Counting the number of solutions to the Erdos-Straus equation on unit fractions

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For any positive integer \$n\$, let \$f(n)\$ denote the number of solutions to the Diophantine equation \$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\$ with \$x,y,z\$ positive integers. The \emph{Erd\H{o}s-Straus conjecture} asserts that \$f(n) > 0\$ for every \$n \geq 2\$. To solve this conjecture, it suffices without loss of generality to consider the case when \$n\$ is a prime \$p\$.
In this paper we consider the question of bounding the sum \$\sum_{p<N} f (p)\$ asymptotically as \$N \to \infty\$, where \$p\$ ranges over primes. Our main result establishes the asymptotic upper and lower bounds \$\$ N \log^2 N \ll \sum_{p \leq N} f(p) \ll N \log^2 N \log \log N.\$\$ In particular, from this bound and the prime number theorem we have \$f(p) = O(\log^3 p \log \log p) \$ for a subset of primes of density arbitrarily close to 1; thus a typical prime has a relatively small number of solutions to the Erd\H{o}s-Straus Diophantine equation.
We also establish some related results on \$f\$ and related quantities, for instance establishing the bound \$f(p) \ll p^{3/5} + O(\frac{1}{\log\log p})\$ for all primes \$p\$.

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