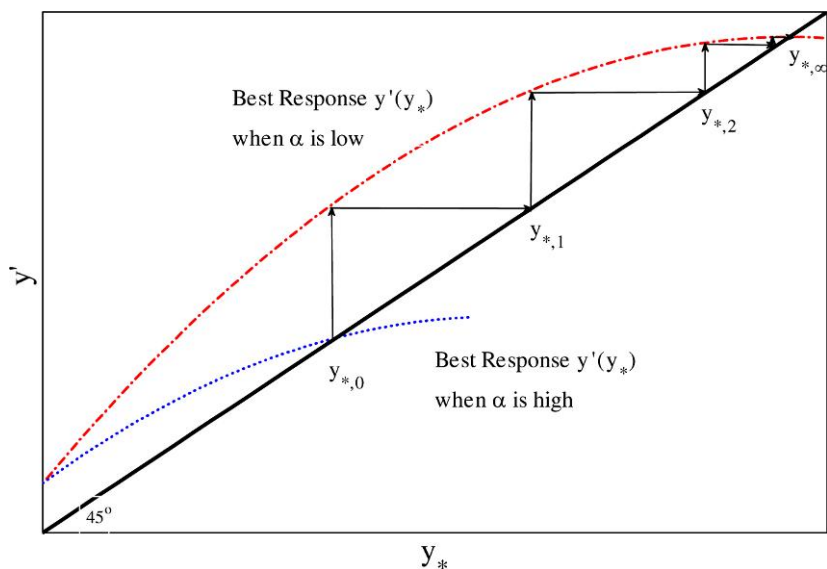
a result, a time-varying fundamental introduces strategic uncertainty to each maturing creditor about rollover decisions of future maturing creditors and thus prevents the emergence of multiple equilibria. Put differently, the current fundamental allows each maturing creditor to assess the firm's future rollover risk. A unique (subgame perfect) equilibrium emerges because anticipation of future creditors' uniquely determined rollover strategy inside the dominance regions allows the creditors to induce their optimal strategy inside the intermediate region. This key insight follows Frankel and Pauzner (2000), who show that in dynamic coordination games with strategic complementarities, random fundamental shocks allow agents to coordinate their asynchronous actions and induce a unique equilibrium.

### 2.3 The rat race in determining rollover threshold

Theorem 1 implies that each maturing creditor will choose to run if and only if a firm's fundamental drops below the equilibrium rollover threshold  $y_*$ . The equilibrium rollover threshold is thus critical to the firm's financial stability. Despite the absence of self-fulfilling multiple equilibria, externality of each creditor's rollover decision can nevertheless lead to a rat race in determining the equilibrium rollover threshold. To illustrate, suppose that initially the liquidation recovery rate of the firm asset is  $\alpha_h$ , and, correspondingly, every creditor uses an equilibrium threshold level  $y_{*,0}$ . Unexpectedly, at a certain time, all creditors find out that the recovery rate drops to a lower level  $\alpha_l < \alpha_h$ . What would the new equilibrium threshold be?

Let's start with a creditor's threshold choice, which depends on others' choices. Suppose that all the other creditors still use the original threshold  $y_{*,0}$ . Then, by solving the HJB Equation in (9), we can derive the creditor's optimal threshold  $y_{*,1}$ , which is higher than  $y_{*,0}$  because the lower liquidation value generates a greater expected loss to the creditor in the event that the firm is forced into a premature liquidation during his contract period. Of course, each creditor will go through this same calculation and choose a new threshold. If all creditors choose the threshold  $y_{*,1}$ , then a creditor's optimal threshold as the best response to  $y_{*,1}$  would be  $y_{*,2}$ , another level even higher than  $y_{*,1}$ . If all creditors choose  $y_{*,2}$ , then each creditor would go through another round of updating. Figure 2 illustrates this updating process until it eventually converges to a fixed point  $y_{*,\infty}$ , the new equilibrium threshold.

The difference between the threshold levels  $y_{*,1}$  and  $y_{*,0}$  represents the necessary safety margin a creditor would demand in response to the reduced asset liquidation value if other creditors' rollover strategies stay the same. This increase in threshold is eventually amplified to a much larger increase  $y_{*,\infty} - y_{*,0}$  through the rat race between creditors. This amplification mechanism is a reflection of the externality of each creditor's running decision on other creditors and directly drives debt runs in our model.



**Figure 2**  
An illustration of the rat race between creditors in choosing rollover thresholds

### 3. Analyzing the Model

We now analyze several determinants of creditors’ equilibrium rollover threshold. We will first calibrate a set of baseline parameters, and then discuss the effects of the firm’s asset illiquidity, fundamental volatility, credit lines, and debt maturity. While the effect of asset illiquidity is well studied in the existing static bank-run models, we emphasize that the effect of fundamental volatility and its joint effects with credit lines and debt maturity are unique to our dynamic setting.

#### 3.1 Model parameters

We first calibrate the parameters of our model to a typical financial firm during the recent financial crisis. These parameters serve as the baseline for our analysis of determinants of debt runs.

The average one-year Treasury rate during the second half of 2008 (July 2008 to December 2008) is 1.56% according to data from the Federal Reserve statistical releases. Motivated by this fact, we choose investors’ discount rate  $\rho = 1.5\%$ .

During the recent financial crisis, many financial firms faced severe risk originated from their holdings of mortgage-backed securities. While these firms also held other securities, it is reasonable to match the firm asset in our model to the characteristics of mortgage-backed securities. Based on data from the Federal Reserve statistical releases, the average mortgage rate from 1995

to 2008 was 6.83%. Thus, we choose the firm asset's constant cash flow rate  $r$  in our model to be 7%. Furthermore, we use the average duration of 30-year fixed-rate mortgages as the basis to determine the asset's average maturity  $1/\phi$ . It is direct to compute that the duration of a 30-year mortgage with a mortgage rate of 7% and a yield of 7% is 12.91. Thus, we choose  $1/\phi = 13$ , which is equivalent to  $\phi = 0.077$ .

The asset's liquidation recovery rate  $\alpha$  has been widely calibrated in the structural credit risk literature. In a recent study, [Chen \(2010\)](#) calibrates industrial firms' liquidation recovery rate  $\alpha$  across nine different aggregate economic states by matching the Moody's aggregate bond default recovery rate in 1982–2008, which is 41.4%. He finds that the recovery rate is around 60% in most states. According to [Moody \(2009\)](#), the default recovery of bonds issued by financial firms in 2008 averaged 35.4%, slightly below the aggregate bond default recovery rate. Thus, we choose  $\alpha = 55\%$ .

As summarized by [Veronesi and Zingales \(2010\)](#), the average asset volatility of a set of financial firms, including Merrill Lynch, Goldman Sachs, Citibank, and others, in October 2008 is 10.1%. Thus, we choose  $\sigma = 20\%$  and  $y_0 = 1.4$ , which imply that the volatility of the firm's asset in our model, which offers a constant cash flow and a risky final payoff, has a volatility of 11%, close to the value reported by Veronesi and Zingales. As assets held by financial firms are mostly tradable and as our model assumes risk-neutral investors, we choose its fundamental growth rate  $\mu = 1.5\%$ , which is consistent with the risk-free rate.

In reality, financial firms use a host of debt contracts, including public bonds, commercial paper, and repo transactions, to finance their investment positions. Instead of matching the firm's rollover frequency in our model to maturities of all of these contracts, it is perhaps reasonable to focus on a set of contracts, asset-backed commercial paper (ABCP), which was widely used by financial firms to finance their holdings of mortgage-backed securities. According to [Covitz, Liang, and Suarez \(2009\)](#), the average maturity of ABCP in March 2007 is 37 days, which implies a rollover frequency of  $1/(37/365) = 9.86$ . Based on this number, we choose the firm's debt rollover frequency  $\delta = 10$ .

In our model, credit lines allow the firm to survive under runs by creditors for an expected period of  $1/(\theta\delta)$ . According to [Cox \(2008\)](#), Bear Stearns lasted for three days under the runs of its creditors and clients before a forced sale to J. P. Morgan in March 2008. Based on the experience of Bear Stearns, we choose  $\theta\delta = 50$  (i.e.,  $\theta = 5$ ), which implies that the firm can survive under runs for an expected period of about one week.

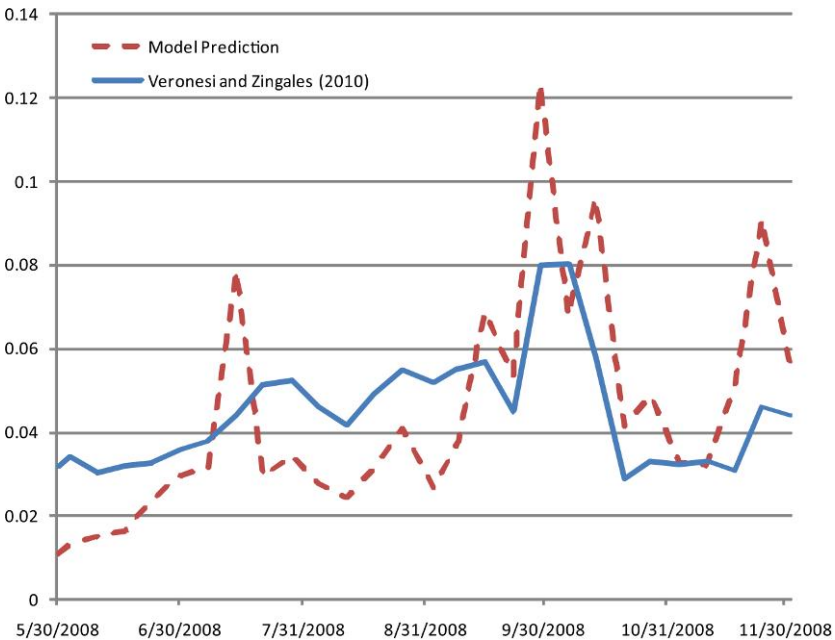
We summarize these parameter values below:

$$\begin{aligned} \rho &= 1.5\%, \quad r = 7\%, \quad \phi = 0.077, \quad \alpha = 55\%, \quad \sigma = 20\%, \quad \mu = 1.5\%, \\ y_0 &= 1.4, \quad \delta = 10, \quad \theta = 5. \end{aligned} \tag{10}$$

We measure creditors' rollover threshold by the firm's fundamental value at  $y_*$ ,  $F(y_*) = \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y_*$ , which is directly comparable to the firm's outstanding liability, 1. Under the baseline parameters given in (10), the equilibrium rollover threshold is  $y_* = 1.025$ , at which the firm's fundamental value is  $F(y_*) = 1.787$ .

To gauge whether the parameters listed in (10) are able to provide reasonable quantitative assessments of a firm's debt run risk, we use the model to predict the default probability of Merrill Lynch during the recent financial crisis. As is well known, Merrill Lynch is a major financial firm and faced severe runs by its creditors during the crisis. The solid line in Figure 3 depicts the one-year default probability of Merrill Lynch constructed by Veronesi and Zingales (2010) based on the market prices of credit default swaps written on Merrill Lynch's public bonds from May 30, 2008, to November 30, 2008. This probability spiked at the end of September 2008, when the failure of Lehman Brothers intensified runs experienced by many other financial firms, including Merrill Lynch.

Given the creditors' equilibrium rollover threshold, it is direct to compute the model implied one-year default probability. We need an estimate of the



**Figure 3**  
**One-year default probability of Merrill Lynch during the recent financial crisis**

The solid line depicts Merrill Lynch's one-year probability constructed by Veronesi and Zingales (2010) based on the market prices of credit default swap written on Merrill Lynch's public bonds. The dashed line depicts default probability predicted by our model based on parameters listed in (10) and the firm's fundamental inferred from its equity price.

firm's time-varying asset fundamental, which we extract from the firm's equity price based on the model-implied mapping between a firm's fundamental value and equity value. We then compute the model implied one-year default probability and plot it as the dashed line in Figure 3. We acknowledge several caveats in this simplistic exercise. First, while we calibrate the firm's current fundamental to its equity price, the model parameters given in (10) are taken from a typical financial firm rather than Merrill Lynch. Second, we ignore fluctuations of market liquidity and fundamental volatility in driving Merrill Lynch's default risk during the crisis. Despite these limitations, Figure 3 shows that the model-predicted default probability matches reasonably well with the CDS-implied probability. The model-predicted probability has a sample mean of 4.4% and a standard deviation of 2.7% during the sample period, while the CDS-implied probability has a mean of 4.5% and a standard deviation of 1.4%. The two series have a correlation of 0.64. Overall, the model predicted probability captures the ballpark of the CDS implied one, albeit more volatile.

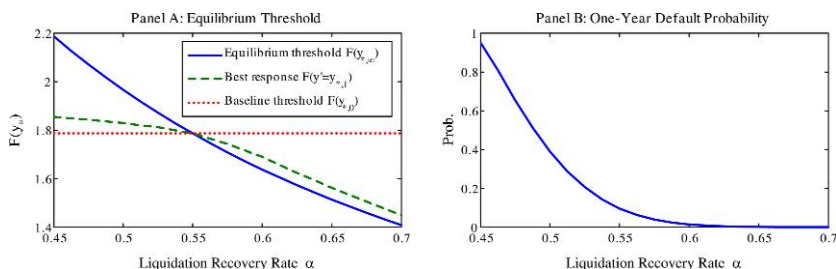
### 3.2 Asset illiquidity

We now examine determinants of creditors' equilibrium rollover threshold. We start with asset illiquidity, which in our model is inversely determined by the firm asset's premature liquidation recovery rate  $\alpha$ . We analyze the effect of  $\alpha$  by varying its value from the baseline level. One can interpret the change as an unexpected shock to market liquidity. In this analysis, we take the firm's debt maturity as given rather than letting it vary with  $\alpha$ . This is because it is difficult to quickly adjust a firm's debt structure in response to a random shock. We will also use a similar strategy to analyze the effect of fundamental volatility.

The prior literature has highlighted the important role of asset illiquidity. In the model of Diamond and Dybvig (1983), concerns about inability of a bank's liquidation value to cover its liability lead to self-fulfilling runs. In the global-games model presented by Rochet and Vives (2004), each creditor's run threshold increases with the bank's asset illiquidity. We can also derive a similar result, as in Proposition 2:

**Proposition 2.** The equilibrium rollover threshold  $y_*$  decreases with the firm's premature liquidation recovery rate  $\alpha$ .

Figure 4 depicts the equilibrium rollover threshold  $F(y_*)$  and the firm's one-year default probability with respect to  $\alpha$  by fixing other model parameters in (10). Following the discussion in Section 2.3, the flat dotted line in Panel A depicts the equilibrium threshold  $F(y_{*,0}) = 1.787$  when  $\alpha$  takes the baseline value 0.55. The thick solid line shows that as  $\alpha$  deviates from the baseline value and decreases from 0.7 to 0.45, the equilibrium threshold  $F(y_{*,\infty})$  rises



**Figure 4**  
**The effect of liquidation recovery rate  $\alpha$**   
 This figure uses the baseline parameters in (10). In Panel A, the rollover threshold is measured in the firm’s fundamental value  $F(y_*)$ . The dotted line is the baseline threshold level,  $F(y_{*,0})$ . The solid line plots the equilibrium threshold  $F(y_{*,\infty})$ , as  $\alpha$  deviates from its baseline value. The dashed line plots a creditor’s best response  $F(y_{*,1})$  to the change in  $\alpha$  while fixing other creditors’ threshold at  $F(y_{*,0})$ . Panel B depicts the firm’s one-year default probability with respect to  $\alpha$ .

monotonically from 1.41 to 2.19.<sup>17</sup> The dashed line is the best response of a creditor to the change in  $\alpha$  in the absence of the rat race with other creditors. Suppose that  $\alpha$  drops unexpectedly from its baseline level 0.55 to 0.5. By solving the HJB Equation in (9) numerically, we find that a creditor will choose an optimal threshold  $F(y_{*,1}) = 1.83$  (on the dashed line) if the other creditors’ rollover threshold is fixed at the baseline level  $F(y_{*,0}) = 1.787$  (the thin solid line). The difference  $F(y_{*,1}) - F(y_{*,0}) = 0.043$  represents the safety margin necessary to compensate the creditor for the increased expected bankruptcy loss in the absence of the rat race. Of course, once we take into account the increased threshold of other creditors, each creditor ends up choosing a higher equilibrium threshold of  $F(y_{*,\infty}) = 1.966$  (on the thick solid line). The difference  $F(y_{*,\infty}) - F(y_{*,1})$  represents the amplification effect of the rat race, which is about four times the effect without the rat race. This decomposition shows that the rat race can dramatically amplify the effect of asset illiquidity.

Panel B of Figure 4 further depicts the firm’s one-year default probability based on the derived equilibrium rollover threshold. As  $\alpha$  drops from 0.55 to 0.5, the default probability rises substantially from 9.65% to 39.03%. Taken together, Figure 4 shows that a small change in asset illiquidity can have a large effect on the firm’s financial stability.

### 3.3 Fundamental volatility

Time-varying fundamentals are an important characteristic of financial firms’ asset holdings. By construction, static models are not particularly suited for analyzing effects of time-varying fundamentals. As discussed in the introduction

<sup>17</sup> Note that  $F(y_{*,\infty})$  is always above 1. As each maturing creditor only holds a partial stake in the firm, it makes sense for him to run and get his money back before the firm’s fundamental value drops below the outstanding liability. This is because he does not internalize the cost of his run on the whole firm.



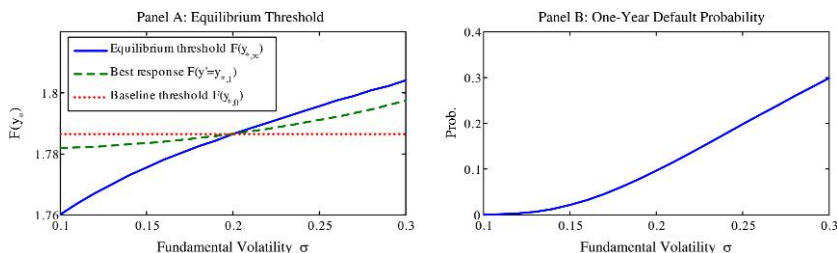
(Footnote 7), the effect of fundamental volatility in our dynamic model is not equivalent to the effect of fundamental uncertainty in the static global-games models, because the latter effect may increase or decrease agents' threshold strategy depending on whether the threshold is above or below their public information. Furthermore, the complication of dealing with dynamic learning in the presence of noisy private information also makes it challenging to extend these models to incorporate time-varying fundamentals (Footnote 8).

In our model, fundamental volatility  $\sigma$  affects a creditor's optimal rollover threshold through several channels. We can intuitively discuss these channels through various terms in the creditor's value function in Equation (8). First, when the firm's fundamental volatility increases, its insolvency risk, which is reflected by the term  $\min(1, y_\tau) \mathbf{1}_{\{\tau=\tau_\phi\}}$ , rises because it becomes more likely that the firm's asset value at the asset maturity is insufficient to pay off its liability. The increased insolvency risk prompts each creditor to use a higher rollover threshold. Second, a higher volatility also increases the firm's rollover risk through the term  $\min(1, L + ly_\tau) \mathbf{1}_{\{\tau=\tau_\theta\}}$  (i.e., other creditors might choose to run and cause the firm to fail before the creditor's debt matures), and thus also motivates the creditor to use a higher threshold. The increased rollover risk essentially represents greater strategic uncertainty about other creditors' rollover decisions. Third, once the creditor's debt matures, he has the option to roll it over and take advantage of the debt's high interest payments if the firm's fundamental is sufficiently strong. Through this embedded option, which is reflected by the term  $\max_{\text{rollover or run}} \{V(y_\tau; y_*), 1\} \mathbf{1}_{\{\tau=\tau_\delta\}}$ , a higher fundamental volatility motivates the creditor to choose a lower rollover threshold. The effect of the embedded option works in an opposite direction to those of the insolvency risk and rollover risk.

Figure 5 illustrates the net effect of these three channels. In Panel A, as  $\sigma$  deviates from its baseline value of 20% and increases from 10% to 30%, the equilibrium rollover threshold  $F(y_*)$  (the solid line) increases from 1.76 to 1.804. We can formally prove in Proposition 3 that the equilibrium threshold increases with  $\sigma$  if the firm's credit lines are sufficiently unreliable, i.e.,  $\theta$  is sufficiently high. Under this condition, the firm would easily fail under a run, and consequently the embedded-option channel becomes dominated by the other two channels. In fact, our numerical exercises show that this result also holds when  $\theta$  takes a modest value.

**Proposition 3.** Suppose that  $\theta$  is sufficiently high. Then, the equilibrium rollover threshold  $y_*$  increases with the firm's fundamental volatility  $\sigma$ .

Panel A of Figure 5 also depicts a creditor's best response  $F(y_{*,1})$  to the change in  $\sigma$  (the dashed line) while fixing other creditors' threshold at the baseline level  $F(y_{*,0}) = 1.787$  when  $\sigma$  takes its baseline level 20%. When  $\sigma$  rises above the baseline value, the increase  $F(y_{*,1}) - F(y_{*,0})$  represents the safety margin that the creditor would demand to protect himself against



**Figure 5**

**The effect of fundamental volatility  $\sigma$**

This figure uses the baseline parameters in (10). Panel A depicts creditors’ equilibrium rollover threshold, measured in the firm’s fundamental value  $F(y_*)$ . The dotted line is the baseline threshold level  $F(y_{*,0})$ , the solid line is the equilibrium threshold  $F(y_{*,\infty})$  as  $\sigma$  deviates from its baseline value, and the dashed line is a creditor’s best response  $F(y_{*,1})$  to the change in  $\sigma$  while fixing other creditors’ threshold at  $F(y_{*,0})$ . Panel B depicts the firm’s one-year default probability based on the equilibrium rollover threshold.

the increased rollover risk in the absence of the rat race among creditors. As  $\sigma$  varies from 10% to 30%,  $F(y_{*,1})$  increases from 1.782 to 1.797. The wider range of the solid line for the equilibrium threshold  $F(y_{*,\infty})$  illustrates a substantial amplification effect of the rat race.

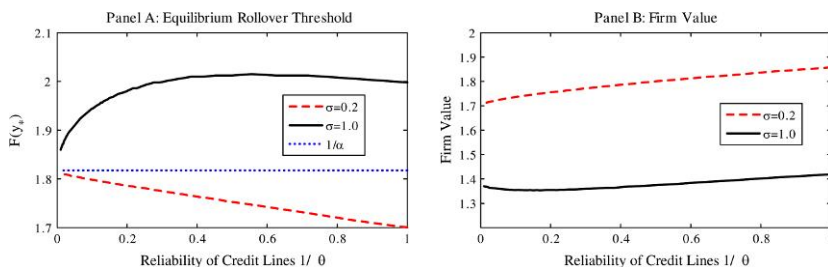
Panel B of Figure 5 also depicts the firm’s one-year default probability against  $\sigma$ . As  $\sigma$  increases from 10% to 30%, the default probability increases from a level near zero to 30%. Overall, Figure 5 demonstrates that an increase in fundamental volatility can significantly increase creditors’ equilibrium rollover threshold and default probability.

**3.4 Credit lines**

In practice, financial firms usually hold (explicit and implicit) credit lines from other financial institutions and the government, which can temporarily sustain them under runs. A common intuition is that the stronger a firm’s credit lines, the less likely creditors will choose to run on the firm. In other words, the creditors’ equilibrium rollover threshold should decrease with the reliability of the firm’s credit lines.

Figure 6 illustrates the equilibrium effects of credit lines. Panel A depicts creditors’ equilibrium rollover threshold with respect to the reliability of the firm’s credit lines  $\frac{1}{\theta}$ . The dotted flat line gives a benchmark level  $F(y_h) = \frac{1}{\alpha}$ . If the firm fails at this fundamental level, its liquidation value is exactly sufficient to pay off the debt holders. The dashed and solid lines correspond to the equilibrium threshold for two different levels of fundamental volatility  $\sigma = 0.2$  and 1.0. If the firm does not have any credit line (i.e.,  $\frac{1}{\theta} \rightarrow 0$ ), the equilibrium threshold collapses to the benchmark level  $F(y_h) = \frac{1}{\alpha}$  regardless of the fundamental volatility.

When fundamental volatility is at the baseline level ( $\sigma = 0.2$ ), the dashed line in Figure 6 shows that the equilibrium threshold decreases with the reliability of credit lines. This is consistent with the aforementioned



**Figure 6**  
**Equilibrium effects of credit lines**

This figure uses the baseline parameters in (10). In Panel A, the creditors’ rollover threshold is measured in the firm’s fundamental value  $F(y_*)$ . The solid and dashed lines plot the equilibrium threshold with respect to the quality of the firm’s credit lines  $1/\theta$  when its fundamental volatility  $\sigma = 1.0$  and  $0.2$ , while the dotted line gives the benchmark level  $F(y_*) = 1/\alpha$ . In Panel B, the solid and dashed lines plot the firm value when  $\sigma = 1.0$  and  $0.2$ , conditional on the creditors’ equilibrium rollover threshold.

common intuition that credit lines deter runs. However, when volatility is high ( $\sigma = 1.0$ ), the solid line is non-monotonic—the rollover threshold initially increases with the reliability of credit lines and then decreases with it. The increasing segment of this relationship is intriguing as it indicates that stronger credit lines can prompt creditors to run earlier.

What drives this intriguing outcome? It is directly related to the firm’s time-varying fundamental. On the one hand, stronger credit lines allow the firm to survive longer under creditors’ runs, which mitigates their incentives to run. On the other hand, the longer the firm survives, the more likely for the firm’s fundamental to deteriorate further below the creditors’ rollover threshold. As a result, the firm’s liquidation value can be substantially lower than what is implied by the threshold level when the firm eventually fails. This volatility effect motivates each creditor to choose a higher rollover threshold. This effect is particularly severe when the firm’s fundamental volatility is high, and can dominate the former survival probability effect. In this case, stronger credit lines prompt creditors to run earlier. Also note that while the volatility level  $\sigma = 1.0$  is high, this level is realistic during crisis periods. For example, the VIX volatility index for the U.S. stock market had reached over 80% during the recent financial crisis.

We can formally derive Proposition 4.

**Proposition 4.** If both  $\sigma$  and  $\theta$  are sufficiently large, the equilibrium rollover threshold  $y_*$  increases with  $1/\theta$ .

Panel B of Figure 6 further depicts the firm’s value with respect to the reliability of credit lines. The firm’s value accounts for the firm’s default probability throughout its life and is calculated by expectation of its future discounted cash flows based on creditors’ equilibrium rollover threshold. The plot shows that when fundamental volatility is high ( $\sigma = 1.0$ ) and the

reliability of credit lines is poor, the firm's value can decrease with the reliability of the credit lines. This pattern shows that the adverse effect of credit lines can significantly affect the firm's value.

In summary, Figure 6 and Proposition 4 demonstrate a delicate effect of credit lines on the firm's financial stability—temporarily keeping it alive under runs can exacerbate incentives to run when fundamental volatility is high. This result is not only directly testable but also offers important policy implications. One can broadly interpret credit lines in our model as measures taken by central banks to temporarily bail out financially distressed financial firms. As political pressure prevents central banks from maintaining their bailout effort for the long term, the short-term nature of any bailout program closely resembles the uncertain credit lines specified in our model. While many policymakers and pundits have advocated for various government bailout programs during the crisis, our model suggests that when fundamental volatility is sufficiently high, a temporary bailout program can be counterproductive as it can exacerbate rather than mitigate creditors' incentives to run.

**3.4.1 Frantic runs.** Figure 6 also illustrates another interesting phenomenon that when fundamental volatility is sufficiently high, creditors choose to run even when the firm fundamental is higher than  $1/\alpha$ . At this level the firm is so well capitalized that it can pay back its liability even after a forced liquidation. We call this phenomenon frantic runs. Recall the synchronous-rollover benchmark considered in Proposition 1. This static benchmark setting closely resembles the Diamond-Dybvig model in that all creditors make simultaneous rollover decisions and face no future rollover risk. Proposition 1 shows that in this static setting runs occur only when the firm fundamental is below  $1/\alpha$ , because in the absence of future rollover risk, this strong fundamental is sufficient to preclude any creditor's concern about other creditors' concurrent rollover decisions. The emergence of frantic runs in our dynamic setting highlights the severe effect of dynamic runs. Despite the firm's strong current fundamental, concerns about the firm's future rollover risk can nevertheless lead creditors to run.<sup>18</sup>

### 3.5 Debt maturity

The heavy use of short-term debt by financial firms is widely regarded as a key source of instability during the recent financial crisis (e.g., Brunnermeier 2009;

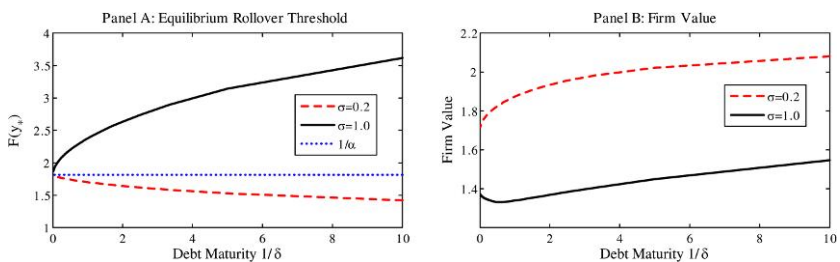
<sup>18</sup> One might argue that as fundamental volatility becomes large, the firm's insolvency risk also rises. To further highlight that the frantic runs are not simply driven by insolvency risk, we have also considered an otherwise identical firm financed by a single large creditor who holds all the debt of the firm and faces no coordination problem with other creditors. We have characterized the large creditor's rollover decision in this benchmark setting in the NBER working paper version of this paper. Under reasonable conditions, the large creditor will always roll over his debt as he internalizes the firm's liquidation cost. Suppose that a small creditor also holds a negligible fraction of the firm's debt along with the large creditor. The small creditor's rollover threshold choice is only affected by the firm's insolvency risk and his embedded option in the firm. Our unreported analysis shows that the firm's insolvency risk does not lead the creditor to run under the parameters used in our illustration.

Krishnamurthy 2010). Short-term debt exposed these firms to rollover risk and contributed to their financial distresses during the crisis. A common argument is that increasing debt maturity can reduce these firms' rollover risk and thus mitigate creditors' incentives to run. Interestingly, our model shows that while this argument holds true in normal periods when volatility is modest, it fails when volatility is sufficiently high.

Consider a reduction in the firm's rollover frequency  $\delta$ , which implies a longer expected debt maturity  $1/\delta$ . From the perspective of a creditor, this can lead to two offsetting effects. On the one hand, the firm needs to roll over its debt at a lower frequency with other creditors and, as a result, when its fundamental drops below their rollover threshold, it is less likely to fail. The reduced rollover pressure works to stabilize the firm, and exactly captures the aforementioned common argument. On the other hand, each creditor also faces a longer lock-in period, during which the firm's fundamental can drop below other creditors' rollover threshold. This internal lock-in effect can motivate the creditor to run and thus destabilizes the firm. The trade-off between these two effects determines whether each creditor increases or decreases his rollover threshold in response to the increased debt maturity.

Figure 7 illustrates the equilibrium effects of varying expected debt maturity  $1/\delta$  from 0 to 10 for two different values of fundamental volatility,  $\sigma = 0.2$  (baseline value) and 1.0. When  $\sigma = 0.2$ , Panel A shows that the equilibrium rollover threshold monotonically decreases with  $1/\delta$ , and Panel B shows that the firm value increases with  $1/\delta$ . These patterns suggest that the reduced rollover pressure effect dominates the internal lock-in effect.

However, when  $\sigma = 1.0$ , Panel A in Figure 7 shows that the equilibrium rollover threshold increases with  $1/\delta$ . This pattern indicates that the internal lock-in effect dominates the reduced rollover pressure effect. Relative to the baseline volatility case, the high volatility exacerbates the effect of a longer



**Figure 7**  
**Equilibrium effects of debt maturity**

This figure uses the baseline parameters in (10). In Panel A, the creditors' rollover threshold is measured in the firm's fundamental value  $F(y_*)$ . The solid and dashed lines plot the equilibrium threshold with respect to the expected debt maturity  $1/\delta$  when its fundamental volatility is  $\sigma = 1.0$  and 0.2, while the dotted line gives the benchmark level  $F(y_*) = 1/\alpha$ . In Panel B, the solid and dashed lines plot the firm value when  $\sigma = 1.0$  and 0.2, conditional on the creditors' equilibrium rollover threshold.

debt maturity on the probability of the firm fundamental dropping below other creditors' rollover threshold during the lock-in period. As the high volatility has no effect on the firm's rollover pressure conditional on its fundamental being below other creditors' rollover threshold, the increased internal lock-in effect can dominate the reduced rollover pressure effect if volatility is sufficiently high. In this case, each creditor chooses a higher rollover threshold in response to a longer debt maturity. Panel B further shows that when  $\sigma = 1.0$ , the firm value initially decreases and then increases with  $1/\delta$ . This non-monotonic pattern again reflects the two offsetting effects of a longer debt maturity.

Formally, we can derive Proposition 5.

**Proposition 5.** If both  $\sigma$  and  $\delta$  are sufficiently large, the equilibrium rollover threshold  $y_*$  increases with expected debt maturity  $1/\delta$ .

Taken together, Figure 7 and Proposition 5 demonstrate that while increasing debt maturity mitigates runs during normal periods when volatility is modest, a longer debt maturity can exacerbate creditors' incentives to run and thus destabilize the firm when volatility is sufficiently high. This surprising result is actually consistent with the recent runs on ABCP in 2007. As documented by Covitz, Liang, and Suarez (2009), some ABCP programs have the option to extend the term of their commercial paper, which effectively gives them longer debt maturity. Interestingly, during the summer of 2007, these programs experienced significantly more severe runs by creditors after controlling for other program characteristics. Given the pervasive presence of high volatility during financial crises, this result cautions against the widely advocated policy of requiring firms to use long-term debt.

## 4. Further Discussion

Our model provides a useful framework to examine various issues related to stability of financial firms and systemic risk in the financial system. We will discuss several of them in this section.

### 4.1 Synchronous vs. asynchronous debt structure

It is a common practice for firms to spread out their debt expirations across time to reduce the liquidity risk of having to roll over large quantities of debt at the same time. This practice avoids the simultaneous coordination problem among creditors, but introduces another dynamic coordination problem. Is this structure superior to a synchronous structure with all debt maturing at the same time? Formally analyzing this issue requires a more elaborate framework to incorporate both synchronous and asynchronous debt structures. While developing such a structure is beyond the scope of this article, our model can nevertheless shed some light on this issue.

As we discussed before, with an asynchronous debt structure frantic runs can occur when fundamental volatility is sufficiently high. In such a case, a strong firm fundamental of  $1/\alpha$  is insufficient to preclude creditors from running, even though it is sufficient in static settings in which creditors make synchronous rollover decisions and face no future rollover risk. This contrast indicates that the asynchronous debt structure can exacerbate the coordination problem among creditors, especially when fundamental volatility is high. Thus, the common practice of spreading out debt expirations is not always superior to the alternative of having all debt maturing at the same time. We will leave more thorough analysis of the firm's optimal debt structure to future research. Many important questions remain. Should firms with higher fundamental volatility choose a synchronous debt structure? Or a combination of staggered short-term debt and synchronous long-term debt?

#### 4.2 Optimal debt maturity

A common argument advocating the use of short-term debt is that it can discipline managers from risk-shifting (e.g., Calomiris and Kahn 1991). However, short-term debt also exposes firms to rollover risk. Cheng and Milbradt (2012) recently extend our model to analyze the trade-off between incentive provision and rollover risk for a firm financed by short-term debt. Specifically, they allow the firm manager to freely switch between two projects: one with high growth rate and low volatility and the other with low growth rate and high volatility. As the first project strictly dominates the latter, investing in the latter is considered risk-shifting. Cheng and Milbradt find that the firm's optimal debt maturity takes an interior solution that avoids excessive rollover risk while providing sufficient incentives for the manager to avoid risk-shifting when the firm's fundamental is healthy. In particular, they illustrate a subtle interaction between risk-shifting and debt runs. While risk-shifting is clearly undesirable to the firm when the firm's fundamental is healthy, it can mitigate creditors' incentives to run once the fundamental drops below creditors' rollover threshold.

#### 4.3 Spillover and systemic risk

One can readily extend our model to include multiple firms holding similar assets to analyze systemic risk triggered by creditors' debt runs on one firm. As these firms face the same downward-sloping demand curve for their assets in an illiquid secondary market, the liquidation recovery rate  $\alpha$  of each firm depends on other firms' liquidation (e.g., Shleifer and Vishny 1992). Suppose that one firm, say Bear Stearns, suffers idiosyncratic negative shocks to its fundamental. As a result, when this firm experiences runs by its creditors and needs to liquidate its asset, the liquidation potentially pushes down the liquidation values of other firms. This in turn increases the losses of other firms' creditors in the event that their firms are forced into liquidation. Thus, through this liquidation-value channel, debt runs spill over to these



firms as their creditors now have greater incentives to run, even if there is no fundamental deterioration in these firms. The possibility of other firms experiencing runs also feeds back to the creditors of the initial firm in distress, creating even greater incentives to run. In this way, a rat race to exit risky debt is underway not just between creditors of one firm, but also between creditors of all firms holding similar assets. Thus, market liquidity evaporates and systemic risk becomes imminent. The presence of such systemic risk implies that when choosing its optimal debt structure and debt maturity each firm needs to not only consider its own characteristics (such as fundamental volatility and asset illiquidity) but also peer characteristics. This is another important direction for future research.

## 5. Conclusion

In this article, we develop a dynamic model of debt runs by creditors on a firm, which invests in an illiquid asset by rolling over staggered short-term debt contracts. Our model highlights the dynamic coordination between creditors who make rollover decisions at different times, and shows that fear of the firm’s future rollover risk could motivate each creditor to preemptively run ahead of others even when the firm is still fundamentally healthy. Our model allows us to characterize the roles of deteriorating fundamentals, asset illiquidity, fundamental volatility, reliability of credit lines, and debt maturity in driving such dynamic runs. In particular, when fundamental volatility is sufficiently high, commonly argued measures such as temporarily keeping the firm alive under runs and increasing debt maturity can exacerbate rather than mitigate runs.

## Appendix

### A.1 Proof of Theorem 1

Using the HJB Equation in (9), we first construct a creditor’s value function by utilizing the fact that in any symmetric equilibrium all creditors (including this creditor) use the same monotone strategy with threshold  $y_*$ . The equilibrium threshold must then be the solution to the equation  $V(y_*, y_*) = 1$ . Of course, individual optimality requires that  $V(y; y_*) > 1$  for  $y > y_*$  and  $V(y; y_*) < 1$  for  $y < y_*$ , a condition that we will verify in Lemma 3. Lemma 4 shows that there does not exist any asymmetric threshold equilibrium.

Depending on whether  $y$  is higher or lower than  $y_*$ , we can derive the HJB Equation (9) in the following two cases:

- If  $y < y_*$ ,

$$\begin{aligned}
 0 = & \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - [\rho + \phi + (\theta + 1) \delta] V(y; y_*) + \phi \min(1, y) \\
 & + \theta \delta \min(L + ly, 1) + r + \delta;
 \end{aligned}
 \tag{A1}$$

- If  $y \geq y_*$ ,

$$0 = \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - (\rho + \phi) V(y; y_*) + \phi \min(1, y) + r. \tag{A2}$$

The value function has to satisfy these two differential equations and be continuous and differentiable at the boundary point  $y_*$ . In solving these differential equations, we need to introduce the two roots to the first fundamental equation for (A1):

$$\frac{1}{2} \sigma^2 x(x - 1) + \mu x - [\rho + \phi + (1 + \theta) \delta] = 0, \tag{A3}$$

which are

$$-\gamma_1 = -\frac{\mu - \frac{1}{2} \sigma^2 + \sqrt{\left(\frac{1}{2} \sigma^2 - \mu\right)^2 + 2 \sigma^2 [\rho + \phi + (1 + \theta) \delta]}}{\sigma^2} < 0, \tag{A4}$$

and

$$\eta_1 = -\frac{\mu - \frac{1}{2} \sigma^2 - \sqrt{\left(\frac{1}{2} \sigma^2 - \mu\right)^2 + 2 \sigma^2 [\rho + \phi + (1 + \theta) \delta]}}{\sigma^2} > 1; \tag{A5}$$

and the two roots to the second fundamental equation for (A2):

$$\frac{1}{2} \sigma^2 x(x - 1) + \mu x - (\rho + \phi) = 0, \tag{A6}$$

which are

$$-\gamma_2 = -\frac{\mu - \frac{1}{2} \sigma^2 + \sqrt{\left(\frac{1}{2} \sigma^2 - \mu\right)^2 + 2 \sigma^2 (\rho + \phi)}}{\sigma^2} < 0, \tag{A7}$$

and

$$\eta_2 = -\frac{\mu - \frac{1}{2} \sigma^2 - \sqrt{\left(\frac{1}{2} \sigma^2 - \mu\right)^2 + 2 \sigma^2 (\rho + \phi)}}{\sigma^2} > 1. \tag{A8}$$

We summarize the constructed value function in Lemma 1.

**Lemma 1.** Given the equilibrium rollover threshold  $y_*$ , the value function of a creditor is given by the following three cases:

1. If  $y_* < 1$ ,

$$V(y; y_*) = \begin{cases} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\phi + \theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + A_1 y^{\eta_1} & \text{when } 0 < y \leq y_* \\ \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y + A_2 y^{-\gamma_2} + A_3 y^{\eta_2} & \text{when } y_* < y \leq 1 \\ \frac{r + \phi}{\rho + \phi} + A_4 y^{-\gamma_2} & \text{when } 1 < y. \end{cases} \tag{A9}$$

The four coefficients  $A_1, A_2, A_3,$  and  $A_4$  are given by

$$\begin{aligned}
 A_1 &= \frac{[H_3\gamma_2 + H_1] - y_*^{-\eta_2} (\gamma_2 H_4 + H_2 y_*)}{(\eta_1 + \gamma_2) y_*^{\eta_1 - \eta_2}}, \\
 A_2 &= \frac{y_*^{\gamma_2}}{\eta_2 + \gamma_2} \left[ \eta_2 H_4 - H_2 y_* + A_1 (\eta_2 - \eta_1) y_*^{\eta_1} \right], \\
 A_3 &= \frac{y_*^{-\eta_2}}{\eta_2 + \gamma_2} \left[ \gamma_2 H_4 + H_2 y_* + A_1 (\eta_1 + \gamma_2) y_*^{\eta_1} \right], \\
 &= \frac{1}{\eta_2 + \gamma_2} [H_3\gamma_2 + H_1], \\
 A_4 &= A_2 - \frac{1}{\eta_2 + \gamma_2} [H_3\eta_2 - H_1],
 \end{aligned}$$

where

$$\begin{aligned}
 H_1 &= -\frac{\phi}{\rho + \phi - \mu}, \\
 H_2 &= \frac{\theta \delta l (\rho + \phi - \mu) - \phi (1 + \theta) \delta}{(\rho + \phi + (1 + \theta) \delta - \mu) (\rho + \phi - \mu)}, \\
 H_3 &= -\frac{\phi \mu}{(\rho + \phi) (\rho + \phi - \mu)}, \\
 H_4 &= \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} - \frac{r}{\rho + \phi} + H_2 y_*.
 \end{aligned}$$

2. If  $1 < y_* \leq \frac{1-L}{I}$ ,

$$\begin{aligned}
 &V(y; y_*) \\
 &= \begin{cases} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\phi + \theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + B_1 y^{\eta_1} & \text{when } y \leq 1, \\ \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + B_2 y^{-\gamma_1} + B_3 y^{\eta_1} & \text{when } 1 < y \leq y_*, \\ \frac{r + \phi}{\rho + \phi} + B_4 y^{-\gamma_2} & \text{when } y_* < y. \end{cases} \tag{A10}
 \end{aligned}$$

The four coefficients  $B_1, B_2, B_3,$  and  $B_4$  are given by

$$\begin{aligned}
 B_1 &= B_3 - \frac{M_2 \gamma_1 + M_1}{\eta_1 + \gamma_1}, \\
 B_2 &= \frac{M_2 \eta_1 - M_1}{\eta_1 + \gamma_1} < 0, \\
 B_3 &= \frac{(\gamma_1 - \gamma_2) B_2 (y_*)^{-\gamma_1} + \gamma_2 M_3 - \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*}{(\eta_1 + \gamma_2) y_*^{\eta_1}}, \\
 B_4 &= \frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} B_2 y_*^{\gamma_2 - \gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*^{\gamma_2 + 1},
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{\eta_1}{\eta_1 + \gamma_2} \left[ \frac{r + \phi}{\rho + \phi} - \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} \right] y_*^{\gamma_2}, \\
 & = \frac{(\eta_1 + \gamma_1) B_2 y_*^{-\gamma_1} - \eta_1 M_3 - \frac{\theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} y_*}{(\eta_1 + \gamma_2) y_*^{-\gamma_2}},
 \end{aligned}$$

where

$$\begin{aligned}
 M_1 &= \frac{\phi}{\rho + \phi + (1 + \theta)\delta - \mu}, \\
 M_2 &= \frac{\phi\mu}{(\rho + \phi + (1 + \theta)\delta)(\rho + \phi + (1 + \theta)\delta - \mu)}, \\
 M_3 &= \frac{r + \phi}{\rho + \phi} - \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} - \frac{\theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} y_*.
 \end{aligned}$$

3. If  $y_* > \frac{1-L}{l}$ ,

$$\begin{aligned}
 & V(y; y_*) \\
 & = \begin{cases} \frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} + \frac{\phi + \theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} y + C_1 y^{\eta_1} & \text{when } y \leq 1, \\ \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} + \frac{\theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} y \\ \quad + C_2 y^{-\gamma_1} + C_3 y^{\eta_1} & \text{when } 1 < y \leq \frac{1-L}{l}, \\ \frac{r + \phi + \theta\delta + \delta}{\rho + \phi + (1 + \theta)\delta} + C_4 y^{-\gamma_1} + C_5 y^{\eta_1} & \text{when } \frac{1-L}{l} < y \leq y_*, \\ \frac{r + \phi}{\rho + \phi} + C_6 y^{-\gamma_2} & \text{when } y > y_*. \end{cases} \quad (A11)
 \end{aligned}$$

The six coefficients  $C_1, C_2, C_3, C_4, C_5,$  and  $C_6$  are given by

$$\begin{aligned}
 C_1 &= C_3 - \frac{K_4 \gamma_1 + K_5}{\eta_1 + \gamma_1}, \\
 C_2 &= \frac{K_4 \eta_1 - K_5}{\eta_1 + \gamma_1}, \\
 C_3 &= C_5 + \frac{K_2 \gamma_1 - K_3 \frac{1-L}{l}}{(\eta_1 + \gamma_1) \left(\frac{1-L}{l}\right)^{\eta_1}}, \\
 C_4 &= C_2 - \frac{K_2 \eta_1 + K_3 \frac{1-L}{l}}{(\eta_1 + \gamma_1) \left(\frac{1-L}{l}\right)^{-\gamma_1}}, \\
 C_5 &= \frac{(\gamma_1 - \gamma_2) C_4 y_*^{-\gamma_1} - \gamma_2 K_1}{(\eta_1 + \gamma_2) y_*^{\eta_1}}, \\
 C_6 &= \frac{(\eta_1 + \gamma_1) C_4 y_*^{-\gamma_1} + \eta_1 K_1}{(\eta_1 + \gamma_2) y_*^{-\gamma_2}},
 \end{aligned}$$

where

$$K_1 = \frac{r + \phi + \theta\delta + \delta}{\rho + \phi + (1 + \theta)\delta} - \frac{r + \phi}{\rho + \phi},$$

$$K_2 = \frac{\theta\delta(1 - L)}{\rho + \phi + (1 + \theta)\delta} - \frac{\theta\delta(1 - L)}{\rho + \phi + (1 + \theta)\delta - \mu},$$

$$K_3 = \frac{\theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu},$$

$$K_4 = \frac{\phi}{\rho + \phi + (1 + \theta)\delta - \mu} - \frac{\phi}{\rho + \phi + (1 + \theta)\delta},$$

$$K_5 = \frac{\phi}{\rho + \phi + (1 + \theta)\delta - \mu}.$$

*Proof.* We can derive the three cases listed above using the same method. For illustration we just solve the first case with  $y_* < 1$ . Depending on the value of  $y$ , we have the following three scenarios:

1. If  $0 < y \leq y_*$ :

$$\frac{\sigma^2}{2}y^2V_{yy} + \mu yV_y - [\rho + \phi + (1 + \theta)\delta]V(y) + (\phi + \theta\delta l)y + r + \theta\delta l = 0.$$

The general solution of this differential equation is given in the first line of Equation (A9) with the coefficient  $A_1$  to be determined by the boundary conditions. Note that to ensure the value of  $V$  is finite as  $y$  approaches zero, we have ruled out another power solution  $y^{-\gamma_1}$  of the equation.

2. If  $y_* < y \leq 1$ :

$$\frac{\sigma^2}{2}y^2V_{yy} + \mu yV_y - (\rho + \phi)V(y) + \phi y + r = 0.$$

The general solution of this differential equation is given in the second line of Equation (A9) with the coefficients  $A_2$  and  $A_3$  to be determined by the boundary conditions.

3. If  $y > 1$ :

$$\frac{\sigma^2}{2}y^2V_{yy} + \mu yV_y - (\rho + \phi)V(y) + r + \phi = 0.$$

The general solution of this differential equation is given in the third line of Equation (A9) with the coefficient  $A_4$  to be determined by the boundary conditions. Note that to ensure the value of  $V$  is finite as  $y$  approaches infinity, we have ruled out another power solution  $y^{\eta_2}$  of the equation.

To determine the four coefficients,  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_4$ , we have four boundary conditions at  $y = y_*$  and 1, i.e., the value function  $V(y)$  must be continuous (value-matching) and differentiable (smooth-pasting) at these two points. Solving these boundary conditions leads to the coefficients given in Lemma 1. ■

Based on the value function derived in Lemma 1, we now show that there exists a unique threshold  $y_*$  that satisfies the equilibrium condition.

**Lemma 2.** There exists a unique  $y_*$  such that

$$V(y_*; y_*) = 1.$$

*Proof.* Define

$$W(y) \equiv V(y; y).$$

We need to show that there is a unique  $y_*$  such that  $W(y_*) = 1$ .

We first show that  $W(y)$  is monotonically increasing when  $y < 1$ . In this case, we can directly extract the value of  $W(y)$  from Equation (A9), which, by neglecting terms independent of  $y$ , is

$$W(y) = \left[ -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 \right] y + \frac{[H_3 \gamma_2 + H_1]}{\eta_1 + \gamma_2} y^{\eta_2}.$$

Note that

$$\begin{aligned} \frac{dW(y)}{dy} &= -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{[H_3 \gamma_2 + H_1]}{\eta_1 + \gamma_2} \eta_2 y^{\eta_2 - 1} \\ &> -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{[H_3 \gamma_2 + H_1]}{\eta_1 + \gamma_2} \eta_2 \\ &= \frac{\eta_1 - 1}{\eta_1 + \gamma_2} (H_2 - H_1) + \frac{\eta_2 - \gamma_2 - 1}{\eta_1 + \gamma_2} H_1 + \frac{\gamma_2 \eta_2}{\eta_1 + \gamma_2} H_3, \end{aligned}$$

where the second inequality is due to the fact that  $H_3 < 0$  and  $H_1 < 0$  (defined in Lemma 1).

In the first term above,

$$H_2 - H_1 = \frac{\theta \delta l + \phi}{\rho + \phi + (1 + \theta) \delta - \mu}$$

is positive according to the parameter restriction in (5). For the second term, note that  $\eta_2 - \gamma_2 - 1 = -2 \frac{\mu}{\sigma^2}$ . Then, after some algebraic substitutions (note that  $\gamma_2 \eta_2 = \frac{2(\rho + \phi)}{\sigma^2}$ ), the sum of the second and third terms is

$$-2 \frac{\mu}{\sigma^2} \frac{1}{\eta_1 + \gamma_2} H_1 + \frac{\gamma_2 \eta_2}{\eta_1 + \gamma_2} H_3 = 0.$$

Thus,  $\frac{dW(y)}{dy} > 0$ .

We now show that  $W(y)$  is monotonically increasing when  $1 < y \leq \frac{1-L}{L}$ . Equation (A10) implies that

$$\begin{aligned} W(y) &= \frac{r + \phi}{\rho + \phi} + B_4 y^{-\gamma_2} \\ &= \frac{M_2 \eta_1 - M_1}{\eta_1 + \gamma_2} y^{-\gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y \\ &\quad + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta}. \end{aligned}$$

We now show  $\eta_1 < \frac{M_1}{M_2} = \frac{\rho + \phi + (1 + \theta) \delta}{\mu}$ . Plugging  $x = \frac{\rho + \phi + (1 + \theta) \delta}{\mu}$  into the first fundamental Equation (A3), we find that the value is positive, which implies that  $\eta_1 < \frac{M_1}{M_2}$ . Therefore,

$M_2\eta_1 - M_1 < 0$ , and the first term is increasing in  $y$ . Because  $\eta_1 > 1$ , the second term is increasing in  $y$ . As a result,  $W(y)$  is increasing in  $y$ .

Similarly, we can show that  $W(y)$  is increasing in  $y$  for  $y > \frac{1-L}{T}$ . Equation (A11) implies that

$$W(y) = \frac{r + \phi}{\rho + \phi} + C_6 y^{-\gamma_2} = \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + (1 + \theta)\delta}{\rho + \phi + (1 + \theta)\delta} + \frac{(\eta_1 + \gamma_1) C_4 y^{-\gamma_1}}{\eta_1 + \gamma_2}.$$

Since  $\frac{K_5}{K_4} = \frac{K_3 \frac{1-L}{T}}{-K_2} = \frac{\rho + \phi + (1 + \theta)\delta}{\mu} = M_1/M_2$ , we have

$$\begin{aligned} \frac{(\eta_1 + \gamma_1) C_4 y^{-\gamma_1}}{\eta_1 + \gamma_2} &= \frac{K_4 \eta_1 - K_5 + \frac{-K_2 \eta_1 - K_3 \frac{1-L}{T}}{\left(\frac{1-L}{T}\right)^{-\gamma_1}}}{\eta_1 + \gamma_2} y^{-\gamma_1} \\ &= \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \left( K_4 y^{-\gamma_1} + (-K_2) \left(\frac{1-L}{T}\right)^{-\gamma_1} \right). \end{aligned}$$

Therefore, because  $\eta_1 - M_1/M_2 < 0$ , as shown in the case of  $1 < y \leq \frac{1-L}{T}$ , we can check that  $K_4 > 0$  and  $-K_2 > 0$ .  $W(y)$  is strictly increasing.

Next, we need to ensure that  $W(0) < 1$ . Equation (A9) implies that

$$W(0) = \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r}{\rho + \phi}.$$

The parameter restriction in (4) ensures that

$$\frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} < 1 \quad \text{and} \quad \frac{r}{\rho + \phi} < 1;$$

thus,  $W(0) < 1$ .

Finally, note that under our parameter restrictions in (4) and (6) we have

$$W(\infty) = \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + (1 + \theta)\delta}{\rho + \phi + (1 + \theta)\delta} > 1.$$

Because  $W(y)$  is continuous and monotonically increasing, and because  $W(0) < 1$  and  $W(\infty) > 1$ , there exists a unique  $y_*$  such that  $W(y_*) = 1$ . ■

Lemma 2 implies that there can be at most one symmetric monotone equilibrium. Next, we verify that a monotone strategy with the threshold level determined in Lemma 2 is indeed optimal for a creditor if every other creditor uses this threshold.

**Lemma 3.** If every other creditor uses a monotone strategy with a threshold  $y_*$  identified in Lemma 2, then the same strategy is also optimal for a creditor.

*Proof.* To show that the value function constructed in Lemma 1 is indeed optimal for a creditor, i.e., the value function solves the HJB Equation (9), we need to verify that  $V(y; y_*) > 1$  for  $y > y_*$  and  $V(y; y_*) < 1$  for  $y < y_*$ . By construction in Lemma 1,  $V(0; y_*) = \frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} < 1$  and  $V(\infty; y_*) = \frac{r + \phi}{\rho + \phi} > 1$ . We just need to show that  $V(y; y_*)$ , as a function of  $y$ , only crosses 1 once at  $y_*$ . Later in this proof we simply write  $V(y; y_*)$  as  $V(y)$ .



We first consider the case where  $y_* < 1$ .

We prove by contradiction. Suppose that  $V(y)$  also crosses 1 at another point below  $y_*$ . Then, there exists  $y_1 < y_* < 1$  such that

$$V(y_1) > V(y_*) = 1, V'(y_1) = 0, \text{ and } V''(y_1) < 0.$$

Using the differential Equation (A1), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + \phi \min(1, y_1) + \theta\delta(L + ly_1) + r + \delta}{\rho + \phi + (\theta + 1)\delta} \\ &< \frac{(\phi + \theta\delta)y_1 + \theta\delta L + r + \delta}{\rho + \phi + (\theta + 1)\delta} < \frac{\phi + \theta\delta l + \theta\delta L + r + \delta}{\rho + \phi + (\theta + 1)\delta} < 1. \end{aligned}$$

The last inequality is implied by the parameter restrictions in (4) and (7). This is a contradiction with  $V(y_1) > 1$ . Thus,  $V(y)$  cannot cross 1 at any  $y$  below  $y_*$ .

Next, we show that  $V(y)$  is monotonic in the region  $y \geq y_*$ . Suppose that  $V(y)$  is non-monotone, then there exist two points  $y_* \leq y_1 < y_2$  such that

$$V(y_1) > V(y_2), V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2).$$

(If, say,  $y_1$  happens to be on the break point 1 where the second derivative is not necessary continuous, then take the point as  $1+$  as  $V''(1+)$  has to be negative. The same caveat applies to the case where  $y_1 = y_*$ .) According to the differential Equation (A2), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1)}{\rho + \phi} \\ &> \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2)}{\rho + \phi} = V(y_2), \end{aligned}$$

which is a contradiction.

We next consider the case where  $y_* \geq 1$ . We do not separate the two cases of  $1 < y_* \leq \frac{1-L}{\Gamma}$  and  $y_* > \frac{1-L}{\Gamma}$ , as the following proof applies to both.

The expression in Equation (A10) or (A11) implies that  $V(y)$  has to approach  $\frac{r+\phi}{\rho+\phi}$  from below (because  $\frac{r+\phi}{\rho+\phi}$  is the debt holder's highest possible payoff), thus  $B_4$  or  $C_6$  is strictly negative. This implies that  $V(y)$  is increasing on  $[y_*, \infty)$ , and

$$V'(y_*) > 0.$$

Now consider the region  $[0, y_*)$ ; it is easy to check that  $V'(0) > 0$ . Therefore, if  $V(y)$  is not monotonic on  $[0, y_*)$ , there must exist two points  $y_1 < y_2$  such that

$$V(y_1) > V(y_2), V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2).$$

According to the HJB Equation, we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1) + \delta[1 + \theta \min(L + ly_1, 1)]}{\rho + \phi + (1 + \theta)\delta} \\ &< \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2) + \delta[1 + \theta \min(L + ly_2, 1)]}{\rho + \phi + (1 + \theta)\delta} \\ &= V(y_2), \end{aligned}$$

which is a contradiction. Thus,  $V(y)$  is also monotonically increasing on  $[0, y_*)$ .

To summarize, we have shown that  $V(y)$  only crosses 1 once at  $y_*$ . Thus, it is optimal for a creditor to roll over his debt if  $y > y_*$  and to run if  $y < y_*$ . ■

Finally, we prove in Lemma 4 that there does not exist any asymmetric monotone equilibrium.

**Lemma 4.** There does not exist any asymmetric monotone equilibrium in which creditors choose different rollover thresholds.

*Proof.* We prove by contradiction. Suppose that there exists an asymmetric monotone equilibrium. Then, there exist at least two groups of creditors who use two different monotone strategies with thresholds  $y_{*,1} < y_{*,2}$ . For creditors who use the threshold  $y_{i,*}$ , we denote their value function as  $V^i(y)$ . At the corresponding thresholds, we must have

$$V^1(y_{1,*}) = V^2(y_{2,*}) = 1.$$

Moreover, we must have

$$V^1(y_{2,*}) = V^2(y_{1,*}) = 1,$$

because each creditor is free to switch between these two strategies. Then, for all  $y \in [y_{1,*}, y_{2,*}]$ , we must have  $V^1(y) = V^2(y) = 1$ . Otherwise the threshold strategies cannot be optimal. This implies that each creditor is indifferent between choosing any threshold in  $[y_{1,*}, y_{2,*}]$ . Denote by  $\zeta(y)$  the measure of creditors who use a threshold lower than  $y \in [y_{1,*}, y_{2,*}]$ . Then,  $V^i$  has to satisfy the HJB Equation in this region:

$$\begin{aligned} \rho V^i(y) = & \mu y V_y + \frac{\sigma^2}{2} y^2 V_{yy} + r + \phi \left[ \min(1, y) - V^i(y) \right] \\ & + \theta \delta \zeta(y) \left[ \min(L + ly, 1) - V^i(y) \right] + \delta \max \left\{ 1 - V^i(y), 0 \right\}. \end{aligned}$$

Since  $V^i(y) = 1$  for any  $y \in [y_{1,*}, y_{2,*}]$ , we have

$$\rho = r + \phi [\min(1, y) - 1] + \theta \delta \zeta(y) [\min(L + ly, 1) - 1].$$

Note that  $\zeta(y)$  is non-decreasing in  $y$  because it is a distribution function. Since both  $\min(1, y)$  and  $\min(L + ly, 1)$  are also non-decreasing in  $y$ , the only possibility that the above equation holds is that  $L + ly > 1$  and  $y > 1$  for  $y \in [y_{1,*}, y_{2,*}]$ . Then,  $\rho = r$  has to hold. This contradicts the parameter restriction that  $\rho < r$  in (4). ■

## A.2 Proof of Proposition 1

As mentioned in the main text, in this modified synchronous setting the firm's debt contracts all expire at time 0. At this time, each creditor decides whether to run or to roll over into a perpetual debt contract lasting until the firm asset matures at  $\tau_\phi$ . If all creditors choose to run, we assume that there is a probability  $\theta_s \in (0, 1)$  that the firm cannot find new creditors to replace the outgoing ones and is forced into a premature liquidation.<sup>19</sup> The current firm fundamental is  $y_0$ .

<sup>19</sup> In this synchronous rollover setting, the liquidation probability parameter  $\theta_s$  has to be inside  $(0, 1)$ , while the liquidation intensity parameter  $\theta$  in the main model can be higher than 1 (conditional on creditors' runs, the liquidation probability over  $(t, t + dt)$  is  $\theta \delta dt$ ).

We first derive a creditor's value function  $U(y)$  if the firm survives the creditors' rollover decisions at time 0 and thus will be able to stay until the asset maturity at  $\tau_\phi$ .  $U(y)$  satisfies the following differential equation:

$$\rho U = \mu y U_y + \frac{1}{2} \sigma^2 y^2 U_{yy} + \phi [\min(1, y) - U] + r.$$

It is direct to solve this differential equation:

$$U(y) = \begin{cases} \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y + D_1 y^{\eta_2} & \text{if } 0 < y < 1 \\ \frac{r+\phi}{\rho+\phi} + D_2 y^{-\gamma_2} & \text{if } y > 1 \end{cases}, \tag{A12}$$

where

$$D_1 = -\frac{\frac{\phi}{\rho+\phi-\mu} + \gamma_2 \frac{\phi\mu}{(\rho+\phi-\mu)(\rho+\phi)}}{\eta_2 + \gamma_2}$$

$$D_2 = \frac{-\frac{\phi}{\rho+\phi-\mu} + \eta_2 \frac{\phi\mu}{(\rho+\phi-\mu)(\rho+\phi)}}{\eta_2 + \gamma_2}.$$

$D_1$  and  $D_2$  are constant and independent of the liquidation recovery parameter  $\alpha$ . Because  $U(y)$  is dominated by the fundamental value of the bank asset,  $U(y) < \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y$ . This implies that  $D_1 < 0$ . In addition, since  $U(\infty) = \frac{r+\phi}{\rho+\phi}$ ,  $D_2 < 0$  and  $U(y)$  approaches  $\frac{r+\phi}{\rho+\phi}$  from below. Therefore,  $U(y)$  is a monotonically increasing function with

$$U(0) = \frac{r}{r+\phi} < 1 \quad \text{and} \quad U(\infty) = \frac{r+\phi}{\rho+\phi} > 1.$$

Then, the intermediate value theorem implies that there exists  $y_l > 0$  such that  $U(y_l) = 1$ .

Define  $y_h \equiv \frac{1-L}{l}$ . According to the parameter restriction (6),  $y_h > 1$ . We impose the following condition so that a premature liquidation is sufficiently costly, i.e.,  $\alpha$  is sufficiently small:

$$\alpha < \frac{\rho + \phi - \mu}{\phi \left[ \left[ -D_2 \frac{\rho+\phi}{r-\rho} \right]^{\frac{1}{\gamma_2}} + \frac{r(\rho+\phi-\mu)}{\phi(\rho+\phi)} \right]}. \tag{A13}$$

This condition is analogous to the parameter restriction (6) in our main model.

Given this condition and that  $\frac{1-L}{l} = \frac{\rho+\phi-\mu}{\phi\alpha} - \frac{r(\rho+\phi-\mu)}{\phi(\rho+\phi)}$ , we have

$$U\left(\frac{1-L}{l}\right) = \frac{r+\phi}{\rho+\phi} + D_2 \left(\frac{1-L}{l}\right)^{-\gamma_2} > 1,$$

which further implies that  $y_l < y_h = \frac{1-L}{l}$ .

Next, we show that if  $y_0 > y_h$ , then it is optimal for a creditor to roll over, even if all the other creditors choose to run (so that the liquidation probability is  $\theta_s$ ). Note that the liquidation value of the bank asset is sufficient to pay off all the creditors because  $L + ly_0 > 1$ . Thus, the creditor's expected payoff from choosing to run is  $\theta_s + (1 - \theta_s) = 1$ . His expected payoff from choosing to roll over is  $\theta_s + (1 - \theta_s) U(y_0)$ , which is higher than the expected payoff from choosing to run.

Next, we show that if  $y_0 < y_l$ , then it is optimal for a creditor to run even if all the other creditors choose to roll over. In this case, the bank will always survive no matter what the individual

creditor's decision is. If he chooses to run, he gets a payoff of 1, while if he chooses to roll over, his continuation value function is  $U(y_0) < 1$ . Thus, it is optimal for the creditor to run.

Finally, we consider the case when  $y_0 \in [y_l, y_h]$ . If all the other creditors choose to roll over, then a creditor's payoff from run is 1, while his continuation value function is  $U(y_0) > 1$ . Thus, it is optimal for him to roll over too. If all the other creditors choose to run, then his expected payoff from run is  $\theta_s(L + ly_0) + (1 - \theta_s)$ . His expected payoff from choosing to roll over is  $(1 - \theta_s)U(y_0)$ , because once the bank is forced into a premature liquidation, the liquidation value of the bank asset is not sufficient to pay off the other outgoing creditors and the creditor who chooses rollover gets zero. Therefore, we need to ensure that  $\theta_s(L + ly_0) > (1 - \theta_s)(U(y_0) - 1)$ . Analogous to the parameter restriction (7) of our main model, we impose a parameter restriction on  $\theta_s$  so that it is sufficiently large:

$$\frac{\theta_s}{1 - \theta_s} > \frac{1}{L} \frac{r - \rho}{\rho + \phi}.$$

Then, because  $U(y_0) - 1 < \frac{r + \phi}{\rho + \phi} - 1 = \frac{r - \rho}{\rho + \phi}$ , we have  $(1 - \theta_s)(U(y_0) - 1) < (1 - \theta_s) \frac{r - \rho}{\rho + \phi} < \theta_s L < \theta_s(L + ly_0)$ . As a result, it is optimal for the creditor to run with other creditors.

### A.3 Proof of Proposition 2

Note that  $y_*$  is determined by the condition that  $W(y_*) = V(y_*, y_*) = 1$ . Theorem 1 implies that if  $y_* > \frac{1-L}{l}$ , it is determined by the following implicit function:

$$1 = W(y_*) = \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \left( K_4 y_*^{-\gamma_1} + (-K_2) \left( \frac{ly_*}{1-L} \right)^{-\gamma_1} \right) + \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta}, \tag{A14}$$

where  $L = \frac{\alpha r}{\rho + \phi}$  and  $l = \frac{\alpha\phi}{\rho + \phi - \mu}$  increase with  $\alpha$ , and  $M_1/M_2$  and  $K_4$  are independent of  $\alpha$ . By the implicit function theorem,  $\frac{dy_*}{d\alpha} = -\frac{\partial W/\partial \alpha}{\partial W/\partial y_*}$ . Since we have shown that  $\partial W/\partial y_* > 0$  in Lemma 2, to prove the claim we need to show that  $\partial W/\partial \alpha > 0$ . There are two terms in  $W$  that involve  $\alpha$ : 1) because  $-K_2 = \frac{\mu\theta\delta(1-L)}{(\rho + \phi + (1 + \theta)\delta)(\rho + \phi + (1 + \theta)\delta)}$ , the second term in the first bracket is proportional to  $-\frac{(1-L)^{1+\gamma_1}}{l^{\gamma_1}}$ , which is increasing in  $\alpha$ ; and 2) the second term  $\frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta}$  in the second line is increasing in  $\alpha$ . Therefore,  $\partial W/\partial \alpha > 0$ , and  $\frac{dy_*}{d\alpha} < 0$ .

When  $1 < y_* \leq \frac{1-L}{l}$ , it is determined by the following implicit function:

$$1 = W(y_*) = \frac{M_2\eta_1 - M_1}{\eta_1 + \gamma_2} y_*^{-\gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} y_* + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta}. \tag{A15}$$

Therefore,

$$\partial W/\partial \alpha = \eta_1 \frac{\theta\delta \frac{r}{\rho + \phi}}{\rho + \phi + (1 + \theta)\delta} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta\delta \frac{\phi}{\rho + \phi - \mu}}{\rho + \phi + (1 + \theta)\delta - \mu} y_* > 0, \tag{A16}$$

which implies  $\frac{dy_*}{d\alpha} > 0$ .

When  $y_* < 1$ , it is determined by the following implicit function:

$$W(y_*) = \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r}{\rho + \phi} + \frac{[H_3\gamma_2 + H_1]}{(\eta_1 + \gamma_2)} y_*^{\eta_2} + \left[ \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta\delta l + \phi}{\rho + \phi + (1 + \theta)\delta - \mu} + \frac{1 + \gamma_2}{(\eta_1 + \gamma_2)} \frac{\phi}{(\rho + \phi - \mu)} \right] y_* = 1,$$

where  $H_3$  and  $H_1$  are independent of  $\alpha$ . Then,

$$\partial W / \partial \alpha = \frac{\eta_1}{\eta_1 + \gamma_2} \frac{\theta\delta \frac{r}{\rho + \phi}}{\rho + \phi + (1 + \theta)\delta} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta\delta \frac{\phi}{\rho + \phi - \mu}}{\rho + \phi + (1 + \theta)\delta - \mu} > 0. \tag{A17}$$

Taken together, the equilibrium rollover threshold  $y_*$  decreases with  $\alpha$ .

### A.4 Proof of Proposition 3

First note that as  $\theta \rightarrow \infty$ ,  $y_* \rightarrow \frac{1-L}{1} > 1$ . The reason follows. As  $\theta \rightarrow \infty$ , the firm fails immediately after creditors start to run on the firm. Thus, the rollover risk term  $\min(1, L + ly_t) \mathbf{1}_{\{\tau = t\theta\}}$  in Equation (8) is replaced by a boundary condition that when  $y = y_*$ ,  $V(y, y_*) = L + ly_*$ . It is direct to see that the equilibrium condition  $V(y_*, y_*) = 1$  implies that  $y_* = \frac{1-L}{1}$  is the unique equilibrium threshold.

Then, by the continuity of  $y_*$  with respect to  $\theta$ , if  $\theta$  is sufficiently high,  $y_* > 1$ . Our numerical exercises also show that this holds true over a wide range of parameter values. Thus, we will focus on showing that  $y_*$  increases with  $\sigma^2$  in the range where  $y_* > 1$ .

Since  $y_*$  is determined by the implicit function  $W(y_*) = V(y_*, y_*) = 1$ , to show that  $y_*$  increases with  $\sigma^2$ , we only need to verify that  $\frac{\partial W(y)}{\partial \sigma^2} < 0$ .

We first note several inequalities. Directly from condition (5), we have  $\frac{\partial \eta_i}{\partial \sigma^2} < 0$  and  $\frac{\partial \gamma_i}{\partial \sigma^2} < 0$  for  $i = 1, 2$ . Moreover, by using the definitions of  $\eta_1$  in (A5) and  $\gamma_2$  in (A7), we can also show that

$$\frac{\partial \left( \frac{\gamma_2}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} < 0. \tag{A18}$$

We now consider the case where  $1 < y_* \leq \frac{1-L}{1}$ . Based on  $W(y)$  given in Equation (A15), we have

$$\begin{aligned} \frac{\partial W(y)}{\partial \sigma^2} &= \frac{\partial \left( \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} M_2 y^{-\gamma_1} + \frac{\partial \left( \frac{-1}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} \frac{\theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} y \\ &+ \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} M_2 y^{-\gamma_1} \ln y \frac{\partial (-\gamma_1)}{\partial \sigma^2} + \frac{\partial \left( \frac{\gamma_2}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} \\ &\times \left( \frac{r + \phi}{\rho + \phi} - \frac{r + \phi + \theta\delta(L + ly) + \delta}{\rho + \phi + (1 + \theta)\delta} \right). \end{aligned}$$

As  $\frac{r + \phi}{\rho + \phi} - \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} > 0$ , inequality in (A18) implies that the last term is negative. Also,  $\frac{\partial \left( \frac{-1}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} < 0$  implies that the second term is negative. Moreover, because  $M_2\eta_1 - M_1 < 0$  (shown in the proof of Lemma 2), and  $\frac{\partial (-\gamma_1)}{\partial \sigma^2} > 0$ , the third term is negative. Finally, note that

when  $\theta$  is sufficiently large,  $\eta_1$  and  $\gamma_2$  are in the order of  $\theta^{0.5}$ . Since  $M_1/M_2 = \frac{\rho + \phi + (1 + \theta)\delta}{\mu}$ , the first part of the second term  $\frac{\partial \left( \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2}$  is approximately equal to  $-\frac{\partial \left( \frac{1}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} M_1/M_2$ , which is negative. Taken together,  $\frac{\partial W(y)}{\partial \sigma^2} < 0$ .

We now consider the case where  $y_* > \frac{1-L}{l}$ . Based on  $W(y)$  in Equation (A14), we have

$$\begin{aligned} \frac{\partial W(y)}{\partial \sigma^2} &= \frac{\partial \left( \frac{\gamma_2}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} \left( \frac{r + \phi}{\rho + \phi} - \frac{r + \phi + (1 + \theta)\delta}{\rho + \phi + (1 + \theta)\delta} \right) + \frac{\partial \left( \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} \\ &\quad \times \left( K_4 y^{-\gamma_1} + (-K_2) \left( \frac{ly}{1-L} \right)^{-\gamma_1} \right) \\ &\quad + \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \frac{\partial \left( K_4 y^{-\gamma_1} + (-K_2) \left( \frac{ly}{1-L} \right)^{-\gamma_1} \right)}{\partial \sigma^2}. \end{aligned}$$

Using arguments similar to those presented in the previous case, it is easy to show that every term in this expression is negative. Thus,  $\frac{\partial W(y)}{\partial \sigma^2} < 0$ . This concludes the proof.

### A.5 Proof of Proposition 4

We first show that if both  $\theta$  and  $\sigma$  are sufficiently large, the equilibrium rollover threshold  $y_* > \frac{1-L}{l}$ . We use contradiction. Suppose that case 2 of Lemma 1 in 5 holds true. Then,

$$\begin{aligned} W \left( \frac{1-L}{l} \right) &= V \left( \frac{1-L}{l}, \frac{1-L}{l} \right) \\ &= \frac{M_2 \eta_1 - M_1}{\eta_1 + \gamma_2} \left( \frac{1-L}{l} \right)^{-\gamma_1} + \frac{-1}{\eta_1 + \gamma_2} \frac{\theta \delta (1-L)}{\rho + \phi + (1 + \theta)\delta - \mu} \\ &\quad + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + (1 + \theta)\delta}{\rho + \phi + (1 + \theta)\delta}. \end{aligned}$$

Note that  $W \left( \frac{1-L}{l} \right) < 1$  is equivalent to

$$\begin{aligned} &\frac{M_1 - M_2 \eta_1}{\eta_1 + \gamma_2} \left( \frac{1-L}{l} \right)^{-\gamma_1} + \frac{1}{\eta_1 + \gamma_2} \frac{\theta \delta (1-L)}{\rho + \phi + (1 + \theta)\delta - \mu} \\ &> \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r - \rho}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r - \rho}{\rho + \phi + (1 + \theta)\delta}. \end{aligned} \tag{A19}$$

If  $\theta$  is sufficiently large, both  $\gamma_1$  and  $\eta_1$  are in the order of  $\theta^{0.5}$ . Then, (A19) is equivalent to  $1 - L > \gamma_2 \frac{r - \rho}{\rho + \phi}$ . Since  $\gamma_2 \rightarrow 0$  as  $\sigma^2$  is large, this condition holds for sufficiently large  $\sigma^2$ . Thus,  $W \left( \frac{1-L}{l} \right) < 1$  if both  $\theta$  and  $\sigma$  are sufficiently large. Since  $W'(y) > 0$ , we must have  $y_* > \frac{1-L}{l}$ . This contradicts with case 2 holding true. Therefore, we will focus on case 3 of Lemma 1.

We use the implicit function theorem to analyze the properties of  $y_*$ . Since  $W(y_*; \theta) = 1$ ,

$$\frac{dy_*}{d\theta} = - \frac{\partial W / \partial \theta}{\partial W / \partial y_*}.$$

Thus,  $\frac{dy_*}{d\theta} < 0$  is equivalent to  $\partial W/\partial\theta > 0$ . Based on case 3 of Lemma 1, we have

$$\begin{aligned} W(y_*; \theta) - 1 &= \frac{r + \phi}{\rho + \phi} + \frac{K_4\eta_1 - K_5 - \left(K_2\eta_1 + K_3 \frac{1-L}{l}\right) \left(\frac{ly_*}{1-L}\right)^{-\gamma_1} + \eta_1 K_1}{\eta_1 + \gamma_2} \\ &= K_4 + \frac{r - \rho}{\rho + \phi + (1 + \theta)\delta} - \frac{K_5}{\eta_1} - \left(K_2 + \frac{K_3}{\eta_1} \frac{1-L}{l}\right) \\ &\quad \times \left(\frac{ly_*}{1-L}\right)^{-\gamma_1}. \end{aligned}$$

We will again use the fact that both  $\gamma_1$  and  $\eta_1$  are in the order of  $\theta^{0.5}$  and  $\gamma_2 \rightarrow 0$  when both  $\theta$  and  $\sigma$  are large. Note that  $K_4$  is in the order of  $\theta^{-2}$ ,  $\frac{K_5}{\eta_1}$  is in the order of  $\theta^{-1.5}$ , and  $K_2 + \frac{K_3}{\eta_1} \frac{1-L}{l}$  is dominated by  $\frac{K_3}{\eta_1} \frac{1-L}{l}$  and is in the order of  $\theta^{-0.5}$ . Since  $W(y_*; \theta) = 1$ , we know that  $\left(\frac{ly_*}{1-L}\right)^{-\gamma_1}$  must be in the order of  $\theta^{-0.5}$ . As a result,

$$\ln\left(\frac{ly_*}{1-L}\right) \propto \frac{\ln\theta}{\gamma_1} \propto \ln\theta \cdot \theta^{-0.5}.$$

Thus, the leading order in  $W(y_*; \theta) - 1$  is  $\frac{r-\rho}{\rho+\phi+(1+\theta)\delta}$  and  $-\left(\frac{K_3}{\eta_1} \frac{1-L}{l}\right) \left(\frac{ly_*}{1-L}\right)^{-\gamma_1}$ . We can now evaluate their derivatives with respect to  $\theta$ . The first term  $\frac{r-\rho}{\rho+\phi+(1+\theta)\delta}$  has a derivative of  $-\theta^{-2}$ . The second term has a derivative of

$$\begin{aligned} &-\frac{\partial}{\partial\theta} \left(\frac{K_3}{\eta_1} \frac{1-L}{l}\right) \cdot \left(\frac{ly_*}{1-L}\right)^{-\gamma_1} - \left(\frac{K_3}{\eta_1} \frac{1-L}{l}\right) \frac{\partial}{\partial\theta} \left[\left(\frac{ly_*}{1-L}\right)^{-\gamma_1}\right] \\ &= -\frac{\partial}{\partial\theta} \left(\frac{K_3}{\eta_1} \frac{1-L}{l}\right) \left(\frac{ly_*}{1-L}\right)^{-\gamma_1} - \left(\frac{K_3}{\eta_1} \frac{1-L}{l}\right) \left(\frac{ly_*}{1-L}\right)^{-\gamma_1} \\ &\quad \times \ln\left(\frac{ly_*}{1-L}\right) \frac{d(-\gamma_1)}{d\theta} > -\left(\frac{K_3}{\eta_1} \frac{1-L}{l}\right) \theta^{-0.5} \ln\left(\frac{ly_*}{1-L}\right) \frac{d(-\gamma_1)}{d\theta} \\ &\propto \theta^{-0.5} \cdot \theta^{-0.5} \cdot \ln\theta \cdot \theta^{-0.5} \cdot \theta^{-0.5}, \end{aligned}$$

which has a higher order than  $\theta^{-2}$ . As a result,  $\partial W/\partial\theta > 0$ , and we have  $\frac{dy_*}{d\theta} < 0$ .

### A.6 Proof of Proposition 5

Note that if both  $\delta$  and  $\sigma$  are sufficiently large,  $\gamma_1$  and  $\eta_1$  are in the order of  $\delta^{0.5}$  and  $K_2, K_3, K_4$ , and  $K_5$  all have the leading term  $\theta\delta$ . Thus, we can follow a similar procedure as that in the proof of Proposition 4. For brevity, we skip reporting the proof here.

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