TWISTED GROSS-ZAGIER THEOREMS

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ABSTRACT. The theorems of Gross-Zagier and Zhang relate the Néron-Tate heights of complex multiplication points on the modular curve $X_0(N)$ (and on Shimura curve analogues) with the central derivatives of automorphic L-function. We extend these results to include certain CM points on modular curves of the form $X(\Gamma_0(M) \cap \Gamma_1(S))$ (and on Shimura curve analogues). These results are motivated by applications to Hida theory which are described in the companion article [15].

1. Introduction

Let χ_0 be a finite order character of the idele class group $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ of $\mathbb{Q}$, and suppose that $f \in S_2(\Gamma_0(N), \chi_0^{-1}, \mathbb{C})$ is a normalized newform of level N and character χ_0^{-1}. In particular we assume that f is an eigenform for all Hecke operators T_n with $(n, N) = 1$. Writing $f = \sum_n b_n q^n$ the L-series of f is defined as the analytic continuation of $L(s, f) = \sum_n b_n n^{-s}$. To compare with the notation used in the body of the article, $L(s, \Pi) = L^*(s + 1/2, f)$ where

$$L^*(s, f) = 2(2\pi)^{-s}\Gamma(s)L(s, f)$$

is the completed L-function of f and Π is the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ attached to f. Let E be a quadratic imaginary field of discriminant $-D$ and let χ be a finite order character of the idele class group $E^\times \backslash \mathbb{A}_E^\times$ whose restriction to $\mathbb{A}^\times$ agrees with χ_0. Factor $N = MS$ in such a way that S is divisible only by primes dividing $\mathrm{N}_{E/\mathbb{Q}}(\mathrm{cond}(\chi))$ and M is relatively prime to $\mathrm{N}_{E/\mathbb{Q}}(\mathrm{cond}(\chi))$. We assume

(a) N and $\mathrm{N}_{E/\mathbb{Q}}(\mathrm{cond}(\chi))$ are each relatively prime to D,
(b) for any prime $p \mid S$ the restriction of χ to $E_p^\times = (E \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ factors through the norm $E_p^\times \to \mathbb{Q}_p^\times$,
(c) $S = \mathrm{cond}(\chi_0)$.

It is easy to see from these hypotheses that $\mathrm{cond}(\chi) = C\mathcal{O}_E$ for some positive integer C which is divisible by S.

Let ω denote the quadratic Dirichlet character attached to E. The L-function of f and the Hecke L-series of χ each admit Euler products over the rational primes. For each prime p the local Eulers factors have the form

$$\begin{aligned} L_p(s, f) &= (1 - \alpha_1 p^{-s})^{-1}(1 - \alpha_2 p^{-s})^{-1} \\ L_p(s, \chi) &= (1 - \beta_1 p^{-s})^{-1}(1 - \beta_2 p^{-s})^{-1} \end{aligned}$$

and we define a new Euler factor

$$L_p(s, \chi, f) = \prod_{1 \le i,j \le 2} (1 - \alpha_i \beta_j p^{-s})^{-1}.$$

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The Rankin-Selberg convolution L-function $L(s,\chi,f)=\prod_p L_p(s,\chi,f)$ has analytic continuation to an entire function of s and satisfies the functional equation

$$L^*(s,\chi,f)=-\omega(M)\cdot(C^2DM)^{2-2s}\cdot L^*(2-s,\chi,f)$$

where

$$L^*(s,\chi,f)=4(2\pi)^{-2s}\Gamma(s)^2L(s,\chi,f).$$

In the notation of the body of the text $L(s,\Pi\times\Pi_\chi)=L^*(s+1/2,\chi,f)$, and so the functional equation follows from the functional equation (2.6) of the Rankin-Selberg kernel and the integral representation of the L-function (2.8).

Assume that every prime divisor of M splits in E. In particular the functional equation forces $L(1,\chi,f)=0$. Let $\mathcal{O}=\mathbb{Z}+C\mathcal{O}_E$ and $\mathcal{O}'=\mathbb{Z}+CS^{-1}\mathcal{O}_E$ be the orders of conductors C and CS^{-1}, respectively, of $\mathcal{O}_E$. Fix an invertible ideal $\mathfrak{M}\subset\mathcal{O}$ such that $\mathcal{O}/\mathfrak{M}\cong\mathbb{Z}/M\mathbb{Z}$ and consider the isogenies of complex elliptic curves

$$\mathbb{C}/\mathcal{O}\xrightarrow{F_M}\mathbb{C}/\mathfrak{M}^{-1}\qquad\qquad \mathbb{C}/\mathcal{O}\xrightarrow{F_S}\mathbb{C}/\mathcal{O}'.$$

These isogenies are cyclic of degree M and S, respectively, and if we pick an arbitrary generator $\pi\in\ker(F_S)$ the triple $Q=(\mathbb{C}/\mathcal{O},\ker(F_M),\pi)$ determines a point on the moduli space $X_\Gamma(\mathbb{C})$ parametrizing complex elliptic curves with

$$\Gamma=\Gamma_0(M)\cap\Gamma_1(S)$$

level structure. We view X_Γ as a scheme over $\mathrm{Spec}(\mathbb{Q})$. Let $\widehat{\mathcal{O}}$ denote the closure of $\mathcal{O}$ in the ring $\mathbb{A}_{E,f}$ of finite adeles of E and let $\theta:\widehat{\mathcal{O}}^\times\to(\mathbb{Z}/S\mathbb{Z})^\times$ denote homomorphism giving the action of $\widehat{\mathcal{O}}^\times$ on $\widehat{\mathcal{O}}'/\widehat{\mathcal{O}}\cong\mathbb{Z}/S\mathbb{Z}$. The character χ has trivial restriction to $\ker(\theta)$, and by the theory of complex multiplication the point Q is rational over the abelian extension of E with class group $E^\times\backslash\mathbb{A}_{E,f}^\times/\ker(\theta)$. Thus we may form the divisor with complex coefficients

$$Q_\chi=\sum_{t\in E^\times\backslash\mathbb{A}_{E,f}^\times/\ker(\theta)}\overline{\chi(t)}\cdot Q^{[t,E]}$$

on $X_\Gamma\times_{\mathbb{Q}}E_\chi$ where $[\cdot,E]$ is the Artin symbol normalized as in [28, §5.2] and E_χ is the abelian extension of E cut out by χ. Assume that χ is nontrivial (otherwise $S=1$ and we are in the case originally considered by Gross and Zagier [14]) so that Q_χ has degree zero and may be viewed as a point in the modular Jacobian $Q_\chi\in J_\Gamma(E_\chi)\otimes_{\mathbb{Z}}\mathbb{C}$. Denote by $\mathbb{T}$ the (semi-simple) $\mathbb{C}$-algebra generated by the Hecke operators $\{T_n\mid(n,N)=1\}$ and the diamond operators $\{\langle d\rangle\mid(d,S)=1\}$ acting on $S_2(\Gamma,\mathbb{C})$. By the Eichler-Shimura theory the algebra $\mathbb{T}$ acts on $J_\Gamma(E_\chi)\otimes_{\mathbb{Z}}\mathbb{C}$ via the Albanese endomorphisms T_{n*} and $\langle d\rangle_*$ as in [22, §2.4]).

The following theorem is a special case of Theorem 5.6.2. When $S=1$ this result is due to Zhang [36, Theorem 6.1]. When $S=1$ and χ is unramified it is due to Gross-Zagier [14].

Theorem A. *Let $Q_{\chi,f}$ denote the projection of Q_χ to the maximal summand of $J_\Gamma(E_\chi)\otimes_{\mathbb{Z}}\mathbb{C}$ on which $\mathbb{T}$ acts through $T_n\mapsto b_n$ and $\langle d\rangle\mapsto\chi_0^{-1}(d)$. Then*

$$L'(1,\chi,f)=0\iff Q_{\chi,f}=0.$$

Remark 1.0.1. The hypotheses (b) and (c) placed on the primes divisors of S are not made for the sake of convenience; rather these hypotheses seem to be closely related to the particular choice of $\Gamma_1(S)$ level structure on $\mathbb{C}/\mathcal{O}$, given by a generator of the kernel of an isogeny to an elliptic curve with complex multiplication by a *different* quadratic order.

Remark 1.0.2. If $\Pi \cong \bigotimes_v \Pi_v$ denotes the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by the adelization of f then the condition (c) above is equivalent to Hypothesis 1.1.1(b) below, with $F = \mathbb{Q}$, $\mathfrak{s} = S\mathbb{Z}$, and $\mathfrak{c} = C\mathbb{Z}$. This follows from the formulas of [27, §12.3] and [25, Theorem 4.6.17].

Throughout the body of the article we work in much greater generality than the situation described above; instead of a classical modular form f as above we work with a Hilbert modular newform ϕ_Π over a totally real field F, and assume that ϕ_Π is either holomorphic of parallel weight 2 or is a Maass form of parallel weight 0. Let χ be a finite order character of the idele class group of a totally imaginary quadratic extension E of F, and assume that the restriction of χ^{-1} to the ideles of F agrees with the central character of the automorphic representation Π generated by ϕ_Π. We assume that Π, χ, and E also satisfy the hypotheses of §1.1 below. The Rankin-Selberg L-function $L(s, \Pi \times \Pi_\chi)$, where Π_χ is the theta series representation associated to χ, is normalized so that the center of symmetry of the functional equation is at $s = 1/2$.

Assume first that ϕ_Π is holomorphic of parallel weight 2. When the sign in the functional equation of $L(s, \Pi \times \Pi_\chi)$ is 1 we prove a formula (Theorem 4.3.3) relating the central value $L(1/2, \Pi \times \Pi_\chi)$ to certain CM-points on a totally definite quaternion algebra over F. In special cases such results go back to Gross's special value formula [10]. Such special value formulas have been used by Bertolini and Darmon to construct anticyclotomic p-adic L-functions for elliptic curves [1], and such L-functions play a central role both in those authors' work on the anticyclotomic Iwasawa main conjecture for elliptic curves [2] as well as in the work of Vatsal [30] and Cornut-Vatsal [6, 5] on the nonvanishing of L-values in towers of ring class fields. We point out also the helpful expository article of Vatsal [31]. When the sign in the functional equation of $L(s, \Pi \times \Pi_\chi)$ is -1 we prove a theorem (Theorem 5.6.2, which includes Theorem A as a special case) which generalizes results of Zhang [36, Theorem 6.1] and Gross-Zagier [14] by relating the central derivative $L'(1/2, \Pi \times \Pi_\chi)$ to the Néron-Tate height of CM-cycles on a Shimura curve over F. Now assume that ϕ_Π is Maass form of parallel weight 0 and that the sign in the functional equation of $L(s, \Pi \times \Pi_\chi)$ is 1. In this case we prove (Theorem 4.4.2) a formula expressing the central value $L(1/2, \Pi \times \Pi_\chi)$ as a weighted sum of the values at CM points of a weight 0 Maass form (related to ϕ_Π by the Jacquet-Langlands correspondence) on a Shimura variety of dimension $[F : \mathbb{Q}]$.

Our methods follow those of Zhang [34, 36] and we freely use his results and calculations when they carry over to our setting without significant change; the reader is advised to keep copies of [34, 36] close at hand. The original contributions are primarily found in §3 and §4.

The primary motivation for this work is to obtain results on the behavior of Selmer groups and L-functions in Hida families. Indeed, the somewhat peculiar point $Q \in X_\Gamma(\mathbb{C})$ defined above plays a central role in the construction of *big Heegner points* [16] in the cohomology of Galois representations for Λ-adic modular forms. Theorem A can be used to verify, in any particular case, the conjectural nonvanishing of these big Heegner points and can also be used to give examples of Hida families of modular forms whose L-functions vanish to exact order one with only finitely many exceptions. The applications to Hida theory and Iwasawa theory of the results contained herein is found in the separate article [15].

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1.1. **Notation and conventions.** The following choices and conventions apply throughout the remainder of the article.

Fix a totally real field F, a CM-extension E/F of relative discriminant $\mathfrak{d}$ and relative different $\mathfrak{D}$, and denote by $\mathbb{A}$ and $\mathbb{A}_E$ the adele rings of F and E, respectively. The integer rings of F and E are denoted $\mathcal{O}_F$ and $\mathcal{O}_E$, respectively, and ω denotes the quadratic character of $\mathbb{A}^\times/F^\times$ corresponding to the extension E/F. If M is any finitely generated $\mathbb{Z}$-module we let $\widehat{M}$ denote its profinite completion. If $\mathfrak{a}$ is any nonzero $\mathcal{O}_F$-ideal, $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})$ denotes the cardinality of $\mathcal{O}_F/\mathfrak{a}$. If v is a real place of F then $|\cdot|_v$ denotes the usual absolute value on $F_v \cong \mathbb{R}$. If v is a finite place then $|\cdot|_v$ is normalized so that for any uniformizing parameter ϖ of F_v, $|\varpi|_v^{-1}$ is the size of the residue field of v. For any $\mathcal{O}_F$-module M and any place v of F, set $M_v = M \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$. For any $x \in \mathbb{A}^\times$ let $x\mathcal{O}_F$ denote the fractional ideal of $\mathcal{O}_F$ determined by $(x\mathcal{O}_F)_v = x_v\mathcal{O}_{F,v}$ for every finite place v.

Fix a finite order character $\chi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$. Let χ_0 denote the restriction of χ to $\mathbb{A}^\times/F^\times$ and let $\mathfrak{C}$ denote the conductor of χ. We abbreviate $\mathrm{N}(\mathfrak{C}) = \mathrm{N}_{E/F}(\mathfrak{C})$. For each place v of F let χ_v denote the restriction of χ to $E_v^\times = (E \otimes_F F_v)^\times$. Let Π be an irreducible infinite dimensional cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of central character χ_0^{-1} and conductor $\mathfrak{n}$, as defined in §2.1. Factor $\mathfrak{n} = \mathfrak{m}\mathfrak{s}$ in such a way that $\mathfrak{m}$ is prime to $\mathrm{N}(\mathfrak{C})$ and $\mathfrak{s}$ is divisible only by primes dividing $\mathrm{N}(\mathfrak{C})$. We assume throughout that $\mathfrak{n}$ and $\mathrm{N}(\mathfrak{C})$ are both prime to $\mathfrak{d}$.

Hypothesis 1.1.1. At times we will assume that Π satisfies the following hypotheses.

(a) For every $v \mid \mathfrak{s}$ there is a character ν_v of $F_v^\times$ such that $\chi_v = \nu_v \circ \mathrm{N}_{E_v/F_v}$. Note that this hypothesis implies that $\mathfrak{C} = \mathfrak{c}\mathcal{O}_E$ for some ideal $\mathfrak{c}$ of $\mathcal{O}_F$.

(b) For every $v \mid \mathfrak{s}$, Π_v is a principal series representation $\Pi(\mu_v, \chi_{0,v}^{-1}\mu_v^{-1})$ of $\mathrm{GL}_2(F_v)$ with μ_v an unramified quasi-character of $F_v^\times$. In particular

$$\mathrm{ord}_v(\mathfrak{s}) = \mathrm{ord}_v(\mathrm{cond}(\chi_0)) \le \mathrm{ord}_v(\mathfrak{c}).$$

These hypotheses will be assumed in §4 and §5 but are not needed for the calculations of §3, or for the calculations of §2 unless otherwise indicated.

2. Automorphic forms and the Rankin-Selberg integral

Let $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ be a nontrivial additive character. Fix an idele $\delta \in \mathbb{A}^\times$ in such a way that for every finite place v of F the restriction to F_v of the additive character $\psi^0 : \mathbb{A} \to \mathbb{C}^\times$ defined by $\psi^0(x) = \psi(\delta^{-1}x)$ has conductor $\mathcal{O}_{F,v}$ and so that for every archimedean place v the restriction of ψ^0 to $F_v \cong \mathbb{R}$ is given by $\psi_v^0(x) = e^{2\pi i x}$. This implies that F has absolute discriminant $D_F = |\delta|^{-1}$. For any finite place v of F we normalize the additive Haar measure dx on F_v in such a way that the volume of $\mathcal{O}_{F,v}$ is equal to $|\delta|_v^{1/2}$, and normalize the multiplicative Haar measure $d^\times x$ on $F_v^\times$ in such a way that the volume of $\mathcal{O}_{F,v}^\times$ is 1. Then dx and $d^\times x$ are related by

$$|\delta|_v^{1/2}(1 - |\varpi|_v) \cdot d^\times x = |x|_v^{-1} \cdot dx \tag{2.1}$$

for any uniformizer ϖ of F_v. On $\mathbb{R}^\times$ we normalize the Haar measure $d^\times x$ by $d^\times x = |x|^{-1} d^{\mathrm{Leb}} x$, where $d^{\mathrm{Leb}} x$ is the usual Lebesgue measure giving $[0,1]$ unit mass. For an archimedean place v the additive Haar measure dx on $F_v \cong \mathbb{R}$ is normalized by $dx = |\delta|_v^{1/2} d^{\mathrm{Leb}} x$. In all cases the Haar measure on the additive group F_v is self-dual with respect to ψ_v. Endow $\mathbb{A}$ and $\mathbb{A}^\times$ with the product measures; the quotient measure on $\mathbb{A}/F$ has total volume 1 by [33, Proposition V.4.7].

Fix $d \in \mathbb{A}^\times$ such that $d\mathcal{O}_F = \mathfrak{d}$ and $d_v = 1$ for $v \mid \infty$. Let S denote the set of places of F dividing $\mathfrak{d}$, and for each $v \in S$ set $h_v = \begin{pmatrix} & 1 \\ -d_v & \end{pmatrix} \in \mathrm{GL}_2(F_v)$, viewed as an element of $\mathrm{GL}_2(\mathbb{A})$ with trivial components away from v. For each subset $T \subset S$ set $h_T = \prod_{v \in T} h_v$ and view h_T as an operator on automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ via $(h_T\phi)(g) = \phi(g h_T)$. For $a \in \mathbb{A}^\times$ define $e_\infty(a) = \prod_{v \mid \infty} e_v(a)$ where

$$e_v(a) = \begin{cases} 2e^{-2\pi a_v} & \text{if } a_v > 0 \\ 0 & \text{otherwise.} \end{cases}$$

for each $v \mid \infty$. Define the usual gamma factors

$$G_1(s) = \pi^{-s/2}\Gamma(s/2) \qquad G_2(s) = 2(2\pi)^{-s}\Gamma(s).$$

2.1. **Automorphic forms.** Let ϕ be an automorphic form on $\mathrm{GL}_2(\mathbb{A})$. Then ϕ admits a Fourier expansion

$$\phi(g) = C_\phi(g) + \sum_{\alpha \in F^\times} W_\phi\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g \right)$$

in which the constant term C_ϕ and Whittaker function W_ϕ (with respect to ψ) are defined by [34, (2.4.3)] and [34, (2.4.4)], respectively. For every $a \in \mathbb{A}^\times$ the *Whittaker coefficient*

$$B(a;\phi) = W_\phi \begin{pmatrix} a\delta^{-1} & \\ & 1 \end{pmatrix}$$

is independent of the choice of ψ, and a simple calculation shows that the Whittaker coefficients of ϕ and $\overline{\phi}$ are related by $B(a;\overline{\phi}) = \overline{B(-a;\phi)}$. The *zeta function* of ϕ is defined as the meromorphic continuation of

$$\begin{aligned} Z(s;\phi) &= |\delta|^{1/2-s} \int_{\mathbb{A}^\times} B(y;\phi) \cdot |y|^{s-1/2} \, d^\times y \\ &= \int_{\mathbb{A}^\times/F^\times} (\phi - C_\phi) \begin{pmatrix} y & \\ & 1 \end{pmatrix} \cdot |y|^{s-1/2} \, d^\times y \end{aligned}$$

in which both integrals are convergent for $\mathrm{Re}(s) \gg 0$. As in [34, §3.5] we say that an automorphic form ϕ of parallel weight 2 is *holomorphic* if its Whittaker coefficient has the form

$$B(a;\phi) = |a|_\infty e_\infty(a) \cdot \widehat{B}(\mathfrak{a};\phi)$$

with $\mathfrak{a} = a\mathcal{O}_F$ for some function $\widehat{B}(\mathfrak{a};\phi)$ on fractional ideals of $\mathcal{O}_F$ which vanishes on non-integral ideals.

Let v be a finite place of F. If $\mathfrak{n}_v$ is an ideal of $\mathcal{O}_{F,v}$ define the habitual congruence subgroup

$$K_1(\mathfrak{n}_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,v}) \;\middle|\; c \in \mathfrak{n}_v, d \in 1 + \mathfrak{n}_v \right\}.$$

For an irreducible, admissible, infinite dimensional representation π_v of $\mathrm{GL}_2(F_v)$ the *conductor* of π_v is the largest ideal $\mathfrak{n}_v$ such that π_v admits a $K_1(\mathfrak{n}_v)$-fixed vector. The space $K_1(\mathfrak{n}_v)$-fixed vectors is then 1-dimensional, and any nonzero vector on this line will be called a *newvector*. If v is an infinite place of F then any π_v as above has a unique line of vectors of minimal non-negative weight for the action of $\mathrm{SO}_2(\mathbb{R})$; a nonzero vector on this line is again called a *newvector*. If $\pi \cong \bigotimes_v \pi_v$ is an irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ then a newvector in π is a product of local newvectors. Such a newvector is unique up to scaling, and we define the *normalized newvector* $\phi_\pi \in \pi$ to be the unique newvector satisfying

$$Z(s, \phi_\pi) = |\delta|^{1/2-s} L(s, \pi).$$

If $\mathfrak{n}$ is an ideal of $\mathcal{O}_F$ set $K_1(\mathfrak{n}) = \prod_v K_1(\mathfrak{n}_v)$, where the product is over all finite places.

Suppose v is a finite place of F, ϕ is an automorphic form which is fixed by the action of $K_1(\mathfrak{n})$, and $(\mathfrak{a}, \mathfrak{n}) = 1$. We define

$$(T_\mathfrak{a}\phi)(g) = \sum_{h \in H(\mathfrak{a})/K_1(\mathfrak{n})} \phi(gh)$$

where $H(\mathfrak{a}_v)$ is the set of elements of $M_2(\mathcal{O}_{F,v})$ whose determinant generates $\mathfrak{a}_v$ and

$$H(\mathfrak{a}) = \prod_{v \nmid \mathfrak{a}} K_1(\mathfrak{n}_v) \cdot \prod_{v \mid \mathfrak{a}} H(\mathfrak{a}_v).$$

If $a \in \mathbb{A}^\times$ satisfies $\mathfrak{a} = a\mathcal{O}_F$ and $a_v = 1$ for $v \mid \infty$ then the Hecke operator $T_\mathfrak{a}$ satisfies [35, Proposition 3.1.4]

$$B(1; T_\mathfrak{a}\phi) = \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a}) \cdot B(a; \phi).$$

2.2. **Eisenstein series.** For any place v of F and any subset $X \subset F_v$ let $\mathbf{1}_X$ denote the characteristic function of X. Let $\mathcal{S}(\mathbb{A}^2)$ denote the space of Schwartz functions on $\mathbb{A}^2$ and fix $\Omega \in \mathcal{S}(\mathbb{A}^2)$. Given a pair $\eta = (\eta_1, \eta_2)$ of quasi-characters of $\mathbb{A}^\times/F^\times$ we define

$$f_{\Omega,\eta,s}(g) = |\det(g)|^s \eta_1(\det(g)) \int_{\mathbb{A}^\times} \Omega\big([0,t] \cdot g\big) |t|^{2s} \eta_1(t) \eta_2(t^{-1})\, d^\times t$$

for s a complex variable and $g \in \mathrm{GL}_2(\mathbb{A})$. Then $f_{\Omega,\eta,s}$ lies in the space of the induced representation $\mathcal{B}(\eta_1|\cdot|^{s-1/2}, \eta_2|\cdot|^{1/2-s})$ of [34, §2.2]. The Eisenstein series defined by the meromorphic continuation of

$$E_{\Omega,\eta,s}(g) = \sum_{\gamma \in B(F)\backslash \mathrm{GL}_2(F)} f_{\Omega,\eta,s}(\gamma g)$$

is an automorphic form with central character $\eta_1\eta_2$. If we set $w_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ then according to [34, §3.3] $E_{\Omega,\eta,s}(g)$ has constant term

$$C_{\Omega,\eta,s}(g) = f_{\Omega,\eta,s}(g) + \int_{\mathbb{A}} f_{\Omega,\eta,s}\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\, dx$$

and Whittaker function

$$W_{\Omega,\eta,s}(g) = \int_{\mathbb{A}} f_{\Omega,\eta,s}\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-x)\, dx.$$

To fix a particular Eisenstein series we let $\mathfrak{r}$ be an $\mathcal{O}_F$-ideal relatively prime to $\mathfrak{d}$ and choose $r \in \mathbb{A}^\times$ so that $r\mathcal{O}_F = \mathfrak{r}$ and $r_v = 1$ for $v \mid \infty$. Define a Schwartz function $\Omega_\mathfrak{r} = \prod \Omega_{\mathfrak{r},v}$ on $\mathbb{A}^2$ by

$$\Omega_{\mathfrak{r},v}(x,y) = \begin{cases} \mathbf{1}_{\mathfrak{r}_v}(x)\mathbf{1}_{\mathcal{O}_{F,v}}(y) & \text{if } v \nmid \mathfrak{d}\infty \\ \omega_v(y)\mathbf{1}_{\mathfrak{d}_v}(x)\mathbf{1}_{\mathcal{O}_{F,v}^\times}(y) & \text{if } v \mid \mathfrak{d} \\ (ix+y)e^{-\pi(x^2+y^2)} & \text{if } v \mid \infty. \end{cases}$$

Taking $\eta = (1,\omega)$ we abbreviate

$$E_{\mathfrak{r},s}(g) = E_{\Omega_\mathfrak{r},\eta,s}(g) \qquad f_{\mathfrak{r},s}(g) = f_{\Omega_\mathfrak{r},\eta,s}(g).$$

Proposition 2.2.1. *Fix $a \in \mathbb{A}^\times$ and set $\mathfrak{a} = a\mathcal{O}_F$. There is a product expansion*

$$B(a;E_{\mathfrak{r},s}) = \prod B_v(a,E_{\mathfrak{r},s})$$

over all places v of F, in which the local factors are given as follows.

(a) *If v is a finite place which does not divide $\mathfrak{d}$ then for any uniformizing parameter ϖ of F_v*

$$B_v(a;E_{\mathfrak{r},s}) = \omega_v(\delta)\cdot|a|_v^s\cdot|\delta|_v^{s-1/2}\sum_{j=0}^{\mathrm{ord}_v(\mathfrak{a}\mathfrak{r}^{-1})}|\varpi^j|_v^{1-2s}\omega_v(\varpi^j).$$

if $\mathrm{ord}_v(\mathfrak{a}) \geq \mathrm{ord}_v(\mathfrak{r})$, and otherwise $B_v(a;E_{\mathfrak{r},s}) = 0$.

(b) *If $v \mid \mathfrak{d}$ then*

$$B_v(a;E_{\mathfrak{r},s}) = \begin{cases} \omega_v(\delta)|ad|_v^s\cdot|\delta d|_v^{s-1/2}\epsilon_v(1/2,\omega_v,\psi_v^0) & \text{if } \mathrm{ord}_v(\mathfrak{a}) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_v(a;h_vE_{\mathfrak{r},1-s}) = \omega_v(-a)|d|_v^{3/2-3s}|\delta|_v^{1-2s}\epsilon_v(1/2,\omega,\psi_v^0)^{-1}\cdot B_v(a;E_{\mathfrak{r},s})$$

where $\epsilon_v(1/2,\omega,\psi_v^0)$ is the usual local epsilon factor as in [19, §3].

(c) *If v is archimedean then*

$$B_v(a;E_{\mathfrak{r},s}) = \omega_v(a\delta)|a|_v^{1-s}|\delta|_v^{s-1/2}\frac{\Gamma(s+1/2)}{\pi^{s+1/2}}V_s(-a_v)$$

where for $t \in \mathbb{R}$

$$V_s(t) = \int_\mathbb{R}\frac{e^{-2\pi itx}}{(i+x)(1+x^2)^{s-1/2}}\,d^{\mathrm{Leb}}x.$$

Proof. For v nonarchimedean these formulas are found in Lemmas 3.3.2 and 3.3.3 of [34]. For v archimedean see [34, Lemma 3.3.4]. At each place our formulas differ from Zhang's by a factor of $\omega_v(-1)$. As $\omega(-1) = 1$ this local factor does not change the value of $B(a;E_{\mathfrak{r},s})$. □

Proposition 2.2.2. *The Eisenstein series $E_{\mathfrak{r},s}(g)$ satisfies the functional equation*

$$E_{\mathfrak{r},s}(g) = E_{\mathfrak{r},1-s}(gh_S)\cdot(-i)^{[F:\mathbb{Q}]}|r\delta|^{2s-1}|d|^{3s-3/2}\omega(r\cdot\det g).$$

Proof. See §3.2 of [34], especially (3.2.1) and Lemmas 3.2.3 and 3.2.4. □

Let $L(s,\omega)=\prod_v L_v(s,\omega)$ be the usual Dirichlet L-function attached to ω, including the gamma factors $L_v(s,\omega)=G_1(s+1)$ for $v\mid\infty$. Writing $L(s,\omega)$ as the quotient of the completed Dedekind ζ-functions of E and F and using the functional equation and residue formulas of [33, VII.6] gives the functional equation

$$L(s,\omega)=|d\delta|^{s-1/2}\cdot L(1-s,\omega) \tag{2.2}$$

and the special value formula

$$L(0,\omega)=\frac{H_E}{H_F}\cdot[\mathcal{O}_E^\times:\mathcal{O}_F^\times]^{-1}\cdot 2^{[F:\mathbb{Q}]-1} \tag{2.3}$$

in which H_F and H_E are the class numbers of F and E, respectively.

Proposition 2.2.3. *Fix $a\in\mathbb{A}^\times$ and set $\alpha=\begin{pmatrix}a\delta^{-1}&\\&1\end{pmatrix}$. For any $T\subset S$*

$$f_{\mathfrak{r},s}(\alpha h_T)=\begin{cases}|a|^s|\delta|^{-s}L(2s,\omega) & \text{if } T=\emptyset\\ 0 & \text{otherwise.}\end{cases}$$

Furthermore if $T=S$ then

$$\begin{aligned}&\int_{\mathbb{A}} f_{\mathfrak{r},s}\left(w_0\begin{pmatrix}1&x\\&1\end{pmatrix}\alpha h_T\right)\,dx\\ &\quad= i^{[F:\mathbb{Q}]}\omega(a\delta)\omega(\mathfrak{r})|r|^{2s-1}|a|^{1-s}|\delta|^{3s-2}|d|^{3(s-1/2)}\cdot L(2-2s,\omega),\end{aligned}$$

and otherwise the integral is 0.

Proof. Let v be a place of F and, if v is finite, let ϖ be a uniformizing parameter of F_v. We may factor $f_{\mathfrak{r},s}=\prod_v f_{\mathfrak{r},s,v}$ where

$$f_{\mathfrak{r},s,v}(g)=|\det(g)|_v^s\int_{F_v^\times}\Omega_{\mathfrak{r},v}\big([0,t]\cdot g\big)|t|_v^{2s}\omega_v(t)\,d^\times t.$$

For any place v one easily computes

$$f_{\mathfrak{r},s,v}(\alpha)=|a|_v^s\cdot|\delta|_v^{-s}\cdot L_v(2s,\omega)$$

and, if $v\in S$,

$$f_{\mathfrak{r},s,v}(\alpha h_v)=|a\delta^{-1}r|_v^s\int_{F_v^\times}\Omega_{\mathfrak{r},v}(-rt,0)|t|_v^{2s}\omega_v(t)\,d^\times t$$

which vanishes as $\Omega_{\mathfrak{r},v}(-rt,0)=0$. This proves the first claim. If v is a finite place with $v\nmid\mathfrak{d}$ then

$$\begin{aligned}&\int_{F_v} f_{\mathfrak{r},s,v}\left(w_0\begin{pmatrix}1&x\\&0\end{pmatrix}\alpha\right)\,dx\\ &\quad= |a\delta^{-1}|_v^s\int_{F_v^\times}\mathbf{1}_{\mathfrak{r}_v}(ta\delta^{-1})\left(\int_{F_v}\mathbf{1}_{\mathcal{O}_{F,v}}(tx)\,dx\right)|t|_v^{2s}\omega_v(t)\,d^\times t\\ &\quad= |a|_v^s|\delta|_v^{1/2-s}\int_{F_v^\times}\mathbf{1}_{\mathfrak{r}_v}(ta\delta^{-1})|t|_v^{2s-1}\omega_v(t)\,d^\times t\\ &\quad= \omega_v(a\delta)|a|_v^{1-s}|\delta|_v^{s-1/2}|r|_v^{2s-1}\omega_v(r)L_v(2s-1,\omega).\end{aligned}$$

If $v\mid\mathfrak{d}$ then by (2.1)

$$\int_{F_v}\mathbf{1}_{\mathcal{O}_{F,v}^\times}(tx)\omega_v(x)\,dx=|\delta|_v^{1/2}(1-|\varpi|_v)\int_{F_v^\times}\mathbf{1}_{\mathcal{O}_{F,v}^\times}(tx)\omega_v(x)|x|_v\,d^\times x.$$

The integral on the right vanishes, and hence so does

$$\begin{aligned}
&\int_{F_v} f_{\mathfrak{r},s,v}\left(w_0\begin{pmatrix}1 & x\\ & 0\end{pmatrix}\alpha\right)\,dx\\
&\quad= |a\delta^{-1}|_v^s\int_{F_v}\int_{F_v^\times}\Omega_{\mathfrak{r},v}(-ta\delta^{-1},-tx)|t|_v^{2s}\omega_v(t)\,d^\times t\,dx\\
&\quad= |a\delta^{-1}|_v^s\int_{F_v^\times}\mathbf{1}_{\mathfrak{d}_v}(ta\delta^{-1})\left(\int_{F_v}\mathbf{1}_{\mathcal{O}_{F,v}^\times}(tx)\omega_v(x)\,dx\right)|t|_v^{2s}\,d^\times t.
\end{aligned}$$

Still assuming $v\mid\mathfrak{d}$,

$$\begin{aligned}
&\int_{F_v} f_{\mathfrak{r},s,v}\left(w_0\begin{pmatrix}1 & x\\ & 0\end{pmatrix}\alpha h_v\right)\,dx\\
&\quad= |ad\delta^{-1}|_v^s\int_{F_v^\times}\left(\int_{F_v}\mathbf{1}_{\mathcal{O}_{F,v}}(tx)\,dx\right)\mathbf{1}_{\mathcal{O}_{F,v}^\times}(ta\delta^{-1})|t|_v^{2s}\omega_v(-a\delta)\,d^\times t\\
&\quad= |a|_v^{1-s}|d|_v^s|\delta|_v^{s-1/2}\omega_v(-a\delta)\int_{F_v^\times}\mathbf{1}_{\mathcal{O}_{F,v}^\times}(ta\delta^{-1})\,d^\times t\\
&\quad= \omega_v(-a\delta)|a|_v^{1-s}|\delta|_v^{s-1/2}|d|_v^s.
\end{aligned}$$

Finally, if v is archimedean then

$$\begin{aligned}
&\int_{F_v} f_{\mathfrak{r},s,v}\left(w_0\begin{pmatrix}1 & x\\ & 0\end{pmatrix}\alpha\right)\,dx\\
&\quad= -|a|_v^s|\delta|_v^{1/2-s}\int_{\mathbb{R}}\int_{\mathbb{R}^\times} t(a\delta^{-1}i+x)e^{-\pi(ta\delta^{-1})^2}e^{-\pi(tx)^2}|t|_v^{2s}\omega_v(t)\,d^\times t\,d^{\mathrm{Leb}}x\\
&\quad= i\cdot\omega_v(-a\delta)|a|_v^{s+1}|\delta|_v^{-1/2-s}\int_{\mathbb{R}^\times}e^{-\pi(ta\delta^{-1})^2}|t|_v^{2s+1}\left(\int_{\mathbb{R}}e^{-\pi(tx)^2}\,d^{\mathrm{Leb}}x\right)\,d^\times t\\
&\quad= i\cdot\omega_v(-a\delta)|a|_v^{s+1}|\delta|_v^{-1/2-s}\int_{\mathbb{R}^\times}e^{-\pi(ta\delta^{-1})^2}|t|_v^{2s}\,d^\times t\\
&\quad= i\cdot\omega_v(-a\delta)|a|_v^{1-s}|\delta|_v^{s-1/2}\pi^{-s}\Gamma(s).
\end{aligned}$$

Putting everything together gives

$$\begin{aligned}
&\int_{\mathbb{A}} f_s\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\alpha h_T\right)\,dx\\
&\quad= \begin{cases} i^{[F:\mathbb{Q}]}\omega(a\delta)\omega(r)|r|^{2s-1}|a|^{1-s}|\delta|^{s-1/2}|d|^s\cdot L(2s-1,\omega) & \text{if } T=S\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

and the second claim now follows from the functional equation (2.2). □

2.3. **Theta series.** As in [17, §12] or [34, §2.2] (see also §12.6.1 and §12.6.5 of [27], and the references therein) there is an irreducible automorphic representation Π_χ of $\mathrm{GL}_2(\mathbb{A})$ of central character $\omega\chi_0$ and conductor $\mathfrak{d}\mathrm{N}(\mathfrak{C})$ characterized by $L(s,\Pi_\chi)=L(s,\chi)$, where the right hand side is the Dirichlet L-function of χ including the gamma factors $L_v(s,\chi)=G_2(s)$ for $v\mid\infty$. Denote by $\theta_\chi\in\Pi_\chi$ the normalized newvector and define

$$\theta(g)=\theta_\chi\left(g\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\right)$$

so that θ has parallel weight -1.

Proposition 2.3.1. *Fix $a \in \mathbb{A}^\times$. The Whittaker coefficient $B(a;\theta)$ admits a product decomposition $B(a;\theta) = \prod_v B_v(a;\theta)$ over all places of F in which the local factors are given as follows. Let v be a place of F, and if v is finite let ϖ be a uniformizing parameter of F_v.*

(a) *If v is finite and inert in K then*

$$B_v(a;\theta) = |a|_v^{1/2} \cdot \begin{cases} \chi_v(\varpi)^{\frac{1}{2}\mathrm{ord}_v(a)} & \text{if } \mathrm{ord}_v(a) \geq 0,\ \mathrm{ord}_v(a) \text{ even},\ \chi_v \text{ unramified} \\ 1 & \text{if } \mathrm{ord}_v(a) = 0,\ \chi_v \text{ ramified} \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If v is finite and splits in K then identify $E_v^\times \cong F_v^\times \times F_v^\times$. Set $\alpha = 0$ if the restriction of χ_v to the first factor is ramified, and $\alpha = \chi_v(\varpi, 1)$ otherwise. Set $\beta = 0$ if the restriction of χ_v to the second factor is ramified, and $\beta = \chi_v(1, \varpi)$ otherwise. Then*

$$B_v(a;\theta) = |a|_v^{1/2} \sum_{\substack{i+j=\mathrm{ord}_v(\mathfrak{a}) \\ i,j\geq 0}} \alpha^i \beta^j.$$

Here we adopt the convention that $0^0 = 1$ in case one or both of α, β is 0.

(c) *If $v \mid \mathfrak{d}$ (so that χ_v is unramified) let ϖ_E denote a uniformizer of E_v. Then*

$$B_v(a;\theta) = |a|_v^{1/2} \cdot \begin{cases} \chi_v(\varpi_E)^{\mathrm{ord}_v(a)} & \text{if } \mathrm{ord}_v(a) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(d) *If v is archimedean then $B_v(a;\theta) = |a|_v^{1/2} e_v(-a)$.*

Proof. When χ_0 is trivial this is a restatement of Lemmas 3.3.6 and 3.3.7 of [34]. The proof of the general case is identical. □

Proposition 2.3.2. *The local Whittaker coefficients of θ satisfy*

$$\begin{aligned} \omega_v(a)B_v(a;\theta) &= B_v(a;\theta) && \text{if } v \nmid \mathfrak{d}\cdot\infty \\ \omega_v(a)B_v(a;\theta) &= -B_v(a;\theta) && \text{if } v \mid \infty \\ \omega_v(a)B_v(a;h_v\theta) &= \chi_v(\mathfrak{D})\epsilon_v(1/2,\omega,\psi_v^0)\cdot B_v(a;\theta) && \text{if } v \mid \mathfrak{d}. \end{aligned}$$

Furthermore θ satisfies the global functional equation

$$\theta(g) = \theta(gh_S)\cdot\omega(\det g)\cdot\overline{\chi}(\mathfrak{D})\cdot(-i)^{[F:\mathbb{Q}]}.$$

Proof. When χ_0 is trivial this is [34, Lemma 3.2.5], and the proof of the general case is identical. □

Lemma 2.3.3. *Let $\chi^*(t) = \chi(\overline{t})$ where $t \mapsto \overline{t}$ is the nontrivial involution of E/F, extended to $\mathbb{A}_E^\times$. The following are equivalent*

(a) *Π_χ is noncuspidal*
(b) *there is a character $\nu : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ such that $\chi = \nu \circ \mathrm{N}$*
(c) *$\chi^* = \chi$.*

Proof. If (b) does not hold then Π_χ is cuspidal by [17, Proposition 12.1]. Conversely, if (b) does hold then comparing L-functions we see that Π_χ is isomorphic to (indeed, is defined as) the principal series $\Pi(\nu,\nu\omega)$, hence is noncuspidal. Thus (a) and (b) are equivalent. The equivalence of (b) and (c) is a consequence of Hilbert's theorem 90. □

Lemma 2.3.4. *Assume that $\mathfrak{C} = \mathcal{O}_E$ and that the equivalent conditions of Lemma 2.3.3 hold. Then*

$$\nu(\det g) \cdot E_{\mathcal{O}_F,1/2}(g) = (-1)^{[F:\mathbb{Q}]}|d|^{1/2}\theta(g) \tag{2.4}$$

where $E_{\mathcal{O}_F,s}$ is the Eisenstein series of §2.2 with $\mathfrak{r} = \mathcal{O}_F$.

Proof. As in the proof of Lemma 2.3.3, Π_χ is isomorphic to $\Pi(\nu, \nu\omega)$, and so is generated by $\nu(\det g)E_{\mathcal{O}_F,1/2}(g)$. As both $\theta(g)$ and $\nu(\det g)E_{\mathcal{O}_F,1/2}(g)$ are $K_1(\mathfrak{d})$-fixed and of parallel weight -1, they must be scalar multiples of one another. To compute the scalar we compute Whittaker coefficients. For any $a \in \mathbb{A}^\times$, comparing Propositions 2.2.1 and 2.3.1 gives

$$B_v(a; E_{\mathcal{O}_F,1/2}) = \overline{\nu}_v(a)\omega_v(a\delta)B_v(a; h_v\theta) \cdot \begin{cases} \overline{\chi}_v(\mathfrak{D})|d|_v^{1/2} & \text{if } v \nmid \infty \\ i & \text{if } v \mid \infty \end{cases}$$

Using Proposition 2.3.1 we see that both sides of (2.4) have the same Whittaker coefficients. □

2.4. **The kernel Θ.** For each $v \in S$ set $\sigma_{s,v} = 1 + \overline{\chi}_v(\mathfrak{D})|d|_v^{1/2-s}h_v$ and define the *symmetrized kernel*

$$\begin{aligned} \Theta_{\mathfrak{r},s}(g) &= \left(\prod_{v\in S}\sigma_{s,v}\right) \cdot \big[\theta(g)E_{\mathfrak{r},s}(g)\big] \\ &= \sum_{T\subset S} \overline{\chi}_T(\mathfrak{D})|d|_T^{1/2-s}\theta(gh_T)E_{\mathfrak{r},s}(gh_T) \end{aligned}$$

where the subscript T indicates the product over all $v \in T$; e.g. $\chi_T = \prod_{v\in T}\chi_v$. For every place v of F define

$$\epsilon_v(s, \mathfrak{r}, \psi) = |\delta|_v^{2s-1} \cdot \begin{cases} -1 & \text{if } v \mid \infty \\ \omega_v(r)|r|_v^{2s-1} & \text{if } v \mid \mathfrak{r} \\ |d|_v^{2s-1} & \text{otherwise} \end{cases} \tag{2.5}$$

and set $\epsilon(s, \mathfrak{r}) = \prod_v \epsilon_v(s, \mathfrak{r}, \psi)$, so that

$$\epsilon(s, \mathfrak{r}) = (-1)^{[F:\mathbb{Q}]}\omega(\mathfrak{r})\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}\mathfrak{r})^{1-2s}D_F^{1-2s}.$$

Combining Propositions 2.2.2 and 2.3.2 gives the relation

$$\theta(g)E_{\mathfrak{r},s}(g) = \epsilon(s, \mathfrak{r})|d|^{s-1/2}\overline{\chi}(\mathfrak{D}) \cdot \theta(gh_S)E_{\mathfrak{r},1-s}(gh_S)$$

and hence

$$\left(\prod_{v\in S}\sigma_{s,v}\right)\big[\theta(g)E_{\mathfrak{r},s}(g)\big] = \epsilon(s, \mathfrak{r})\left(\prod_{v\in S}\overline{\chi}_v(\mathfrak{D})|d|_v^{s-1/2}\sigma_{s,v}h_v\right)\big[\theta(g)E_{\mathfrak{r},1-s}(g)\big].$$

For $v \in S$ the operator h_v^2 acts as $\chi_{0,v}(\mathfrak{d}) = \chi_v(\mathfrak{D})^2$ on automorphic forms of central character χ_0. Thus we may replace the expression $\overline{\chi}_v(\mathfrak{D})|d|_v^{s-1/2}\sigma_{s,v}h_v$ with $\sigma_{1-s,v}$ to arrive at the functional equation

$$\Theta_{\mathfrak{r},s}(g) = \epsilon(s, \mathfrak{r}) \cdot \Theta_{\mathfrak{r},1-s}(g). \tag{2.6}$$

As in [34, §3.3], multiplying the Fourier expansions of $\theta(g)$ and $E_{\mathfrak{r},s}(g)$ shows that the product $\theta(g) \cdot E_{\mathfrak{r},s}(g)$ has constant term

$$\mathbf{C}_{\mathfrak{r},s}(g) = C_\theta(g)C_{\mathfrak{r},s}(g) + \sum_{\substack{\eta,\xi\in F^\times \\ \eta+\xi=0}} W_\theta\left(\begin{pmatrix}\eta & \\ & 1\end{pmatrix}g\right)W_{\mathfrak{r},s}\left(\begin{pmatrix}\xi & \\ & 1\end{pmatrix}g\right)$$

and Whittaker function

$$\begin{aligned}\mathbf{W}_{\mathfrak{r},s}(g) &= C_\theta(g)W_{\mathfrak{r},s}(g) + C_{\mathfrak{r},s}(g)W_\theta(g)\\ &\quad + \sum_{\substack{\eta,\xi\in F^\times\\ \eta+\xi=1}} W_\theta\left(\begin{pmatrix}\eta & \\ & 1\end{pmatrix} g\right) W_{\mathfrak{r},s}\left(\begin{pmatrix}\xi & \\ & 1\end{pmatrix} g\right).\end{aligned}$$

From the Fourier expansion of $\theta(g)E_{\mathfrak{r},s}(g)$ and the definition of the symmetrized kernel we find the decomposition

$$B(a;\Theta_{\mathfrak{r},s}) = A_0(a;\Theta_{\mathfrak{r},s}) + A_1(a;\Theta_{\mathfrak{r},s}) + \sum_{\substack{\eta,\xi\in F^\times\\ \eta+\xi=1}} B(a,\eta,\xi;\Theta_{\mathfrak{r},s}) \tag{2.7}$$

in which the terms on the right hand side are defined by

$$\begin{aligned}A_0(a;\Theta_{\mathfrak{r},s}) &= \sum_{T\subset S} \overline{\chi}_T(\mathfrak{D})|d|_T^{1/2-s} W_\theta(\alpha h_T)C_{\mathfrak{r},s}(\alpha h_T)\\ A_1(a;\Theta_{\mathfrak{r},s}) &= \sum_{T\subset S} \overline{\chi}_T(\mathfrak{D})|d|_T^{1/2-s} C_\theta(\alpha h_T)W_{\mathfrak{r},s}(\alpha h_T)\\ B(a,\eta,\xi;\Theta_{\mathfrak{r},s}) &= \sum_{T\subset S} \overline{\chi}_T(\mathfrak{D})|d|_T^{1/2-s} B(\eta a;h_T\theta)B(\xi a;h_T E_{\mathfrak{r},s})\end{aligned}$$

where we have abbreviated $\alpha = \begin{pmatrix} a\delta^{-1} & \\ & 1\end{pmatrix}$. If we define

$$\begin{aligned}&B_v(a,\eta,\xi;\Theta_{\mathfrak{r},s})\\ &= B_v(\eta a;\theta)\cdot\begin{cases} B_v(\xi a;E_{\mathfrak{r},s}) & \text{if } v\nmid\mathfrak{d}\\ B_v(\xi a;E_{\mathfrak{r},s}) + \omega_v(-\eta\xi)|d\delta|_v^{2s-1}B_v(\xi a;E_{\mathfrak{r},1-s}) & \text{if } v\mid\mathfrak{d}\end{cases}\end{aligned}$$

then the local functional equations of Propositions 2.2.1 and 2.3.1 imply the factorization

$$B(a,\eta,\xi;\Theta_{\mathfrak{r},s}) = \prod_v B_v(a,\eta,\xi;\Theta_{\mathfrak{r},s}).$$

Lemma 2.4.1. *For every place v of F, every $a\in\mathbb{A}^\times$, and every $\eta,\xi\in F^\times$,*

$$B_v(a,\eta,\xi;\Theta_{\mathfrak{r},s}) = \omega_v(-\eta\xi)\epsilon_v(s,\mathfrak{r},\psi)\cdot B_v(a,\eta,\xi;\Theta_{\mathfrak{r},1-s}).$$

Proof. This follows from direct examination of the explicit formulas of Propositions 2.2.1 and 2.3.1. For $v\mid\infty$ one also uses the functional equation satisfied by $V_s(t)$ found in [14, Proposition IV.3.3 (c)]. □

Proposition 2.4.2. *Suppose $\eta,\xi\in F^\times$, $\eta+\xi=1$, and $\omega_v(-\eta\xi) = \epsilon_v(1/2,\mathfrak{r},\psi)$. Fix $a\in\mathbb{A}^\times$ and abbreviate, here and later, $\Theta_{\mathfrak{r}} = \Theta_{\mathfrak{r},1/2}$.*

(a) *If v is a finite place which is split in E then*

$$B_v(a,\eta,\xi;\Theta_{\mathfrak{r}}) = |a|_v|\eta\xi|_v^{1/2}\omega_v(\delta)\big(\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{r}^{-1})+1\big)\sum_{\substack{i+j=\mathrm{ord}_v(\eta\mathfrak{a})\\ i,j\geq 0}}\alpha^i\beta^j$$

if $\mathrm{ord}_v(\eta\mathfrak{a})$ and $\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{r}^{-1})$ are nonnegative, and is 0 otherwise. Here α and β are as in Proposition 2.3.1.

(b) *Suppose v is a finite place which is inert in E. If χ_v is unramified then*

$$B_v(a,\eta,\xi;\Theta_{\mathfrak{r}}) = |a|_v|\eta\xi|_v^{1/2}\omega_v(\delta)\chi_v(\varpi)^{\frac{1}{2}\mathrm{ord}_v(\eta a)}$$

if $\mathrm{ord}_v(\eta\mathfrak{a})$ *and* $\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{r}^{-1})$ *are even and nonnegative, and is* 0 *otherwise. If χ_v is ramified then*

$$B_v(a,\eta,\xi;\Theta_{\mathfrak{r}}) = |a|_v|\eta\xi|_v^{1/2}\omega_v(\delta)$$

if $\mathrm{ord}_v(\eta\mathfrak{a}) = 0$ *and* $\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{r}^{-1})$ *is even and nonnegative, and is* 0 *otherwise.*

(c) *If $v \mid \mathfrak{d}$ then*

$$B_v(a,\eta,\xi;\Theta_{\mathfrak{r}}) = 2\chi_v(\varpi_E)^{\mathrm{ord}_v(\eta a)}\omega_v(\delta)|\eta\xi d|_v^{1/2}|a|_v\epsilon_v(1/2,\omega_v,\psi_v^0)$$

if $\mathrm{ord}_v(\eta\mathfrak{a})$ *and* $\mathrm{ord}_v(\xi\mathfrak{a})$ *are nonnegative, and is* 0 *otherwise.*

(d) *If v is archimedean then*

$$B_v(a,\eta,\xi;\Theta_{\mathfrak{r}}) = 2i|\eta\xi|_v^{1/2}|a|_v\omega_v(\delta)\cdot e_v(-a).$$

Proof. This follows from Propositions 2.2.1 and 2.3.1. For $v \mid \infty$ one also uses the special value formula for $V_{1/2}(t)$ found in [14, Proposition IV.3.3 (d)], which implies

$$B_v(a;E_{\mathfrak{r},1/2}) = -i|a|_v^{1/2}\omega_v(\delta)\cdot e_v(-a).$$

□

2.5. **The Rankin-Selberg L-function.** Recall the automorphic representation Π of $\mathrm{GL}_2(\mathbb{A})$ of §1.1 and assume Hypothesis 1.1.1. Fix a Haar measure on $\mathrm{GL}_2(\mathbb{A}_f)$ and let Z denote the center of GL_2. Setting $F_\infty = F\otimes_{\mathbb{Q}}\mathbb{R}$ we identify

$$\mathrm{GL}_2(F_\infty)/Z(F_\infty)\mathrm{SO}_2(F_\infty) \cong \mathcal{H}^{[F:\mathbb{Q}]}$$

in the usual way, where $\mathcal{H} = \mathbb{C}-\mathbb{R}$ is the union of the upper and lower half-planes equipped with the hyperbolic volume form $y^{-2}dxdy$. Suppose F_0 and F_1 are two automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ for which $F_0\overline{F_1}$ is a square integrable function on $\mathrm{GL}_2(F)\backslash\mathcal{H}^{[F:\mathbb{Q}]}\times\mathrm{GL}_2(\mathbb{A}_f)/Z(\mathbb{A}_f)$. If $K\subset\mathrm{GL}_2(\mathbb{A}_f)$ is a compact open subgroup we define the Petersson inner product of level K

$$\langle F_0,F_1\rangle_K = \mathrm{Vol}(K)^{-1}\int_{\mathrm{GL}_2(F)\backslash\mathcal{H}^{[F:\mathbb{Q}]}\times\mathrm{GL}_2(\mathbb{A}_f)/Z(\mathbb{A}_f)} F_0\overline{F_1}$$

where the quotient measure is induced by the Haar measure on $Z(\mathbb{A}_f)$ giving $\widehat{\mathcal{O}}_F^\times$ volume 1. For any $b\in\mathbb{A}^\times$ with trivial archimedean components set $R_b = \begin{pmatrix} b^{-1} & \\ & 1\end{pmatrix}$ and view R_b as an operator on automorphic forms by $(R_b\phi)(g) = \phi(gR_b)$. Let $\mathfrak{b}$ be an ideal of $\mathcal{O}_F$ dividing $\mathfrak{d}\mathfrak{c}^2\mathfrak{s}^{-1}$ and fix $b\in\mathbb{A}^\times$ with trivial archimedean components and $b\mathcal{O}_F = \mathfrak{b}$. Let $L(s,\Pi\times\Pi_\chi)$ be the Rankin-Selberg L-function defined as in [34, §2.5] (see also [27, §12.6.2] and the references therein).

Proposition 2.5.1. *Let $\phi_\Pi\in\Pi$ be the normalized newvector and set $\mathfrak{r} = \mathfrak{m}\mathfrak{c}^2$. Assume that Π_v is a discrete series of weight 2 for each $v\mid\infty$. Then*

$$\mathrm{Vol}(K_0(\mathfrak{d}\mathfrak{r}))^{-1}\int \phi_\Pi(gR_b)\theta(g)E_{\mathfrak{r},s}(g)\ dg = |\delta|^{1/2-s}|b|^{s-1}B(b;\theta)L(s,\Pi\times\Pi_\chi).$$

Proof. Hypothesis 1.1.1 implies that for every finite place v either Π_v or $\Pi_{\chi,v}$ is a principal series. Hence the claim follows from Propositions 2.5.1 and 2.5.2 of [34]. □

Under the notation and assumptions of Proposition 2.5.1, a direct calculation as in [34, Lemma 3.1.2] gives

$$\langle R_b\phi_\Pi, \overline{\Theta}_{\mathfrak{r},s}\rangle_{K_0(\mathfrak{d}\mathfrak{r})} = L(s,\Pi\times\Pi_\chi)\cdot|\delta|^{1/2-s}\prod_{v|\mathfrak{d}\mathfrak{c}}\gamma_{s,v}(b) \tag{2.8}$$

where

$$\gamma_{s,v}(b) = |b|_v^{-1/2}B_v(b;\theta)\begin{cases}|b|_v^{s-1/2}+|b|_v^{1/2-s} & \text{if } v\mid\mathfrak{d}\\ 1 & \text{if } v\mid\mathfrak{c}.\end{cases}$$

2.6. **Central derivatives and holomorphic projection.** Throughout 2.6 we assume that $\epsilon(1/2,\mathfrak{r})=-1$. For any $\eta,\xi\in F^\times$ with $\eta+\xi=1$ define the *difference set*

$$\mathrm{Diff}_\mathfrak{r}(\eta,\xi) = \{\text{places } v \text{ of } F \mid \omega_v(-\eta\xi)\neq\epsilon_v(1/2,\mathfrak{r},\psi)\}.$$

Note that the cardinality of $\mathrm{Diff}_\mathfrak{r}(\eta,\xi)$ is odd, and that Lemma 2.4.1 implies that $B_v(a,\eta,\xi,\Theta_\mathfrak{r})=0$ for each $v\in\mathrm{Diff}_\mathfrak{r}(\eta,\xi)$. In particular $B(a,\eta,\xi;\Theta_\mathfrak{r})=0$. Note also that $\mathrm{Diff}_\mathfrak{r}(\eta,\xi)$ contains only places which are nonsplit in E, as v split implies that both $\omega_v(-\eta\xi)$ and $\epsilon_v(1/2,\mathfrak{r},\psi)$ are equal to 1. Define

$$\Theta'_\mathfrak{r}(g) = \frac{d}{ds}\Theta_{\mathfrak{r},s}(g)\bigg|_{s=1/2}$$

and, with notation as in (2.7), abbreviate

$$\begin{aligned}A_i(a;\Theta'_\mathfrak{r}) &= \frac{d}{ds}A_i(a;\Theta_{\mathfrak{r},s})\big|_{s=1/2}\\ B(a,\eta,\xi,\Theta'_\mathfrak{r}) &= \frac{d}{ds}B(a,\eta,\xi;\Theta_{\mathfrak{r},s})\big|_{s=1/2}\end{aligned}$$

and similarly with $B(\cdot)$ replaced by $B_v(\cdot)$. For t a positive real number define

$$q_0(t) = \int_1^\infty e^{-xt}\,d^\times x.$$

Proposition 2.6.1. *If $w\in\mathrm{Diff}_\mathfrak{r}(\eta,\xi)$ then*

$$B(a,\eta,\xi;\Theta'_\mathfrak{r}) = B_w(a,\eta,\xi,\Theta'_\mathfrak{r})\cdot\prod_{v\neq w}B_v(a,\eta,\xi;\Theta_\mathfrak{r}).$$

The value of $B_w(a,\eta,\xi,\Theta'_\mathfrak{r})$ is given as follows.

(a) *Suppose $w\nmid\infty$ is inert in E. If χ_w is unramified then*

$$B_w(a,\eta,\xi,\Theta'_\mathfrak{r}) = \omega_w(\delta)|\eta\xi|_w^{1/2}|a|_w\log|\xi ar^{-1}\varpi|_w\chi_w(\varpi)^{\frac12\mathrm{ord}_v(a\eta)}$$

if $\mathrm{ord}_w(\eta a)$ is even and nonnegative and $\mathrm{ord}_w(\xi ar^{-1})$ is odd and nonnegative; otherwise the left hand side is 0. If χ_w is ramified then

$$B_w(a,\eta,\xi,\Theta'_\mathfrak{r}) = \omega_w(\delta)|\eta\xi|_w^{1/2}|a|_w\log|\xi ar^{-1}\varpi|_w$$

if $\mathrm{ord}_w(\eta a)=0$ and $\mathrm{ord}_w(\xi ar^{-1})$ is odd and nonnegative; otherwise the left hand side is 0.

(b) *If $w\nmid\infty$ is ramified in E then*

$$B_w(a,\eta,\xi,\Theta'_\mathfrak{r}) = 2\omega_w(\delta)|\eta\xi|_w^{1/2}|a|_w|d|_w^{1/2}\chi_w(\varpi_E)^{\mathrm{ord}_w(\eta\mathfrak{a})}\cdot\epsilon_w(\omega,\psi_w^0)\cdot\log|\xi ad|_w$$

if $\mathrm{ord}_w(\eta\mathfrak{a})$ and $\mathrm{ord}_w(\xi\mathfrak{a})$ are nonnegative; otherwise the left hand side is 0.

(c) *If* $w \mid \infty$ *then*

$$B_w(a,\eta,\xi,\Theta'_{\mathfrak{r}}) = -4i\omega_w(\delta)|\eta\xi|_w^{1/2}|a|_w e^{2\pi a_w} q_0(4\pi a_w \xi_w)$$

if $\eta_w a_w < 0$ *and* $\xi_w a_w > 0$*; otherwise the left hand side is* 0*.*

Proof. The first claim follows from Lemma 2.4.1 and the remaining claims follow from the formulas of Propositions 2.2.1 and 2.3.1, together with the equality

$$\frac{d}{ds}V_s(t)\bigg|_{s=1/2} = -2\pi i e^{-2\pi t} q_0(-4\pi t)$$

for $t < 0$, which is found in [14, Proposition IV.3.3(e)]. □

Remark 2.6.2. It follows from Lemma 2.4.1 and the first claim of Proposition 2.6.1 that $B(a,\eta,\xi;\Theta'_{\mathfrak{r}})$ vanishes unless $\mathrm{Diff}_{\mathfrak{r}}(\eta,\xi)$ consists of a single place, necessarily nonsplit in E.

Let $\Phi_{\mathfrak{r}}(g)$ be the holomorphic projection of $\overline{\Theta'_{\mathfrak{r}}(g)}$. Thus $\Phi_{\mathfrak{r}}$ is the unique holomorphic cusp form on $\mathrm{GL}_2(\mathbb{A})$ of parallel weight 2 such that $\langle\phi,\Phi_{\mathfrak{r}}\rangle_K = \langle\phi,\overline{\Theta'_{\mathfrak{r}}}\rangle_K$ for any cusp form ϕ and any compact open subgroup K. If the representation Π of §2.5 is discrete of weight 2 at every archimedean place then (2.8) implies

$$\langle\phi_\Pi,\Phi_{\mathfrak{r}}\rangle_{K_0(\mathfrak{d}\mathfrak{r})} = 2^{|S|}L'(1/2,\Pi\times\Pi_\chi).$$

We now describe the coefficients $\widehat{B}(\mathfrak{a},\Phi_{\mathfrak{r}})$ as in [34, §3.5] (see also [35, §6.4]). If w is a finite place of F define

$$\widehat{B}^w(\mathfrak{a};\Phi_{\mathfrak{r}}) = (-2i)^{[F:\mathbb{Q}]}\omega_\infty(\delta)\sum_{\eta,\xi}|\eta\xi|_\infty^{1/2}\cdot\overline{B_w(a,\eta,\xi;\Theta'_{\mathfrak{r}})}\prod_{v\nmid w\infty}\overline{B_v(a,\eta,\xi;\Theta_{\mathfrak{r}})} \tag{2.9}$$

where the sum is over all $\eta,\xi\in F^\times$ with $\eta+\xi=1$ and $\mathrm{Diff}_{\mathfrak{r}}(\eta,\xi)=\{w\}$. This sum is finite and is 0 for all but finitely many w. For $t,\sigma\in\mathbb{R}$ with $\sigma>0$ define

$$M_\sigma(t) = \begin{cases}\int_1^\infty \frac{-d^{\mathrm{Leb}}x}{x(1-tx)^{1+\sigma}} & \text{if } t<0\\ 0 & \text{otherwise.}\end{cases}$$

If $w \mid \infty$ then we set

$$\widehat{B}^w(\sigma,\mathfrak{a};\Phi_{\mathfrak{r}}) = (-2i)^{[F:\mathbb{Q}]}\omega_\infty(\delta)\sum_{\eta,\xi}|\eta\xi|_\infty^{1/2}M_\sigma(\xi_w)\cdot\prod_{v\nmid\infty}\overline{B_v(a,\eta,\xi;\Theta_{\mathfrak{r}})} \tag{2.10}$$

where the sum is over all $\eta,\xi\in F^\times$ with $\eta+\xi=1$ and $\mathrm{Diff}_{\mathfrak{r}}(\eta,\xi)=\{w\}$.

Proposition 2.6.3. *The Fourier coefficient* $\widehat{B}(\mathfrak{a};\Phi_{\mathfrak{r}})$ *decomposes as*

$$\widehat{B}(\mathfrak{a};\Phi_{\mathfrak{r}}) = A(\mathfrak{a}) + D(\mathfrak{a}) + \sum_{w\nmid\infty}\widehat{B}^w(\mathfrak{a};\Phi_{\mathfrak{r}}) + \mathrm{const}_{\sigma\to 0}\sum_{w\mid\infty}\widehat{B}^w(\sigma,\mathfrak{a};\Phi_{\mathfrak{r}})$$

in which $A(\mathfrak{a})$ *is a derivation of* $\Pi_{\overline{\chi}}\otimes|\cdot|^{1/2}$ *and* $D(\mathfrak{a})$ *is a sum of derivations of principal series in the sense of* [34, Definition 3.5.3].

Proof. When χ_0 is trivial this is exactly [34, Proposition 3.5.5], and the proof when χ_0 is nontrivial is exactly the same. □

2.7. **The weight zero kernel.** We define an automorphic form $\Theta^*_{\mathfrak{r},\mathfrak{s}}$ in exactly the same way as $\Theta_{\mathfrak{r},s}$ but replacing θ by θ_χ everywhere in the construction of §2.4. Thus

$$\Theta^*_{\mathfrak{r},s}(g) = \left(\prod_{v\in S}\sigma_{s,v}\right)\cdot\left[\theta_\chi(g)E_{\mathfrak{r},s}(g)\right]$$

is a nonholomorphic form of parallel weight 0. Using the relation

$$B_v(a;\theta_\chi) = \begin{cases} B_v(a;\theta) & \text{if } v\nmid\infty \\ B_v(-a;\theta) & \text{if } v\mid\infty\end{cases}$$

and repeating the arguments of §2.4 we find that the weight zero kernel satisfies the functional equation

$$\Theta^*_{\mathfrak{r},s}(g) = (-1)^{[F:\mathbb{Q}]}\epsilon(s,\mathfrak{r})\cdot\Theta^*_{\mathfrak{r},1-s}(g)$$

and admits a decomposition

$$B(a;\Theta^*_{\mathfrak{r},s}) = A_0(a;\Theta^*_{\mathfrak{r},s}) + A_1(a;\Theta^*_{\mathfrak{r},s}) + \sum_{\substack{\eta,\xi\in F^\times\\ \eta+\xi=1}} B(a,\eta,\xi;\Theta^*_{\mathfrak{r},s})$$

in which A_0 and A_1 are defined exactly as in §2.4 but with θ replaced by θ_χ. There is a further product decomposition

$$B(a,\eta,\xi;\Theta^*_{\mathfrak{r},s}) = \prod_v B_v(a,\eta,\xi;\Theta^*_{\mathfrak{r},s})$$

where for $v\nmid\infty$ one has $B_v(a,\eta,\xi;\Theta^*_{\mathfrak{r},s}) = B_v(a,\eta,\xi;\Theta_{\mathfrak{r},s})$ while for $v\mid\infty$

$$B_v(a,\eta,\xi;\Theta^*_{\mathfrak{r},s}) = \begin{cases} -4i|a|_v|\eta\xi|_v^{1/2}\omega_v(\delta)e^{-2\pi a_v(1-2\xi_v)} & \text{if } \omega_v(-\eta\xi)=1, \xi_v a_v<0 \\ 0 & \text{otherwise.}\end{cases}$$

Assume that the representation Π of §1.1 satisfies Hypothesis 1.1.1 and is a weight 0 principal series for every archimedean v. The Rankin-Selberg L-function $L(s,\Pi\times\Pi_\chi)$ is defined exactly as in §2.5, but with the archimedean factors now given by [36, (5.4)]. With notation as in Proposition 2.5.1 one again has the integral representation of the Rankin-Selberg L-function

$$\langle R_b\phi_\Pi, \overline{\Theta^*_{\mathfrak{r},s}}\rangle_{K_0(\mathfrak{d}\mathfrak{r})} = L(s,\Pi\times\Pi_\chi)\cdot|\delta|^{1/2-s}\prod_{v\mid\mathfrak{d}\mathfrak{c}}\gamma_{s,v}(b) \tag{2.11}$$

exactly as in (2.8).

2.8. **The quasi-new line.** Suppose the representation Π of §1.1 satisfies Hypothesis 1.1.1 and is unitary. Set $\mathfrak{r}=\mathfrak{m}\mathfrak{c}^2$. Fix a place v of F dividing $\mathfrak{d}\mathfrak{c}$ and a uniformizer ϖ of F_v. As Π_v has conductor $\mathfrak{s}_v=\mathfrak{n}_v$, [34, Proposition 2.3.1] implies that the space of $K_1(\mathfrak{r}_v)$ fixed vectors of Π_v is finite dimensional with basis

$$\{R_{\varpi^k}\phi_{\Pi,v}\mid 0\le k\le \mathrm{ord}_v(\mathfrak{r}\mathfrak{s}^{-1})\}$$

where $\phi_{\Pi,v}$ is any newvector in Π_v and R_b is as in §2.5. Define a linear functional Λ_v on this finite dimensional vector space by the condition

$$\Lambda_v(R_{\varpi^k}\phi_{\Pi,v}) = \gamma_{\frac12,v}(\varpi^k)$$

where, in the notation of (2.8),

$$\gamma_{\frac12,v}(b) = |b|_v^{-1/2}B_v(b;\theta)\begin{cases} 2 & \text{if } v\mid\mathfrak{d} \\ 1 & \text{if } v\mid\mathfrak{c}.\end{cases}$$

Definition 2.8.1. If $v \mid \mathfrak{d}\mathfrak{c}$ then the *quasi-new line* in Π_v is the orthogonal complement, in the space of $K_1(\mathfrak{r}_v)$ fixed vectors, of the kernel of Λ_v. If $v \nmid \mathfrak{d}\mathfrak{c}$ then the *quasi-new line* is defined to be the span of the newvectors in Π_v, i.e. the line of $K_1(\mathfrak{m}_v) = K_1(\mathfrak{r}_v)$ fixed vectors. The quasi-new line in $\Pi = \bigotimes_v \Pi_v$ is the tensor product of the local quasi-new lines, and a *quasi-newform* in Π is any nonzero vector on the quasi-new line.

Proposition 2.8.2. *Assume that either* Π *or* Π_χ *is cuspidal and that* Π_v *is discrete of weight* 2 *at each archimedean* v*. The projection of* $\overline{\Theta_{\mathfrak{r}}(g)}$ *to* Π *lies on the quasi-new line; if, in addition,* $\epsilon(1/2, \mathfrak{r}) = -1$ *then the projection of* $\Phi_{\mathfrak{r}}(g)$ *to* Π *lies on the quasi-new line. If we instead assume that* Π *has weight* 0 *at every archimedean place then the projection of* $\overline{\Theta^*_{\mathfrak{r}}(g)}$ *to* Π *lies on the quasi-new line.*

Proof. There is an evident global characterization of the quasi-new line in Π: for each $\mathfrak{b} \mid \mathfrak{r}\mathfrak{s}^{-1}$ fix $b \in \mathbb{A}^\times$ with $b\mathcal{O}_F = \mathfrak{b}$. The set $\{R_b\phi_\Pi \mid \mathfrak{b} \text{ divides } \mathfrak{r}\mathfrak{s}^{-1}\}$ is a basis for the space of $K_1(\mathfrak{r})$-fixed vectors in Π, and the quasi-new line is the orthogonal complement (in the $K_1(\mathfrak{r})$-fixed vectors) of the kernel of the linear functional Λ defined by

$$\Lambda(R_b\phi_\Pi) = \prod_{v \mid \mathfrak{d}\mathfrak{c}} \gamma_{\frac{1}{2},v}(b).$$

In the weight 2 case (2.8) implies that the projection of $\overline{\Theta}_{\mathfrak{r}}$ to Π is orthogonal to any form in the kernel of Λ, hence lies on the quasi-new line. If $\epsilon(1/2, \mathfrak{r}) = -1$ then $L(1/2, \Pi \times \Pi_\chi) = 0$ and again (2.8) shows that the projection of $\Phi_{\mathfrak{r}}$ to Π lies on the quasi-new line. In the weight 0 case one uses (2.11) in place of (2.8). □

3. CM cycles on quaternion algebras

Let B be a quaternion algebra over F and assume that there is an embedding $E \to B$, which we fix once and for all. Let T and G denote the algebraic groups over F determined by

$$T(A) = (E \otimes_F A)^\times \qquad G(A) = (B \otimes_F A)^\times$$

for any F-algebra A, and let Z denote the center of G. We denote by N both the norm $T \to Z$ and the reduced norm $G \to Z$. Let $t \mapsto \overline{t}$ be the involution of $T(\mathbb{A})$ induced by the nontrivial Galois automorphism of E/F.

3.1. **Preliminaries.** Define $B^+ = E$ and $B^- = \{b \in B \mid bt = \overline{t}b \; \forall t \in E\}$. It follows from the Noether-Skolem theorem that B^- is nontrivial, and from this one deduces that $B = B^+ \oplus B^-$ with each summand free of rank one as a left E-module. For any $\gamma \in G(F)$ the two invariants

$$\eta = \frac{\mathrm{N}(\gamma^+)}{\mathrm{N}(\gamma)} \qquad \xi = \frac{\mathrm{N}(\gamma^-)}{\mathrm{N}(\gamma)} \tag{3.1}$$

where $\gamma^\pm$ denote the projection of γ to $B^\pm$, depend only on the double coset $T(F)\gamma T(F)$ and not on γ itself. A simple calculation shows that all elements of B^- are trace-free and that $\mathrm{N}(\gamma) = \mathrm{N}(\gamma^+) + \mathrm{N}(\gamma^-)$. For any place v of F let $B_v^\pm = B^\pm \otimes_F F_v$. We say that γ is *degenerate* if $\{\eta, \zeta\} = \{0, 1\}$ (i.e. if $\gamma \in B^+ \cup B^-$), and that γ is *nondegenerate* otherwise. Of course we may make similar definitions for $\gamma \in G(F_v)$ for v any place of F.

Lemma 3.1.1. *The function $\gamma \mapsto (\eta, \xi)$ defines an injection*

$$T(F)\backslash G(F)/T(F) \to F \times F.$$

The image of this injection is the union of $\{(1,0),(0,1)\}$ and the set of pairs (η,ξ) such that $\eta, \xi \neq 0$, $\eta + \xi = 1$, and for every place v of F

$$\omega_v(-\eta\xi) = \begin{cases} 1 & \text{if } B_v \text{ is split} \\ -1 & \text{otherwise.} \end{cases} \tag{3.2}$$

Proof. This is stated without proof in [34, §4.1]. We leave the injectivity as an easy exercise, and sketch a proof of the second claim. Choose a generator ϵ for B^- as a left E-module and write $E = F[\sqrt{\Delta}]$. Then B has as an F-basis $\{1, \sqrt{\Delta}, \epsilon, \sqrt{\Delta}\cdot\epsilon\}$, or, in the standard notation (as in [4, Example A.2]), $B \cong \left(\frac{\Delta, -\mathrm{N}(\epsilon)}{F}\right)$. It follows that the right hand side of (3.2) is equal to the Hilbert symbol

$$(\Delta, -\mathrm{N}(\epsilon))_v = \omega_v(-\mathrm{N}(\epsilon)).$$

On the other hand it is easy to see that for any nondegenerate $\gamma \in G(F)$ we have $\omega_v(\eta\xi) = \omega_v(\mathrm{N}(\epsilon))$, so that (η,ξ) satisfies (3.2). The condition $\eta+\xi = 1$ is clear from the additivity of N with respect to the decomposition $B = B^+ \oplus B^-$ noted earlier. Conversely, given a pair $\eta, \xi \in F^\times$ satisfying (3.2) and $\eta + \xi = 1$ we must have $(\Delta, -\mathrm{N}(\epsilon))_v = (\Delta, -\eta\xi)_v$ for every place v. It follows from the Hasse-Minkowski theorem that there are $x, y \in F$ such that

$$\xi\eta^{-1}\mathrm{N}(\epsilon)^{-1} = x^2 - y^2\Delta.$$

Taking $\gamma = 1 + (x + y\sqrt{\Delta})\epsilon$ shows that (η,ξ) arises from a nondegenerate γ. Any degenerate γ generates either B^+ or B^- as a left E-module and so has image either $(1,0)$ or $(0,1)$, respectively. □

Lemma 3.1.2. *For any nondegenerate $\gamma \in G(F)$ and any place v of F set*

$$\tau_v(\gamma) = \omega_v(\delta)|\eta\xi|_v^{1/2}\chi_v(\eta)\overline{\chi}_v(\gamma^+)\epsilon_v(1/2, \omega, \psi_v^0).$$

Then $\prod_v \tau_v(\gamma) = 1$ where the product is over all places of F. If v is an archimedean place then $\tau_v(\gamma) = \omega_v(\delta)\cdot i \cdot |\eta\xi|_v^{1/2}$.

Proof. The functional equation (2.2) and [19, Corollary 4.4] imply $\epsilon(s,\omega) = |d\delta|^{s-1/2}$ while [19, (3.29)] gives

$$|\delta|_v^{s-1/2}\omega_v(\delta)\epsilon_v(s,\omega,\psi_v^0) = \epsilon_v(s,\omega,\psi_v).$$

From this it is clear that $\prod_v \tau_v(\gamma) = 1$. If v is archimedean then $\epsilon(s,\omega,\psi_v^0) = i$ by [19, Proposition 3.8(iii)]. As χ_v is the trivial character, the final claim follows. □

3.2. **Heights of CM-cycles.** If $U \subset G(\mathbb{A}_f)$ is a compact open subgroup we define the set of *CM points* of level U

$$C_U = T(F)\backslash G(\mathbb{A}_f)/U.$$

By a *CM-cycle* of level U we mean a compactly supported (i.e. finitely supported) function on C_U. There is a unique left $T(\mathbb{A}_f)$-invariant measure on C_U with the property that

$$\int_{G(\mathbb{A}_f)/U} f(g)\, dg = \int_{C_U} \sum_{t \in T(F)/(Z(F)\cap U)} f(tg)\, dg$$

for every locally constant compactly supported function f on $G(\mathbb{A}_f)/U$, where the measure on $G(\mathbb{A}_f)/U$ gives every coset volume one. The measure on C_U assigns to each double coset $T(F)gU$ a volume equal to the inverse of

$$[T(F)\cap gUg^{-1} : Z(F)\cap U].$$

Given compact open subgroups $U\subset V$ the measures on C_U and C_V are related by

$$\int_{C_V}\sum_{h\in V/U} f(gh)\ dg = \frac{\lambda_U}{\lambda_V}\int_{C_U} f(g)\ dg \tag{3.3}$$

for any CM-cycle f of level U, where $\lambda_U = [\mathcal{O}_F^\times : \mathcal{O}_F^\times\cap U]$ and similarly with U replaced by V.

Given a $T(F)$ bi-invariant function m on $G(F)$ define a function k_U^m on $G(\mathbb{A}_f)\times G(\mathbb{A}_f)$ by

$$k_U^m(x,y) = \sum_{\gamma\in G(F)/(Z(F)\cap U)} \mathbf{1}_U(x^{-1}\gamma y)\cdot m(\gamma) \tag{3.4}$$

where $\mathbf{1}_U$ is the characteristic function of U. We will address the convergence of this sum as the need arises; for the moment assume that the sum converges absolutely for every x,y. Note that k_U^m descends to a function on $C_U\times C_U$. If P,Q are CM-cycles of level U define the *height pairing* in level U with multiplicity m

$$\langle P,Q\rangle_U^m = \int_{C_U\times C_U} P(x)\cdot k_U^m(x,y)\cdot\overline{Q(y)}\ dx\ dy. \tag{3.5}$$

As in [34, (4.1.9)] a simple calculation shows that there is a decomposition

$$\langle P,Q\rangle_U^m = \sum_{\gamma\in T(F)\backslash G(F)/T(F)} \langle P,Q\rangle_U^\gamma\cdot m(\gamma) \tag{3.6}$$

where for every $\gamma\in G(F)$

$$\langle P,Q\rangle_U^\gamma = \int_{C_U}\sum_{\delta\in T(F)\backslash T(F)\gamma T(F)} P(\delta y)\overline{Q(y)}\ dy$$

is the *linking number* of P and Q at γ.

Abbreviate $U_Z = U\cap Z(\mathbb{A}_f)$ and $U_T = U\cap T(\mathbb{A}_f)$ and suppose now that U is small enough that χ is trivial on U_T. We will say that a CM-cycle P of level U is *χ-isotypic* if for all $t\in T(\mathbb{A}_f)$ and $g\in G(\mathbb{A}_f)$ we have $P(tg)=\chi(t)P(g)$.

Lemma 3.2.1. *Set $\chi^*(t)=\chi(\bar{t})$. Suppose P and Q are χ-isotypic CM-cycles of level U and that Q is supported on the image of $T(\mathbb{A}_f)\to C_U$. If $\gamma\in G(F)$ is degenerate then*

$$\langle P,Q\rangle_U^\gamma = \overline{Q(1)}\cdot\frac{[T(\mathbb{A}_f):T(F)U_T]}{[T(F)\cap U : Z(F)\cap U]}\begin{cases} P(\gamma) & \text{if } (\eta,\xi)=(1,0)\\ P(\gamma) & \text{if } (\eta,\xi)=(0,1) \text{ and } \chi^*=\chi\\ 0 & \text{if } (\eta,\xi)=(0,1) \text{ and } \chi^*\neq\chi.\end{cases}$$

If γ is nondegenerate then

$$\langle P,Q\rangle_U^\gamma = \overline{Q(1)}\cdot[Z(\mathbb{A}_f):Z(F)U_Z]\sum_{t\in Z(\mathbb{A}_f)\backslash T(\mathbb{A}_f)/U_T} P(t^{-1}\gamma t).$$

Proof. First suppose that γ is degenerate. Then γ normalizes $T(F)$ and so

$$\begin{aligned}\langle P,Q\rangle_U^\gamma &= \int_{C_U} P(\gamma y)\overline{Q(y)}\ dy \\ &= \int_{T(F)\backslash T(\mathbb{A}_f)/U_T} P(y^{-1}\gamma y)\overline{Q(1)}\ dy.\end{aligned}$$

If $(\eta,\xi)=(1,0)$ then $\gamma\in T(F)$ leaving

$$\langle P,Q\rangle_U^\gamma = \mathrm{Vol}(T(F)\backslash T(\mathbb{A}_f)/U_T)\cdot P(\gamma)\overline{Q(1)}.$$

If $(\eta,\xi)=(0,1)$ then $\gamma y=\overline{y}\gamma$ for every $y\in T(\mathbb{A}_f)$, leaving

$$\langle P,Q\rangle_U^\gamma = P(\gamma)\overline{Q(1)}\cdot\int_{T(F)\backslash T(\mathbb{A}_f)/U_T}\chi(y)^{-1}\chi^*(y)\ dy.$$

In either case the first claim follows. Now suppose that γ is nondegenerate. The nondegeneracy of γ implies that $\gamma^{-1}T(F)\gamma\cap T(F)=Z(F)$ and so

$$\begin{aligned}\langle P,Q\rangle_U^\gamma &= \int_{T(F)\backslash T(\mathbb{A}_f)/U_T}\sum_{\delta\in T(F)\backslash T(F)\gamma T(F)} P(y^{-1}\delta y)\overline{Q(1)}\ dy \\ &= \int_{T(F)\backslash T(\mathbb{A}_f)/U_T}\sum_{t\in T(F)/Z(F)} P(y^{-1}\gamma t y)\overline{Q(1)}\ dy \\ &= \overline{Q(1)}\int_{Z(F)\backslash T(\mathbb{A}_f)/U_T} P(y^{-1}\gamma y)\ dy\end{aligned}$$

where the measure on $Z(F)\backslash T(\mathbb{A}_f)/U_T$ gives each coset volume 1. The second claim follows. □

In particular, if the $U=\prod_v U_v$ and $P=\prod_v P_v$ of Lemma 3.2.1 are factorizable and γ is nondegenerate then there is a decomposition

$$\langle P,Q\rangle_U^\gamma = \overline{Q(1)}\cdot[Z(\mathbb{A}_f):Z(F)U_Z]\cdot\prod_v O_U^\gamma(P_v) \tag{3.7}$$

where the product is over all finite places of F and

$$O_U^\gamma(P_v) = \sum_{t\in F_v^\times\backslash E_v^\times/U_{T,v}} P_v(t^{-1}\gamma t) \tag{3.8}$$

is the *orbital integral* of P_v at γ, where we abbreviate $U_{T,v}=E_v^\times\cap U_v$.

The remainder of §3 is devoted to the computations of orbital integrals for specific CM-cycles, and we fix the following data throughout §3.3 and §3.4. Let v be a finite place of F and fix $\epsilon_v\in B_v^\times$ such that $E_v\epsilon_v=B_v^-$. We assume that $\mathrm{N}(\epsilon_v)\in\mathcal{O}_{F,v}$ and let $\mathfrak{e}$ be an ideal of $\mathcal{O}_F$ satisfying $\mathfrak{e}_v=\mathrm{N}(\epsilon_v)\mathcal{O}_{F,v}$. Define an order of B_v by

$$R_v=\mathcal{O}_{E,v}+\mathcal{O}_{E,v}\epsilon_v.$$

Fix a uniformizing parameter $\varpi\in F_v$.

3.3. **Local calculations at primes away from** $\mathrm{N}(\mathfrak{C})$**.** Assume that $v \nmid \mathrm{N}(\mathfrak{C})$ and set $U_v = R_v^\times$. Define a function on $G(F_v)/U_v$ by

$$P_{\chi,v}(g) = \sum_{t \in E_v^\times/\mathcal{O}_{E,v}^\times} \chi_v(t)\mathbf{1}_{U_v}(t^{-1}g).$$

For each ideal $\mathfrak{a} \subset \mathcal{O}_F$ set $H(\mathfrak{a}_v) = \{h \in R_v \mid \mathrm{N}(h)\mathcal{O}_{F,v} = \mathfrak{a}_v\}$ and define another function on $G(F_v)/U_v$

$$\begin{aligned} P_{\chi,\mathfrak{a},v}(g) &= \sum_{h \in H(\mathfrak{a}_v)/U_v} P_{\chi,v}(gh) \\ &= \chi_v(\mathfrak{a}) \sum_{t \in E_v^\times/\mathcal{O}_{E,v}^\times} \chi_v(t)\mathbf{1}_{H(\mathfrak{a}_v)}(t^{-1}g). \end{aligned}$$

For each nondegenerate $\gamma \in G(F_v)$ we wish to compute the orbital integral

$$O_U^\gamma(P_{\chi,\mathfrak{a},v}) = \sum_{t \in F_v^\times \backslash E_v^\times/\mathcal{O}_{E,v}^\times} P_{\chi,\mathfrak{a},v}(t^{-1}\gamma t). \tag{3.9}$$

Proposition 3.3.1. *Suppose v is inert in E and $\gamma \in G(F_v)$ is nondegenerate. Then (3.9) is nonzero if and only if* $\mathrm{ord}_v(\eta\mathfrak{a})$ *and* $\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{c}^{-1})$ *are both even and nonnegative. When this is the case*

$$O_U^\gamma(P_{\chi,\mathfrak{a},v}) = \overline{\chi}_v(\eta)\chi_v(\gamma^+)\chi_v(\varpi)^{\frac{\mathrm{ord}_v(\eta\mathfrak{a})}{2}}.$$

Proof. Suppose $\gamma^+ = 1$, so that $\gamma = 1 + \beta\epsilon_v$ with $\beta \in E_v^\times$. The expression (3.9) reduces to

$$\begin{aligned} O_U^\gamma(P_{\chi,\mathfrak{a},v}) &= P_{\chi,\mathfrak{a},v}(\gamma) \\ &= \chi_v(\mathfrak{a}) \sum_{k=-\infty}^{\infty} \chi_v(\varpi)^k \mathbf{1}_{H(\mathfrak{a}_v)}(\varpi^{-k}\gamma) \end{aligned}$$

Using $\mathrm{ord}_v(\eta) = -\mathrm{ord}_v(\mathrm{N}(\gamma))$ we see that the only possible contribution to the inner sum is for k satisfying $2k = -\mathrm{ord}_v(\eta\mathfrak{a})$. Thus we may assume that $\mathrm{ord}_v(\eta\mathfrak{a})$ is even, leaving

$$\begin{aligned} O_U^\gamma(P_{\chi,\mathfrak{a},v}) &= \chi_v(\mathfrak{a})\chi_v(\varpi)^{-\frac{1}{2}\mathrm{ord}_v(\eta\mathfrak{a})}\mathbf{1}_{H(\mathfrak{a}_v)}(\varpi^{\frac{1}{2}\mathrm{ord}_v(\eta\mathfrak{a})}\gamma) \\ &= \overline{\chi(\eta)}\chi_v(\varpi)^{\frac{1}{2}\mathrm{ord}_v(\eta\mathfrak{a})}\mathbf{1}_{R_v}(\varpi^{\frac{1}{2}\mathrm{ord}_v(\eta\mathfrak{a})}\gamma) \end{aligned}$$

which is nonzero if and only if

$$\varpi^{\frac{1}{2}\mathrm{ord}_v(\eta\mathfrak{a})}(1 + \beta\epsilon_v) \in \mathcal{O}_{E,v} + \mathcal{O}_{E,v}\epsilon_v.$$

Thus $O_U^\gamma(P_{\chi,\mathfrak{a},v})$ is nonzero if and only if both

$$\mathrm{ord}_v(\eta\mathfrak{a}) \geq 0 \qquad \mathrm{ord}_v(\eta\mathfrak{a}) \geq -\mathrm{ord}_v(\mathrm{N}(\beta))$$

hold. The observation that

$$\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{c}^{-1}) = \mathrm{ord}_v(\mathfrak{a}) + \mathrm{ord}_v(\mathrm{N}(\beta)) - \mathrm{ord}_v(\mathrm{N}(\gamma)) = \mathrm{ord}_v(\eta\mathfrak{a}) + \mathrm{ord}_v(\mathrm{N}(\beta)),$$

together with $\mathrm{ord}_v(\mathrm{N}(\beta)) \in 2\mathbb{Z}$ completes the proof when $\gamma^+ = 1$. For the general case simply note that if γ is replaced by $t\gamma$ with $t \in E_v^\times$ then both sides of the stated equality are multiplied by $\chi_v(t)$. Thus it suffices to prove the claim for a single element of $E_v^\times\gamma$. □

Remark 3.3.2. In the proof of Proposition 3.3.1 it sufficed to treat the case $\gamma^+ = 1$. This will remain true in all remaining computations of orbital integrals in §3.3 and §3.4. We will continue to state the results for arbitrary γ, but in the proofs we will assume that $\gamma^+ = 1$.

Proposition 3.3.3. *Suppose v is ramified in E and $\gamma \in G(F_v)$ is nondegenerate. Then (3.9) is nonzero if and only if $\mathrm{ord}_v(\eta\mathfrak{a})$ and $\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{c}^{-1})$ are both nonnegative. When this is the case*

$$O_U^\gamma(P_{\chi,\mathfrak{a},v}) = 2 \cdot \overline{\chi_v(\eta)}\chi_v(\gamma^+)\chi_v(\varpi_E)^{\mathrm{ord}_v(\eta\mathfrak{a})}$$

for any uniformizer $\varpi_E \in E_v$.

Proof. Write $\gamma = 1 + \beta\epsilon_v$ with $\beta \in E_v^\times$. Equation (3.9) reduces to

$$\begin{aligned} O_U^\gamma(P_{\chi,\mathfrak{a},v}) &= P_{\chi,\mathfrak{a},v}(\gamma) + P_{\chi,\mathfrak{a},v}(\varpi_E^{-1}\gamma\varpi_E) \\ &= \chi_v(\mathfrak{a}) \sum_{k=-\infty}^{\infty} \chi_v(\varpi_E)^{-k} \big[\mathbf{1}_{H(\mathfrak{a}_v)}(\varpi_E^k\gamma) + \mathbf{1}_{H(\mathfrak{a}_v)}(\varpi_E^{k-1}\gamma\varpi_E)\big]. \end{aligned}$$

The only possible contribution to the final sum is the term $k = \mathrm{ord}_v(\eta\mathfrak{a})$, leaving

$$\begin{aligned} &O_U^\gamma(P_{\chi,\mathfrak{a},v}) \\ &= \chi_v(\mathfrak{a})\chi_v(\varpi_E)^{-\mathrm{ord}_v(\eta\mathfrak{a})}\big[\mathbf{1}_{R_v}(\varpi_E^{\mathrm{ord}_v(\eta\mathfrak{a})}\gamma) + \mathbf{1}_{H(\mathfrak{a}_v)}(\varpi_E^{\mathrm{ord}_v(\eta\mathfrak{a})-1}\gamma\varpi_E)\big]. \end{aligned}$$

The remainder of the proof is exactly as the proof of Proposition 3.3.1. □

Proposition 3.3.4. *Suppose v is split in E and $\gamma \in G(F_v)$ is nondegenerate. Then (3.9) is nonzero if and only if $\mathrm{ord}(\eta\mathfrak{a})$ and $\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{c}^{-1})$ are both nonnegative. When this is the case*

$$O_U^\gamma(P_{\chi,\mathfrak{a},v}) = \overline{\chi_v(\eta)}\chi_v(\gamma^+) \cdot (1 + \mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{c}^{-1})) \sum_{\substack{i+j=\mathrm{ord}_v(\eta\mathfrak{a}) \\ i,j\geq 0}} \alpha^i\beta^j$$

where, under the identification $E_v^\times \cong F_v^\times \times F_v^\times$,

$$\alpha = \chi_v(\varpi, 1) \qquad \beta = \chi_v(1, \varpi).$$

Proof. Write $\gamma = 1 + \beta\epsilon_v$ with $\beta \in E_v^\times$, so that

$$\mathrm{ord}_v(\eta) = -\mathrm{ord}_v(\mathrm{N}(\gamma)) \quad \text{and} \quad \mathrm{ord}_v(\xi\mathfrak{c}^{-1}) = \mathrm{ord}_v(\eta) + \mathrm{ord}_v(\mathrm{N}(\beta)).$$

For any $t \in T(F_v)$

$$P_{\chi,\mathfrak{a},v}(t^{-1}\gamma t) = \chi_v(\mathfrak{a}) \sum_{s \in E_v^\times/\mathcal{O}_{E,v}^\times} \overline{\chi}_v(s) \cdot \mathbf{1}_{H(\mathfrak{a}_v)}(st^{-1}\gamma t),$$

and the only terms in the final sum which may contribute are from those s satisfying $\mathrm{ord}_v(\mathrm{N}(s)) = \mathrm{ord}_v(\eta a)$. Fix an isomorphism $\mathcal{O}_{E,v} \cong \mathcal{O}_{F,v} \times \mathcal{O}_{F,v}$ and set $e_{i,j} = (\varpi^i, \varpi^j)$. Then

$$P_{\chi,\mathfrak{a},v}(t^{-1}\gamma t) = \chi_v(\mathfrak{a}) \sum_{i+j=\mathrm{ord}_v(\eta\mathfrak{a})} \alpha^{-i}\beta^{-j}\mathbf{1}_{R_v}(e_{i,j}t^{-1}\gamma t). \tag{3.10}$$

The set $\{e_{k,0} \mid k \in \mathbb{Z}\}$ is a complete set of coset representatives for $F_v^\times\backslash E_v^\times/\mathcal{O}_{E,v}^\times$, and

$$e_{k,0}^{-1} \cdot \gamma \cdot e_{k,0} = e_{k,0}^{-1}(1 + \beta\epsilon_v)e_{k,0} = 1 + e_{-k,k}\beta\epsilon_v.$$

Combining (3.9) and (3.10) therefore gives

$$O_U^\gamma(P_{\chi,\mathfrak{a},v}) = \chi_v(\mathfrak{a}) \sum_{i+j=\mathrm{ord}_v(\eta\mathfrak{a})} \alpha^{-i}\beta^{-j} \sum_{k=-\infty}^{\infty} \mathbf{1}_{R_v}(e_{i,j}(1+e_{-k,k}\beta\epsilon_v)). \tag{3.11}$$

The inner sum counts the number of k such that

$$e_{i,j} + e_{i-k,j+k}\beta\epsilon_v \in \mathcal{O}_{E,v} + \mathcal{O}_{E,v}\epsilon_v.$$

When $i+j = \mathrm{ord}_v(\eta\mathfrak{a})$ the condition $e_{i,j} \in \mathcal{O}_{E,v}$ is equivalent to $0 \le i, j \le \mathrm{ord}_v(\eta a)$, and so the outer sum may be restricted to $i, j \ge 0$. The inner sum then counts the number of k such that

$$e_{i-k,j+k}\beta \in \mathcal{O}_{E,v}.$$

Replacing β by an $\mathcal{O}_{E,v}^\times$-multiple does not change the number of such k, and so we may assume that $\beta = e_{s,t}$ for some $s, t \in \mathbb{Z}$. The inner sum of (3.11) is then equal to

$$\begin{aligned} \#\{k \in \mathbb{Z} \mid i-k+s \ge 0,\ j+k+t \ge 0\} &= i+j+s+t+1 \\ &= \mathrm{ord}_v(\eta\mathfrak{a}) + \mathrm{ord}_v(\mathrm{N}(\beta)) + 1 \\ &= \mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{e}^{-1}) + 1 \end{aligned}$$

if $\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{e}^{-1}) \ge 0$, and is equal to 0 otherwise. Thus (3.11) reduces to

$$O_U^\gamma(P_{\chi,\mathfrak{a},v}) = \chi_v(\mathfrak{a})(\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{e}^{-1}) + 1) \sum_{\substack{i+j=\mathrm{ord}_v(\eta\mathfrak{a}) \\ i,j\ge 0}} \alpha^{-i}\beta^{-j}$$

when $\mathrm{ord}_v(\xi\mathfrak{a}\mathfrak{e}^{-1}) \ge 0$. Using $\chi_v(\eta\mathfrak{a}) = \alpha^{i+j}\beta^{i+j}$, the proposition follows. □

Corollary 3.3.5. *Suppose $v \nmid \mathrm{N}(\mathfrak{C})$, $\gamma \in G(F_v)$ is nondegenerate, and $\mathfrak{r}$ is an ideal of $\mathcal{O}_F$ with $\mathfrak{r}_v = \mathfrak{e}_v$. Then*

$$|a|_v |d|_v^{1/2} \tau_v(\gamma) \cdot O_U^\gamma(P_{\chi,\mathfrak{a},v}) = B_v(a, \eta, \xi; \Theta_\mathfrak{r})$$

where $\tau_v(\gamma)$ is as in Lemma 3.1.2.

Proof. Propositions 3.3.1, 3.3.3, and 3.3.4 give explicit formulas for the left hand side, while Proposition 2.4.2 gives explicit formulas for the right hand side. □

We now turn to the calculation of $P_{\chi,\mathfrak{a},v}(1)$ and $P_{\chi,\mathfrak{a},v}(\epsilon_v)$.

Lemma 3.3.6. *Suppose that v is inert in E. Then*

$$P_{\chi,\mathfrak{a},v}(1) = \begin{cases} \chi_v(\varpi)^{\frac{1}{2}\mathrm{ord}_v(\mathfrak{a})} & \text{if } \mathrm{ord}_v(\mathfrak{a}) \text{ is even and nonnegative} \\ 0 & \text{otherwise} \end{cases}$$

$$P_{\chi,\mathfrak{a},v}(\epsilon_v) = \begin{cases} \chi_v(\mathfrak{e})\chi_v(\varpi)^{\frac{1}{2}\mathrm{ord}_v(\mathfrak{a}\mathfrak{e}^{-1})} & \text{if } \mathrm{ord}_v(\mathfrak{a}\mathfrak{e}^{-1}) \text{ is even and nonnegative} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Exactly as in Proposition 3.3.1

$$P_{\chi,\mathfrak{a},v}(g) = \chi_v(\mathfrak{a}) \sum_{k=-\infty}^{\infty} \chi_v(\varpi)^{-k} \mathbf{1}_{H(\mathfrak{a}_v)}(\varpi^k g).$$

If $g = 1$ then $\mathrm{ord}_v(\mathrm{N}(g)) = 0$ and the only contribution to the final sum is when $2k = \mathrm{ord}_v(\mathfrak{a})$. Thus we may assume that $\mathrm{ord}_v(\mathfrak{a})$ is even, leaving

$$P_{\chi,\mathfrak{a},v}(1) = \chi_v(\varpi)^{\frac{1}{2}\mathrm{ord}_v(a)} \mathbf{1}_{R_v}(\varpi^{\frac{1}{2}\mathrm{ord}_v(a)})$$

which proves the first claim. If $g = \epsilon_v$ then $\mathrm{ord}_v(\mathrm{N}(g)) = \mathrm{ord}_v(\mathfrak{e})$ and the only contribution to the above sum is for k satisfying $2k + \mathrm{ord}_v(\mathfrak{e}) = \mathrm{ord}_v(\mathfrak{a})$. Thus we may assume $\mathrm{ord}_v(\mathfrak{a}\mathfrak{e}^{-1})$ is even, leaving

$$P_{\chi,\mathfrak{a},v}(\epsilon_v) = \chi_v(\mathfrak{a})\chi_v(\varpi)^{-\frac{1}{2}\mathrm{ord}_v(\mathfrak{a}\mathfrak{e}^{-1})}\mathbf{1}_{R_v}(\varpi^{\frac{1}{2}\mathrm{ord}_v(\mathfrak{a}\mathfrak{e}^{-1})}\epsilon_v)$$

which proves the second claim. □

Lemma 3.3.7. *Suppose that v is ramified in E, and let ϖ_E be a uniformizer of E_v. Then*

$$\begin{aligned} P_{\chi,\mathfrak{a},v}(1) &= \begin{cases} \chi_v(\varpi_E)^{\mathrm{ord}_v(\mathfrak{a})} & \text{if } \mathrm{ord}_v(\mathfrak{a}) \geq 0 \\ 0 & \text{otherwise} \end{cases} \\ P_{\chi,\mathfrak{a},v}(\epsilon_v) &= \begin{cases} \chi_v(\mathfrak{e})\chi_v(\varpi_E)^{\mathrm{ord}_v(\mathfrak{a}\mathfrak{e}^{-1})} & \text{if } \mathrm{ord}_v(\mathfrak{a}\mathfrak{e}^{-1}) \geq 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. The proof is nearly identical to that of Lemma 3.3.6, and the details are omitted. □

Lemma 3.3.8. *Suppose that v is split in E, and let α and β be as in Proposition 3.3.4. Then*

$$\begin{aligned} P_{\chi,\mathfrak{a},v}(1) &= \sum_{\substack{i+j=\mathrm{ord}_v(\mathfrak{a}) \\ i,j\geq 0}} \alpha^i\beta^j \\ P_{\chi,\mathfrak{a},v}(\epsilon_v) &= \chi_v(\mathfrak{e}) \sum_{\substack{i+j=\mathrm{ord}_v(\mathfrak{a}\mathfrak{e}^{-1}) \\ i,j\geq 0}} \alpha^i\beta^j. \end{aligned}$$

Proof. On the right hand side of

$$P_{\chi,\mathfrak{a},v}(g) = \chi_v(\mathfrak{a}) \sum_{t\in E_v^\times/\mathcal{O}_{E,v}^\times} \overline{\chi}_v(t)\mathbf{1}_{H(\mathfrak{a}_v)}(tg)$$

the only terms which may contribute are from those t satisfying

$$\mathrm{ord}_v(\mathrm{N}(t)) = \mathrm{ord}_v(\mathfrak{a}) - \mathrm{ord}_v(\mathrm{N}(g)).$$

Fix an isomorphism $\mathcal{O}_{E,v} \cong \mathcal{O}_{F,v} \times \mathcal{O}_{F,v}$ and set $e_{i,j} = (\varpi^i, \varpi^j)$. Then

$$P_{\chi,\mathfrak{a},v}(g) = \chi_v(\mathfrak{a}) \sum_{i+j=\mathrm{ord}_v(a)-\mathrm{ord}_v(\mathrm{N}(g))} \alpha^{-i}\beta^{-j}\mathbf{1}_{R_v}(e_{i,j}g).$$

The lemma follows easily from this equality, using $\alpha\beta = \chi_v(\varpi)$. □

Corollary 3.3.9. *Suppose v does not divide* $\mathrm{N}(\mathfrak{C})$ *and that $a \in \mathbb{A}^\times$ satisfies $a\mathcal{O}_F = \mathfrak{a}$. Then*

$$P_{\chi,\mathfrak{a},v}(1) = |a|_v^{-1/2}B_v(a;\theta).$$

If we pick $e \in \mathbb{A}^\times$ such that $e\mathcal{O}_F = \mathfrak{e}$ then

$$P_{\chi,\mathfrak{a},v}(\epsilon_v) = \chi_v(e)|e|_v^{1/2}|a|_v^{-1/2}B_v(ae^{-1};\theta)$$

Proof. Compare Lemmas 3.3.6, 3.3.7, and 3.3.8 with Proposition 2.3.1. □

3.4. **Local calculations at primes dividing** $\mathrm{N}(\mathfrak{C})$**.** Let v be a finite place of F dividing $\mathrm{N}(\mathfrak{C})$ (in particular $v \nmid \mathfrak{d}$). Assume that

$$\mathrm{ord}_v(\mathrm{N}(\mathfrak{C})) \le \mathrm{ord}_v(\mathfrak{e}) \tag{3.12}$$

and let $U_v \subset R_v^\times$ be the kernel of the homomorphism $R_v^\times \to (\mathcal{O}_{E,v}/\mathfrak{C}_v)^\times$ given by $x + y\epsilon_v \mapsto x$. Define a function $P_{\chi,v}$ on $G(F_v)$ by

$$P_{\chi,v}(g) = \sum_{t \in E_v^\times/U_{T,v}} \chi_v(t)\mathbf{1}_{U_v}(t^{-1}g).$$

For each nondegenerate $\gamma \in G(F_v)$ we wish to compute the orbital integral

$$O_U^\gamma(P_{\chi,v}) = \sum_{t \in F_v^\times \backslash E_v^\times/U_{T,v}} P_{\chi,v}(t^{-1}\gamma t). \tag{3.13}$$

In accordance with Remark 3.3.2 we will state the results for any nondegenerate γ but will assume in the proofs that $\gamma^+ = 1$ and write $\gamma = 1 + \beta\epsilon_v$ with $\beta \in E_v^\times$.

Proposition 3.4.1. *Suppose v is inert in E and $\gamma \in G(F_v)$ is nondegenerate. Then (3.13) is nonzero if and only if $\mathrm{ord}_v(\eta) = 0$ and $\mathrm{ord}_v(\xi\mathfrak{e}^{-1})$ is even and nonnegative. When this is the case*

$$O_U^\gamma(P_{\chi,v}) = [\mathcal{O}_{E,v}^\times : \mathcal{O}_{F,v}^\times U_{T,v}] \cdot \chi_v(\gamma^+).$$

Proof. In this case (3.13) gives

$$\begin{aligned} O_U^\gamma(P_{\chi,v}) &= \sum_{t \in \mathcal{O}_{F,v}^\times \backslash \mathcal{O}_{E,v}^\times/U_{T,v}} P_{\chi,v}(t^{-1}\gamma t) \\ &= \sum_{t \in \mathcal{O}_{F,v}^\times \backslash \mathcal{O}_{E,v}^\times/U_{T,v}} \sum_{s \in E_v^\times/U_{T,v}} \chi_v(s)\mathbf{1}_{U_v}(s^{-1}t^{-1}\gamma t). \end{aligned}$$

As $U_v = U_{T,v} + \mathcal{O}_{E,v}\epsilon_v$, the only way that $s^{-1}t^{-1}\gamma t = s^{-1}(1 + t^{-1}\bar{t}\beta\epsilon_v)$ can lie in U_v is if $s \in U_{T,v}$. Therefore only the term $s = 1$ contributes to the inner sum, leaving

$$O_U^\gamma(P_{\chi,v}) = \sum_{t \in \mathcal{O}_{F,v}^\times \backslash \mathcal{O}_{E,v}^\times/U_{T,v}} \mathbf{1}_{U_v}(1 + t^{-1}\bar{t}\beta\epsilon_v).$$

If $\mathrm{ord}_v(\mathrm{N}(\beta)) \ge 0$ then every term in the sum is 1, and otherwise every term is 0. As

$$\mathrm{ord}_v(\xi\mathfrak{e}^{-1}) = \mathrm{ord}_v(\eta) + \mathrm{ord}_v(\mathrm{N}(\beta))$$

the condition $\mathrm{ord}_v(\mathrm{N}(\beta)) \ge 0$ is equivalent to $\mathrm{ord}_v(\xi\mathfrak{e}^{-1}) \ge \mathrm{ord}_v(\eta)$, and using $\eta + \xi = 1$ and $\mathrm{ord}_v(\mathfrak{e}) > 0$

$$\mathrm{ord}_v(\xi\mathfrak{e}^{-1}) \ge \mathrm{ord}_v(\eta) \iff \mathrm{ord}_v(\eta) = 0 \text{ and } \mathrm{ord}_v(\xi\mathfrak{e}^{-1}) \ge 0.$$

□

Proposition 3.4.2. *Suppose v is split in E and $\gamma \in G(F_v)$ is nondegenerate. Then (3.13) is nonzero if and only if $\mathrm{ord}_v(\eta) = 0$ and $\mathrm{ord}_v(\xi\mathfrak{e}^{-1}) \ge 0$. When this is the case*

$$O_U^\gamma(P_{\chi,v}) = [\mathcal{O}_{E,v}^\times : \mathcal{O}_{F,v}^\times U_{T,v}] \cdot \chi_v(\gamma^+)(1 + \mathrm{ord}_v(\xi\mathfrak{e}^{-1})).$$

Proof. Using the notation of Proposition 3.3.4, so that $e_{i,j} = (\varpi^i, \varpi^j)$, for any $t \in T(F_v)$ we have

$$\begin{aligned} P_{\chi,v}(t^{-1}\gamma t) &= \sum_{s \in E_v^\times / U_{T,v}} \overline{\chi}_v(s) \cdot \mathbf{1}_{U_v}(st^{-1}\gamma t) \\ &= \sum_{i,j \in \mathbb{Z}} \sum_{s \in \mathcal{O}_{E,v}^\times / U_{T,v}} \overline{\chi}_v(se_{i,j}) \cdot \mathbf{1}_{U_v}(se_{i,j}(1 + t^{-1}\overline{t}\beta\epsilon_v)) \end{aligned}$$

As $U_v = U_{T,v} + \mathcal{O}_{E,v}\epsilon_v$, only terms for which $se_{i,j} \in U_{T,v}$ can contribute to the inner sum, and so the only nonzero term can be the one with $i = j = 0$ and $s \in U_{T,v}$. This leaves

$$P_\chi(t^{-1}\gamma t) = \mathbf{1}_{U_v}(1 + t^{-1}\overline{t}\beta\epsilon_v)$$

and so (3.13) becomes

$$\begin{aligned} O_U^\gamma(P_{\chi,v}) &= \sum_{t \in F_v^\times \backslash E_v^\times / U_{T,v}} \mathbf{1}_{U_v}(1 + t^{-1}\overline{t}\beta\epsilon_v) \\ &= \sum_{k=-\infty}^{\infty} \sum_{t \in \mathcal{O}_{F,v}^\times \backslash \mathcal{O}_{E,v}^\times / U_{T,v}} \mathbf{1}_{U_v}(1 + e_{-k,k}t^{-1}\overline{t}\beta\epsilon_v) \\ &= [\mathcal{O}_{E,v}^\times : \mathcal{O}_{F,v}^\times U_{T,v}] \cdot \sum_{k=-\infty}^{\infty} \mathbf{1}_{U_v}(1 + e_{-k,k}\beta\epsilon_v). \end{aligned}$$

Every term in the final sum is 0 unless the quantity

$$\mathrm{N}(1 + e_{-k,k}\beta\epsilon_v) = \mathrm{N}(\gamma) = \eta^{-1}$$

lies in $\mathcal{O}_F^\times$. Thus we may assume $\mathrm{ord}_v(\eta) = 0$, so that the sum simply counts the number of k for which $e_{-k,k}\beta \in \mathcal{O}_{E,v}$. Multiplying β by an element of $\mathcal{O}_{E,v}^\times$, we may assume that $\beta = e_{s,t}$ for some $s, t \in \mathbb{Z}$. The k for which $e_{-k,k}\beta \in \mathcal{O}_{E,v}$ holds are then precisely those for which $s - k \geq 0$ and $t + k \geq 0$, and there are

$$s + t + 1 = \mathrm{ord}_v(\mathrm{N}(\beta)) + 1 = \mathrm{ord}_v(\xi\mathfrak{e}^{-1}) + 1$$

such k if $\mathrm{ord}_v(\xi\mathfrak{e}^{-1}) \geq 0$, and no such k otherwise. □

Corollary 3.4.3. *Suppose v divides $\mathrm{N}(\mathfrak{C})$ and γ is nondegenerate. Then*

$$\tau_v(\gamma) \cdot O_U^\gamma(P_{\chi,v}) = [\mathcal{O}_{E,v}^\times : \mathcal{O}_{F,v}^\times U_{T,v}] \cdot B_v(1, \eta, \xi; \Theta_\mathfrak{r})$$

where $\tau_v(\gamma)$ is as in Lemma 3.1.2 and $\mathfrak{r}_v = \mathfrak{e}_v$.

Proof. Propositions 3.4.1 and 3.4.2 give explicit formulas for the left hand side while Proposition 2.4.2 gives explicit formulas for the right hand side. □

Lemma 3.4.4. *We have the equalities $P_{\chi,v}(1) = 1$ and $P_{\chi,v}(\epsilon_v) = 0$.*

Proof. Clearly $P_{\chi,v}(1) = 1$ simply by definition of $P_{\chi,v}$. On the other hand

$$P_{\chi,v}(\epsilon_v) = \sum_{t \in T(F_v)/U_{T,v}} \chi_v(t^{-1})\mathbf{1}_{U_v}(t\epsilon_v).$$

If this sum is nonzero then $t\epsilon_v \in R_v^\times$ for some $t \in T(F_v)$. But this would imply both $\mathrm{N}(t\epsilon_v) \in \mathcal{O}_{F,v}^\times$ and $t\epsilon_v \in \mathcal{O}_{E,v}\epsilon_v$, which implies $\mathrm{ord}_v(\mathrm{N}(\epsilon_v)) \leq 0$. But $\mathrm{ord}_v(\mathrm{N}(\epsilon_v)) = \mathrm{ord}_v(\mathfrak{e}) > 0$ by (3.12), a contradiction. □

Corollary 3.4.5. *Choose* $e \in \mathbb{A}^\times$ *with* $e\mathcal{O}_F = \mathfrak{c}$. *Then*

$$P_{\chi,v}(1) = B_v(1;\theta) \qquad P_{\chi,v}(\epsilon_v) = \chi_v(e)|e|_v^{1/2} B_v(e^{-1};\theta).$$

Proof. Compare Lemma 3.4.4 with Proposition 2.3.1. □

4. Central values

Suppose the representation Π of §1.1 satisfies Hypothesis 1.1.1. Recall that Π has conductor $\mathfrak{n} = \mathfrak{m}\mathfrak{s}$ and that χ has conductor $\mathfrak{C} = \mathfrak{c}\mathcal{O}_E$ for some $\mathcal{O}_F$-ideal $\mathfrak{c}$. Let B be a quaternion algebra over F satisfying

$$B_v \text{ is split} \iff \epsilon_v(1/2, \mathfrak{r}, \psi) = 1 \tag{4.1}$$

for every *finite* place v of F, where $\mathfrak{r} = \mathfrak{m}\mathfrak{c}^2$ and the local epsilon factor is defined by (2.5). This implies that the reduced discriminant of B divides $\mathfrak{m}$ and, as E_v is a field whenever B_v is nonsplit, that there is an embedding $E \to B$ which we fix. For the moment we do not specify the behavior of B at archimedean places. Let G and T be the algebraic groups over F defined at the beginning of §3. For any ideal $\mathfrak{b} \subset \mathcal{O}_F$ let $\mathcal{O}_\mathfrak{b} = \mathcal{O}_F + \mathfrak{b}\mathcal{O}_E$ denote the order of $\mathcal{O}_E$ of conductor $\mathfrak{b}$.

4.1. **Special CM cycles.** We construct two particular compact open subgroups $U \subset V$ of $G(\mathbb{A}_f)$ and two special CM-cycles Q_χ and P_χ of level V and U, respectively. It is ultimately the cycle Q_χ in which we are interested, but the local orbital integrals (3.8) of cycles of level V seem too difficult to compute directly. The subgroup U is chosen to make these orbital integrals more readily computable (indeed, they have already been computed in §3.3 and 3.3).

Lemma 4.1.1. *For every finite place* v *there is an order in* B_v *of reduced discriminant* $\mathfrak{m}_v$ *which contains* $\mathcal{O}_{E,v}$. *Such an order is unique up to* $E_v^\times$*-conjugacy.*

Proof. If v is inert in E then (4.1) implies that

$$\mathrm{ord}_v(\mathfrak{m}) \equiv \mathrm{ord}_v(\mathrm{disc}(B_v)) \pmod 2$$

where $\mathrm{disc}(B_v)$ is the reduced discriminant of B_v. Thus the lemma follows from [11, Proposition 3.4]. □

If v is a place of F dividing $\mathfrak{c}$ then, in particular, $v \nmid \mathfrak{d}\mathfrak{m}$ and $B_v \cong M_2(F_v)$. Let W_v denote a two dimensional F_v-vector space on which B_v acts on the left. As W_v is free of rank one over E_v, we may choose $w_0 \in W_v$ such that $W_v = E_v \cdot w_0$. For each rank two $\mathcal{O}_{F,v}$-submodule $\Lambda_v \subset W_v$ set

$$\mathcal{O}(\Lambda_v) = \{b \in B_v \mid b \cdot \Lambda_v \subset \Lambda_v\},$$

a maximal order of B_v. As $\mathfrak{s} \mid \mathfrak{c}$ by Hypothesis 1.1.1 we may consider the two lattices in W_v

$$L'_v = \mathcal{O}_{\mathfrak{c},v} w_0 \qquad L_v = \mathcal{O}_{\mathfrak{c}\mathfrak{s}^{-1},v} w_0.$$

Choose a global order $S \subset B$ such that $S_v = \mathcal{O}(L_v) \cap \mathcal{O}(L'_v)$ for every place $v \mid \mathfrak{c}$ and such that for every finite place $v \nmid \mathfrak{c}$, S_v has reduced discriminant $\mathfrak{m}_v$ and contains $\mathcal{O}_{E,v}$ (which can be done by Lemma 4.1.1). The group $\widehat{S}^\times$ acts on $\prod_{v \mid \mathfrak{c}} L_v/L'_v \cong \mathcal{O}_F/\mathfrak{s}$ through a homomorphism $\vartheta : \widehat{S}^\times \to (\mathcal{O}_F/\mathfrak{s})^\times$, and we define V to be the kernel of ϑ. One should regard $V \subset G(\mathbb{A}_f)$ as a quaternion analogue of the congruence subgroup $K_0(\mathfrak{m}) \cap K_1(\mathfrak{s})$. Define a CM-cycle of level V

$$Q_\chi(g) = \begin{cases} \chi(t) & \text{if } g = tv \text{ for some } t \in T(\mathbb{A}_f),\ v \in V \\ 0 & \text{otherwise.} \end{cases}$$

For this definition to make sense we need to know that χ is trivial on $T(\mathbb{A}_f) \cap V$. This is immediate from the following

Lemma 4.1.2. *We have $\widehat{\mathcal{O}}_{\mathfrak{c}}^{\times} = T(\mathbb{A}_f) \cap \widehat{S}^{\times}$, and $\chi_0 \circ \vartheta$ and χ have the same restriction to $\widehat{\mathcal{O}}_{\mathfrak{c}}^{\times}$.*

Proof. For $v \nmid \mathfrak{c}$ a finite place of F, $\mathcal{O}_{\mathfrak{c},v} \subset S_v$. As $\mathcal{O}_{\mathfrak{c},v}$ is a maximal order in E_v we must therefore have $\mathcal{O}_{\mathfrak{c},v} = E_v \cap S_v$. For $v \mid \mathfrak{c}$ it follows from $\mathcal{O} = \{x \in E_v \mid x\mathcal{O} \subset \mathcal{O}\}$ for any order $\mathcal{O} \subset E_v$ that

$$\mathcal{O}_{\mathfrak{c},v} = \mathcal{O}_{\mathfrak{c},v} \cap \mathcal{O}_{\mathfrak{c}\mathfrak{s}^{-1},v} = E_v \cap \mathcal{O}(L_v) \cap \mathcal{O}(L'_v) = E_v \cap S_v,$$

proving the first claim. For the second claim, if $v \nmid \mathfrak{s}$ then both ϑ_v and χ_v are trivial on $\mathcal{O}_{\mathfrak{c},v}^{\times} = \mathcal{O}_{F,v}^{\times}(1 + \mathfrak{c}\mathcal{O}_{E,v})^{\times}$. If $v \mid \mathfrak{s}$ then $\vartheta_v : \mathcal{O}_{\mathfrak{c},v}^{\times} \to (\mathcal{O}_{F,v}/\mathfrak{s}_v)^{\times}$ is given by $\vartheta_v(x(1+cy)) = x$ for $x \in \mathcal{O}_{F,v}^{\times}$, $y \in \mathcal{O}_{E,v}$, and $c \in \mathcal{O}_{F,v}$ satisfying $c\mathcal{O}_{F,v} = \mathfrak{c}_v$. Thus

$$(\chi_{0,v} \circ \vartheta)(x(1+cy)) = \chi_{0,v}(x) = \chi_v(x) = \chi_v(x(1+cy)).$$

□

Lemma 4.1.3. *For every finite place v there is an $\epsilon_v \in B_v$ satisfying*

(a) $E_v \epsilon_v = B_v^-$
(b) $\mathrm{ord}_v(\mathrm{N}(\epsilon_v)) = \mathrm{ord}_v(\mathfrak{r})$
(c) *If $v \nmid \mathfrak{c}$ then $\epsilon_v \in S_v$*
(d) *if $v \mid \mathfrak{c}$ then $\epsilon_v w_0 \in \mathfrak{c}\mathcal{O}_{E,v} w_0$.*

Proof. First fix an ϵ_v which generates B_v^- as a left E_v-module. If v is split or ramified in E then we may multiply ϵ_v on the left by an element of $E_v^{\times}$ to ensure that (b) holds. If v is inert in E then it follows from the proof of Lemma 3.1.1 that $\omega_v(\mathrm{N}(\epsilon_v))$ is 1 if B_v is split and is -1 if B_v is ramified. Condition (4.1) then implies that $\omega_v(\mathrm{N}(\epsilon_v)) = \omega_v(\mathfrak{r})$, and so again we may multiply ϵ_v on the left by an element of $E_v^{\times}$ so that (b) holds. Assume now that $v \nmid \mathfrak{c}$ and define an order $R_v = \mathcal{O}_{E,v} + \mathcal{O}_{E,v}\epsilon_v$. An easy calculation shows that R_v has reduced discriminant $\mathfrak{d}_v \mathfrak{m}_v$, and so may be enlarged to an order R'_v of reduced discriminant $\mathfrak{m}_v$. By Lemma 4.1.1 $tR'_v t^{-1} = S_v$ for some $t \in E_v^{\times}$. Replacing ϵ_v by $t\epsilon_v t^{-1} = t\bar{t}^{-1}\epsilon_v$ we find that (c) holds. Now assume that $v \mid \mathfrak{c}$. As W_v is free of rank one over E_v there is an $x \in E_v$ such that $\epsilon_v \cdot w_0 = x \cdot w_0$, and it follows that $\mathrm{N}(\epsilon_v)w_0 = -\epsilon_v^2 w_0 = -\mathrm{N}(x)w_0$. Therefore $\mathrm{ord}_v(\mathfrak{c}^2) = \mathrm{ord}_v(\mathrm{N}(x))$. If v is inert in E then this implies $x \in \mathfrak{c}\mathcal{O}_{E,v}$ and hence (d) holds. If v is split in E then we need not have $x \in \mathfrak{c}\mathcal{O}_{E,v}$, but there is some $t \in E_v^{\times}$ satisfying $\mathrm{N}(t) = 1$ and $tx \in \mathfrak{c}\mathcal{O}_{E,v}$. Replacing ϵ_v by $t\epsilon_v$ we again find that (d) holds. □

Let $R \subset B$ be a global order such that $R_v = \mathcal{O}_{E,v} + \mathcal{O}_{E,v}\epsilon_v$ at every finite place v, with ϵ_v satisfying the properties of Lemma 4.1.3. There is a natural $\mathcal{O}_F$-algebra homomorphism $R \to \mathcal{O}_E/\mathfrak{c}\mathcal{O}_E$ defined by $b \mapsto b^+$ (with notation as in §3.1), and the kernel of the induced homomorphism $\widehat{R}^{\times} \to (\mathcal{O}_E/\mathfrak{c}\mathcal{O}_E)^{\times}$ will be denoted U. Define a CM-cycle of level U

$$P_\chi(g) = \begin{cases} \chi(t) & \text{if } g = tu \text{ for some } t \in T(\mathbb{A}_f),\ u \in U \\ 0 & \text{otherwise} \end{cases}$$

so that $P_\chi = \prod_v P_{\chi,v}$ where the function $P_{\chi,v}$ on $G(F_v)/U_v$ agrees with that constructed in §3.3 and §3.4 (with $\mathfrak{e} = \mathfrak{r} = \mathfrak{m}\mathfrak{c}^2$). The compact open subgroups and

CM-cycles constructed above satisfy $U \subset V$ and

$$[V_T : U_T] \cdot Q_\chi(g) = \sum_{h \in V/U} P_\chi(gh). \tag{4.2}$$

For each ideal $\mathfrak{a}$ prime to $\mathfrak{c}$ we have, from §3.3 and §3.4, a CM-cycle of level U defined as the product

$$P_{\chi,\mathfrak{a}}(g) = \prod_{v \mid \mathfrak{a}} P_{\chi,\mathfrak{a},v}(g_v) \prod_{v \nmid \mathfrak{a}} P_{\chi,v}(g_v).$$

If $\mathfrak{a}$ is prime to $\mathfrak{d}\mathfrak{r}$ then R_v is a maximal order for each $v \mid \mathfrak{a}$. and we define the Hecke operator $T_\mathfrak{a}$ on CM-cycles of level U

$$(T_\mathfrak{a} P)(g) = \sum_{h \in H(\mathfrak{a})/U} P(gh),$$

where $H(\mathfrak{a}) = \prod_{v \mid \mathfrak{a}} H(\mathfrak{a}_v) \cdot \prod_{v \nmid \mathfrak{a}} U_v$ and $H(\mathfrak{a}_v)$ was defined in §3.3 for $v \mid \mathfrak{a}$. One then has the relation $T_\mathfrak{a} P_\chi = P_{\chi,\mathfrak{a}}$.

For the remainder of §4 the letters U and V will be used exclusively for the compact open subgroups constructed above.

4.2. **Toric newvectors and the Jacquet-Langlands correspondence.** Let $\mathrm{Ram}(B)$ denote the set of places of F at which B is nonsplit and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. If π_v is square-integrable for every $v \in \mathrm{Ram}(B)$ then there is a unique infinite-dimensional automorphic representation π' of $G(\mathbb{A})$ such that for every $v \notin \mathrm{Ram}(B)$, $\pi_v \cong \pi'_v$ as representations of $G(F_v) \cong \mathrm{GL}_2(F_v)$. We then say that π is the *Jacquet-Langlands lift* of π'. There are many references for the Jacquet-Langlands correspondence including [7, 8, 17, 18, 21]

Lemma 4.2.1. *With* Π *the automorphic representation fixed at the beginning of §4, if* $v \in \mathrm{Ram}(B)$ *is a nonarchimedean place then either*

(a) $\mathrm{ord}_v(\mathfrak{m}) = 1$ *and* Π_v *is a twist of the Steinberg representation by an unramified character*
(b) *or* $\mathrm{ord}_v(\mathfrak{m}) > 1$ *and* Π_v *is supercuspidal.*

In particular Π_v *is square integrable.*

Proof. If $v \in \mathrm{Ram}(B)$ is nonarchimedean then (4.1) implies that $\mathrm{ord}_v(\mathfrak{m}) = \mathrm{ord}_v(\mathfrak{n})$ is odd and Π_v has unramified central character. The lemma now follows from standard formulas for the conductor of irreducible admissible representations as in [27, (12.3.9.1)] □

For the remainder of §4.2 we assume that Π is cuspidal and that either Π_v is a weight 2 discrete series at each archimedean v and B is totally definite, or that Π_v is a weight 0 principal series at each archimedean v and B is totally indefinite. In either case it follows from Lemma 4.2.1 that Π_v is square integrable for each $v \in \mathrm{Ram}(B)$ and so Π is the Jacquet-Langlands lift of some Π'.

Definition 4.2.2. For any place v of F we define a *newvector* $\phi \in \Pi'_v$ to be a nonzero vector such that

(a) if v is a nonarchimedean place then ϕ is V_v-fixed,
(b) If v is an archimedean place and we are in the weight 0 case above, then ϕ is fixed by the action of $E_v^\times \cong \mathbb{R}^\times \cdot \mathrm{SO}_2(\mathbb{R})$,

(c) if v is an archimedean place and we are in weight 2 case then we impose no condition on ϕ.

A *newvector* in $\Pi' \cong \bigotimes_v \Pi'_v$ is a product of local newvectors.

Lemma 4.2.3. *Up to scaling there is a unique newvector in* Π'.

Proof. It suffices to prove existence and uniqueness everywhere locally. If v is archimedean this is clear (in the weight 2 case Π'_v is the one-dimensional trivial representation of $G(F_v)$ by [18, Lemma 4.2(2)]), so assume that v is nonarchimedean. If B_v is split then there is an isomorphism $B_v \cong M_2(F_v)$ which identifies $V_v \cong K_0(\mathfrak{m}_v) \cap K_1(\mathfrak{s}_v)$, and so the claim follows from the theory of newvectors for $\mathrm{GL}_2(F_v)$ as in §2.1. If B_v is nonsplit then (4.1) implies that $v \mid \mathfrak{m}$ and $v \nmid \mathfrak{c}$. As $V_v = S_v^\times$ with S_v an order of B_v of discriminant $\mathfrak{m}_v$ containing $\mathcal{O}_{E,v}$, the claim is a special case of [11, Proposition 6.4]. □

Definition 4.2.4. For any place v of F let $E_v^\times$ act on Π'_v via the embedding $T(F_v) \to G(F_v)$. We define a *toric newvector* $\phi \in \Pi'_v$ to be a nonzero vector such that

(a) if $v \nmid \mathfrak{d}\mathfrak{r}$ then ϕ is a newvector,
(b) if $v \mid \mathfrak{d}$ then ϕ is U_v-fixed and satisfies $t \cdot \phi = \overline{\chi}_v(t) \cdot \phi$ for every $t \in E_v^\times$,
(c) if $v \mid \mathfrak{r}$ then ϕ is U_v-fixed and satisfies $t \cdot \phi = \overline{\chi}_v(t) \cdot \phi$ for every $t \in \mathcal{O}_{E,v}^\times$.

A *toric newvector* in $\Pi' \cong \bigotimes \Pi'_v$ is a product of local toric newvectors.

Lemma 4.2.5. *Up to scaling there is a unique toric newvector in* Π'.

Proof. Again it suffices to prove the claim everywhere locally. If $v \nmid \mathfrak{d}\mathfrak{r}$ then the claim is a restatement of Lemma 4.2.3. If $v \mid \mathfrak{d}$ then χ_v has the form $\chi_v = \nu_v \circ \mathrm{N}$ for some unramified character ν_v of $F_v^\times$. By a theorem of Waldspurger [34, Theorem 2.3.2] the representation $\Pi'_v \otimes \nu_v$ has a unique line of $E_v^\times$-fixed vectors, and by a theorem of Gross-Prasad [34, Theorem 2.3.3] this line is also fixed by the unit group of any maximal order of B_v containing $\mathcal{O}_{E,v}$. As R_v may be enlarged to such an order, the $E_v^\times$-fixed vectors in $\Pi'_v \otimes \nu_v$ are also fixed by $U_v = R_v^\times$. It follows that Π'_v has a unique line of U_v-fixed vectors on which $E_v^\times$ acts through χ_v^{-1}.

If $v \mid \mathfrak{m}$ then $R_v = S_v$ (as $R_v \subset S_v$ and both have reduced discriminant $\mathfrak{m}_v$), $U_v = V_v$, and a toric newvector is just a nonzero V_v-fixed vector; again the claim follows from Lemma 4.2.3. If $v \mid \mathfrak{c}$ but $v \nmid \mathfrak{s}$ then χ_v is trivial on $\mathcal{O}_{F,v}^\times$, and so we may find a character χ'_v of $E_v^\times$ which is trivial on $F_v^\times$ but agrees with χ_v on $\mathcal{O}_{E,v}^\times$. By [34, Theorem 2.3.5] (Zhang's Γ is our $R_v^\times = \mathcal{O}_{E,v}^\times U_v$) there is a unique line of U_v-fixed vectors in Π'_v on which $\mathcal{O}_{E,v}^\times$ acts through $\overline{\chi}'_v$, and thus a unique toric newvector in Π'_v. If $v \mid \mathfrak{s}$ then $\Pi'_v \cong \Pi_v$ is a principal series $\Pi_v \cong \Pi(\mu_v, \chi_{0,v}^{-1}\mu_v^{-1})$ and $\chi_v = \nu_v \circ \mathrm{N}$ for some character ν_v of $F_v^\times$ of conductor $\mathfrak{c}$ (both claims by Hypothesis 1.1.1). It follows that $\Pi'_v \otimes \nu_v$ has trivial central character and conductor $\mathfrak{c}_v^2$. As R_v has reduced discriminant $\mathfrak{c}_v^2$ and contains $\mathcal{O}_{E,v}$ there is a unique line of $R_v^\times$-fixed vectors in $\Pi'_v \otimes \nu_v$ by [34, Theorem 2.3.3]. As $R_v^\times = \mathcal{O}_{E,v}^\times \cdot U_v$ we find that $\Pi'_v \otimes \nu_v$ has a unique line of U_v-fixed vectors on which $\mathcal{O}_{E,v}^\times$ acts through the trivial character, and the claim now follows from the observation that $\mathrm{N}(U_v) \subset 1 + \mathfrak{c}_v \subset \ker(\nu_v)$. □

4.3. **Central values for holomorphic forms.** In addition to Hypothesis 1.1.1 we assume that Π_v is a discrete series of weight 2 for every archimedean place v, and that $\epsilon(1/2, \mathfrak{r}) = 1$. Let B be the (unique up to isomorphism) totally definite quaternion algebra over F satisfying (4.1) for all finite places of F. Taking m to be the constant function 1 on $G(F)$, let $k_U(x,y)$ be the function on $C_U \times C_U$ defined by (3.4) and let $\langle P, Q\rangle_U$ be the associated height pairing on CM-cycles of level U defined by (3.5). According to [31, §7.2] the sum defining $k_U(x,y)$ is actually finite. Recall that we have set $\mathfrak{r} = \mathfrak{m}\mathfrak{c}^2$ and abbreviate $\Theta_\mathfrak{r} = \Theta_{\mathfrak{r},1/2}$.

Proposition 4.3.1. *Fix $a \in \mathbb{A}^\times$ and assume that $\mathfrak{a} = a\mathcal{O}_F$ is prime to $\mathfrak{c}$. Then*

$$\frac{H_F}{\lambda_U}[\widehat{\mathcal{O}}_E^\times : U_T]\cdot B(-a;\Theta_\mathfrak{r}) = 2^{[F:\mathbb{Q}]}|d|^{1/2}|a|\langle P_{\chi,\mathfrak{a}}, P_\chi\rangle_U \cdot e_\infty(a)$$

where, as in §3.2, H_F is the class number of F and $\lambda_U = [\mathcal{O}_F^\times : \mathcal{O}_F^\times \cap U]$.

Proof. Suppose $\gamma \in G(F)$ is nondegenerate and let η and ξ be defined by (3.1). Then Corollaries 3.3.5 and 3.4.3 show that

$$[\mathcal{O}_{E,v}^\times : \mathcal{O}_{F,v}^\times U_{T,v}]\cdot B_v(a,\eta,\xi;\Theta_\mathfrak{r}) = \tau_v(\gamma)\cdot |a|_v|d|_v^{1/2}\cdot O_U^\gamma(P_{\chi,\mathfrak{a},v})$$

for every finite place v of F. By (3.7)

$$\langle P_{\chi,\mathfrak{a}}, P_\chi\rangle_U^\gamma = [Z(\mathbb{A}_f) : Z(F)U_Z]\cdot \prod_{v\nmid\infty} O_U^\gamma(P_{\chi,\mathfrak{a},v})$$

By the final claims of Proposition 2.4.2 and Lemma 3.1.2, for v an archimedean place

$$B_v(a,\eta,\xi;\Theta_\mathfrak{r}) = 2\tau_v(\gamma)|a|_v e_v(-a).$$

Combining these equalities gives

$$\frac{H_F}{\lambda_U}[\widehat{\mathcal{O}}_E^\times : U_T]\cdot B(a,\eta,\xi;\Theta_{\mathfrak{r},1/2}) = 2^{[F:\mathbb{Q}]}|d|^{1/2}|a|\langle P_{\chi,\mathfrak{a}}, P_\chi\rangle_U^\gamma \cdot e_\infty(-a).$$

By Lemma 2.4.1, given $\eta,\xi \in F^\times$ with $\eta+\xi = 1$ we have $B(a,\eta,\xi;\Theta_\mathfrak{r}) = 0$ unless $\omega_v(-\eta\xi) = \epsilon_v(1/2,\mathfrak{r},\psi)$ for every place v of F. Combining (4.1) with Lemma 3.1.1 we find that $B(a,\eta,\xi;\Theta_\mathfrak{r}) = 0$ unless the pair η,ξ is of the form (3.1) for some $\gamma \in G(F)$. Therefore

$$\begin{aligned}&\frac{H_F}{\lambda_U}[\widehat{\mathcal{O}}_E^\times : U_T]\cdot \sum_{\substack{\eta,\xi\in F^\times\\ \eta+\xi=1}} B(-a,\eta,\xi;\Theta_\mathfrak{r})\\ &= 2^{[F:\mathbb{Q}]}|d|^{1/2}|a| \sum_{\substack{\gamma\in T(F)\backslash G(F)/T(F)\\ \gamma \text{ nondegenerate}}} \langle P_{\chi,\mathfrak{a}}, P_\chi\rangle_U^\gamma \cdot e_\infty(a).\end{aligned} \tag{4.3}$$

It remains to compare the linking numbers at the two degenerate choices of γ (i.e. $\gamma \in B^\pm$) with the degenerate terms $A_0(a;\Theta_\mathfrak{r})$ and $A_1(a;\Theta_\mathfrak{r})$ of (2.7). First suppose $\gamma = \epsilon^\circ$ where ϵ° satisfies $B^- = E\epsilon^\circ$, so that $(\eta,\xi) = (0,1)$. Let $z \in \mathbb{A}_E^\times$ be such that $\epsilon_v^\circ = z_v\epsilon_v$ for every finite place v. If $\chi \neq \chi^*$ then both $A_1(a;\Theta_\mathfrak{r})$ and $\langle P_{\chi,\mathfrak{a}}, P_\chi\rangle_U^\gamma$ vanish, by Lemmas 2.3.3 and 3.2.1, respectively. We therefore assume that $\chi = \chi^*$. If χ is ramified then $B_v(a;E_{\mathfrak{r},s}) = 0$ for any $v \mid \mathfrak{c}$ by Proposition 2.2.1 and the inequality $\mathrm{ord}_v(\mathfrak{a}\mathfrak{r}^{-1}) = -\mathrm{ord}_v(\mathfrak{r}) < 0$. Abbreviating $\alpha = \begin{pmatrix} a\delta^{-1} & \\ & 1\end{pmatrix}$, it follows that $W_{\mathfrak{r},s}(\alpha h_T) = 0$ for any $T \subset S$ and so $A_1(a;\Theta_\mathfrak{r}) = 0$. Similarly if χ is ramified then $P_{\chi,\mathfrak{a}}(\epsilon^\circ) = 0$ by Lemma 3.4.4, and so also $\langle P_{\chi,\mathfrak{a}}, P_\chi\rangle_{U,\gamma} = 0$ by

Lemma 3.2.1. We therefore assume that χ is unramified. By (2.2), Proposition 2.2.3, and Lemma 2.3.4, $C_\theta(\alpha h_T) = 0$ unless $T = \emptyset$ or S, and so

$$\begin{aligned} A_1(a;\Theta_{\mathfrak{r}}) &= \sum_{T\subset S} \overline{\chi}_T(\mathfrak{D}) C_\theta(\alpha h_T) W_{\mathfrak{r},1/2}(\alpha h_T) \\ &= B(a; E_{\mathfrak{r},1/2}) C_\theta(\alpha) + \overline{\chi}(\mathfrak{D}) B(a; h_S E_{\mathfrak{r},1/2}) C_\theta(\alpha h_S) \\ &= 2\cdot B(a; E_{\mathfrak{r},1/2}) C_\theta(\alpha) \end{aligned}$$

where we have used Propositions 2.2.2 and 2.3.2 for the third equality. Again using Proposition 2.2.3 and Lemma 2.3.4 we find

$$\begin{aligned} C_\theta(\alpha) &= (-1)^{[F:\mathbb{Q}]}\nu(a\delta^{-1})|ad^{-1}\delta^{-1}|^{1/2} L^*(1,\omega) \\ B(a; E_{\mathfrak{r},1/2}) &= |r|^{1/2} B(ar^{-1}; E_{\mathcal{O}_F,1/2}) \\ &= (-1)^{[F:\mathbb{Q}]}|dr|^{1/2}\overline{\nu}(ar^{-1}\delta^{-1}) B(ar^{-1};\theta) \end{aligned}$$

where $r\mathcal{O}_F = \mathfrak{r}$ for $r \in \mathbb{A}^\times$ with $r_v = 1$ at each archimedean v. Therefore

$$A_1(a;\Theta_{\mathfrak{r}}) = 2\nu(\mathfrak{r})|ar\delta^{-1}|^{1/2} B(ar^{-1};\theta) L(1,\omega).$$

On the other hand using Corollary 3.3.9, Lemma 3.4.4, and

$$\chi(rz) = \nu(r)^2\nu(\mathrm{N}(z)) = \nu(r)^2\nu(\mathrm{N}(\epsilon^\circ)^{-1}) = \nu(r)$$

we find

$$P_{\chi,\mathfrak{a}}(\epsilon^\circ)\cdot e_\infty(-a) = \nu(\mathfrak{r})|r|^{1/2}|a|^{-1/2} B(ar^{-1};\theta)$$

and now (2.2), (2.3), and Lemma 3.2.1 imply that (for $\gamma = \epsilon^\circ$)

$$H_F\lambda_U^{-1}[\widehat{\mathcal{O}}_E^\times : U_T]\cdot A_1(-a;\Theta_{\mathfrak{r}}) = 2^{[F:\mathbb{Q}]}|d|^{1/2}|a|\langle P_{\chi,\mathfrak{a}}, P_\chi\rangle_U^\gamma \cdot e_\infty(a). \tag{4.4}$$

A similar, but easier, argument also shows that (4.4) continues to hold if $\gamma = 1$ and A_1 is replaced by A_0. The theorem follows from this together with equation (4.3), equation (3.6), and the decomposition (2.7). □

We now construct a pairing $[P,Q]$ on CM-cycles of level U taking values in the space of automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ as in [34, (4.4.5)]. Endow the (finite) set $S_U = G(F)\backslash G(\mathbb{A}_f)/U$ with the measure determined by

$$\int_{S_U} \sum_{\gamma\in T(F)\backslash G(F)} P(\gamma g)\ dg = \int_{C_U} P(g)\ dg$$

for any CM-cycle P of level U. For each $\mathfrak{a}$ prime to $\mathfrak{d}\mathfrak{r}$ there is a Hecke operator $(T_{\mathfrak{a}}\phi)(g) = \sum \phi(gh)$ on $L^2(S_U)$ where the sum is over $h \in H(\mathfrak{a})/U$ as in §4.1. For any $\phi \in L^2(S_U)$ we have

$$\int_{S_U} k_U(x,y)\phi(y)\ dy = \phi(x)$$

and it follows that that there is a decomposition $k_U(x,y) = \sum_{i=1}^{\ell} f_i'(x)\overline{f_i'(y)}$ where $\{f_1',\ldots,f_\ell'\}$ is any orthonormal basis for $L^2(S_U)$. We choose this basis in such a way that each f_i' is a simultaneous eigenvector for every $T_{\mathfrak{a}}$ with $(\mathfrak{a},\mathfrak{d}\mathfrak{r}) = 1$. The Jacquet-Langlands correspondence implies that for each f_i' there is a (not necessarily unique) holomorphic automorphic form f_i of weight 2 on $\mathrm{GL}_2(\mathbb{A})$ fixed by $K_1(\mathfrak{d}\mathfrak{r})$ having the same Hecke eigenvalues as f_i'. Indeed, if f_i' generates an infinite dimensional representation π' of $G(\mathbb{A})$ then take f_i to be a newvector in the Jacquet-Langlands lift of π'. If f_i' generates a finite dimensional representation of $G(\mathbb{A})$ then $f_i'(g) = \mu(\mathrm{N}(g))$ with μ some character of $\mathbb{A}^\times/F^\times$, and one takes f_i to be an Eisenstein series constructed from a function in the induced representation $\mathcal{B}(\mu|\cdot|^{1/2}, \mu|\cdot|^{-1/2})$. We may, and do, assume that $\widehat{B}(\mathcal{O}_F, f_i) = 1$ for every i. For any CM-cycles P and Q of level U we define a parallel weight 2, holomorphic, $K_1(\mathfrak{d}\mathfrak{r})$-fixed automorphic form on $\mathrm{GL}_2(\mathbb{A})$

$$[P, Q] = \sum_{i=1}^{\ell} \left(\int_{C_U \times C_U} P(x) f_i'(x) \overline{f_i'(y) Q(y)} \; dx \; dy \right) f_i.$$

This form satisfies $\widehat{B}(\mathcal{O}_F, [P, Q]) = \langle P, Q \rangle_U$ and, for any ideal $\mathfrak{a}$ relatively prime to $\mathfrak{d}\mathfrak{r}$, $T_{\mathfrak{a}} \cdot [P, Q] = [P, T_{\mathfrak{a}} Q]$. Set $\Psi = [P_\chi, P_\chi]$, an automorphic form of central character χ_0^{-1} satisfying

$$\widehat{B}(\mathcal{O}_F; T_{\mathfrak{a}} \Psi) = \langle P_\chi, P_{\chi,\mathfrak{a}} \rangle_U. \tag{4.5}$$

Let Π' be the automorphic representation of $G(\mathbb{A})$ whose Jacquet-Langlands lift is Π, let $\phi_{\Pi'}^{\chi}$ be the toric newvector in Π' normalized by $\int_{S_U} |\phi_{\Pi'}^{\chi}|^2 = 1$ and let $\Psi|_{\Pi}$ denote the projection of Ψ to Π. We may choose the basis $\{f_i'\}$ so that $\phi_{\Pi'}^{\chi} = f_1'$. If we set $\mathcal{P}_\chi(g) = \sum_\gamma P_\chi(\gamma g)$ where the sum is over $\gamma \in T(F) \backslash G(F)$ then

$$\widehat{B}(\mathcal{O}_F; \Psi|_{\Pi}) = \sum_{\substack{1 \le i \le \ell \\ \pi_i = \pi_1}} \left| \int_{S_U} \mathcal{P}_\chi(t) f_i'(t) \; dt \right|^2.$$

The projection of $\overline{\mathcal{P}}_\chi$ to π_1' is a toric newvector, hence a scalar multiple of f_1', and so only the term $i = 1$ contributes to the sum. It follows that

$$\widehat{B}(\mathcal{O}_F; \Psi|_{\Pi}) = \left| \int_{C_U} P_\chi(t) \phi_{\Pi'}^{\chi}(t) \; dt \right|^2. \tag{4.6}$$

Proposition 4.3.2. *Let $\phi_{\Pi}^{\#}$ be the orthogonal projection of the normalized newform $\phi_{\Pi} \in \Pi$ to the quasi-new line (defined in §2.8). Then*

$$\begin{aligned} & 2^{|S|} H_F \lambda_U^{-1} [\widehat{\mathcal{O}}_E^{\times} : U_T] \widehat{B}(\mathcal{O}_F; \phi_{\Pi}^{\#}) L(1/2, \Pi \times \Pi_\chi) \\ = \;& |d|^{1/2} 2^{[F:\mathbb{Q}]} ||\phi_{\Pi}^{\#}||^2_{K_0(\mathfrak{d}\mathfrak{r})} \cdot \left| \int_{C_U} P_\chi(t) \phi_{\Pi'}^{\chi}(t) \; dt \right|^2 \end{aligned}$$

in which S is the set of prime divisors of $\mathfrak{d}$.

Proof. Let $\overline{\Theta}_{\mathfrak{r}}|_{\Pi}$ and $\Psi|_{\Pi}$ denote the projections of $\overline{\Theta}_{\mathfrak{r}}$ and Ψ to Π. Combining Proposition 4.3.1 and (4.5) gives

$$H_F \lambda_U^{-1} [\widehat{\mathcal{O}}_E^{\times} : U_T] \cdot \widehat{B}(\mathcal{O}_F; T_{\mathfrak{a}} \overline{\Theta}_{\mathfrak{r}}) = 2^{[F:\mathbb{Q}]} |d|^{1/2} \widehat{B}(\mathcal{O}_F, T_{\mathfrak{a}} \Psi)$$

for all $\mathfrak{a}$ prime to $\mathfrak{d}\mathfrak{r}$. The action of the operators $T_{\mathfrak{a}}$ with $(\mathfrak{a}, \mathfrak{d}\mathfrak{r}) = 1$ on the space of all $K_1(\mathfrak{d}\mathfrak{r})$-fixed, holomorphic, parallel weight two automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ of central character χ_0^{-1} generates a semi-simple $\mathbb{C}$-algebra, and it follows from this and strong multiplicity one that there is a polynomial e_{Π} in the Hecke operators $T_{\mathfrak{a}}$ such that $\overline{\Theta}_{\mathfrak{r}}|_{\Pi} = e_{\Pi} \cdot \overline{\Theta}_{\mathfrak{r}}$ and $\Psi|_{\Pi} = e_{\Pi} \cdot \Psi$. We therefore deduce that

$$H_F \lambda_U^{-1} [\widehat{\mathcal{O}}_E^{\times} : U_T] \cdot \widehat{B}(\mathcal{O}_F; \overline{\Theta}_{\mathfrak{r}}|_{\Pi}) = 2^{[F:\mathbb{Q}]} |d|^{1/2} \widehat{B}(\mathcal{O}_F; \Psi|_{\Pi}). \tag{4.7}$$

Under the decomposition $\Pi \cong \bigotimes_v \Pi_v$ the newform ϕ_{Π} is decomposable as a pure tensor $\phi_{\Pi} = \otimes \phi_{\Pi,v}$. In the notation of §2.8 $\Lambda_v(\phi_{\Pi,v}) \neq 0$ for $v \mid \mathfrak{d}\mathfrak{c}$, and so $\phi_{\Pi,v}$ has nontrivial projection to the quasi-new line in Π_v. It follows that $\phi_{\Pi}^{\#} \neq 0$. The form $\overline{\Theta}_{\mathfrak{r}}|_{\Pi}$ lies on the quasi-new line of Π by Proposition 2.8.2, and so if $\widehat{B}(\mathcal{O}_F; \phi_{\Pi}^{\#}) = 0$

then also $\widehat{B}(\mathcal{O}_F;\overline{\Theta}_{\mathfrak{r}}|_\Pi)=0$. Using (4.6) and (4.7) we then see that both sides of the stated equality are 0. Therefore we may assume $\widehat{B}(\mathcal{O}_F;\phi_\Pi^\#)\neq 0$ so that

$$\overline{\Theta}_{\mathfrak{r}}|_\Pi=\frac{\widehat{B}(\mathcal{O}_F;\overline{\Theta}_{\mathfrak{r}}|_\Pi)}{\widehat{B}(\mathcal{O}_F;\phi_\Pi^\#)}\cdot\phi_\Pi^\#.$$

Combining this with (2.8) (with $b=1$) gives

$$\begin{aligned}\widehat{B}(\mathcal{O}_F;\overline{\Theta}_{\mathfrak{r}}|_\Pi)\cdot||\phi_\Pi^\#||^2_{K_0(\mathfrak{d}\mathfrak{r})} &= \widehat{B}(\mathcal{O}_F;\overline{\Theta}_{\mathfrak{r}}|_\Pi)\cdot\langle\phi_\Pi,\phi_\Pi^\#\rangle_{K_0(\mathfrak{d}\mathfrak{r})}\\ &= \widehat{B}(\mathcal{O}_F;\phi_\Pi^\#)\cdot\langle\phi_\Pi,\overline{\Theta}_{\mathfrak{r}}\rangle_{K_0(\mathfrak{d}\mathfrak{r})}\\ &= 2^{|S|}\widehat{B}(\mathcal{O}_F;\phi_\Pi^\#)L(1/2,\Pi\times\Pi_\chi).\end{aligned}$$

The claim now follows from (4.6) and (4.7). □

Theorem 4.3.3. *Let $\phi_\Pi\in\Pi$ be the normalized newvector (in the sense of §2.1) and let $\phi_{\Pi'}\in\Pi'$ be the newvector (in the sense of Definition 4.2.2) normalized by $\int_{S_V}|\phi_{\Pi'}|^2=1$. Then*

$$\frac{L(1/2,\Pi\times\Pi_\chi)}{||\phi_\Pi||^2_{K_0(\mathfrak{n})}}=\frac{2^{[F:\mathbb{Q}]}}{H_{F,\mathfrak{s}}\sqrt{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}\mathfrak{c}^2)}}\cdot\left|\int_{C_V}Q_\chi(t)\phi_{\Pi'}(t)\ dt\ \right|^2.$$

where $H_{F,\mathfrak{s}}=[Z(\mathbb{A}_f):F^\times V_Z]$ is the order of the ray class group of conductor $\mathfrak{s}$.

Proof. The proof is postponed until §4.6. □

4.4. **Central values for Maass forms.** In addition to Hypothesis 1.1.1 we assume that Π_v is a weight zero principal series for every archimedean place v, and that $\epsilon(1/2,\mathfrak{r})=(-1)^{[F:\mathbb{Q}]}$. Thus the weight 0 kernel of §2.7 satisfies $\Theta^*_{\mathfrak{r},s}=\Theta^*_{\mathfrak{r},1-s}$. Let B be the (unique up to isomorphism) totally indefinite quaternion algebra over F satisfying (4.1) for every finite place v. Let $\mathbb{S}=\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ and set $F_\infty=F\otimes_\mathbb{Q}\mathbb{R}$. As F_∞ is naturally an $\mathbb{R}$-algebra,

$$T_{/F_\infty}=T\times_{\mathrm{Spec}(F)}\mathrm{Spec}(F_\infty)\qquad G_{/F_\infty}=G\times_{\mathrm{Spec}(F)}\mathrm{Spec}(F_\infty)$$

are naturally algebraic groups over $\mathbb{R}$. Fixing an embedding of real algebraic groups $\mathbb{S}\to T_{/F_\infty}$ the embedding $T\to G$ determines an embedding $x_0:\mathbb{S}\to G_{/F_\infty}$, and we let X denote the $G(F_\infty)$-conjugacy class of x_0. As $T(F_\infty)$ is the stabilizer of x_0 we may identify $X\cong G(F_\infty)/T(F_\infty)$. Writing $\mathcal{H}=\mathbb{C}-\mathbb{R}$ and choosing an isomorphism $G(F_\infty)\cong\mathrm{GL}_2(\mathbb{R})^{[F:\mathbb{Q}]}$, we may fix a point in $\mathcal{H}^{[F:\mathbb{Q}]}$ whose stabilizer under the action of $G(F_\infty)$ is $T(F_\infty)$. This allows us to identify $X\cong\mathcal{H}^{[F:\mathbb{Q}]}$. Endowing $\mathcal{H}$ with the usual hyperbolic volume form $y^{-2}dxdy$ we obtain a measure on X. Define

$$S_U=G(F)\backslash X\times G(\mathbb{A}_f)/U$$

endowed with the quotient measure induced from that on $G(\mathbb{A}_f)/U$ giving each coset volume 1. The map $G(\mathbb{A}_f)\to X\times G(\mathbb{A}_f)$ defined by $g\mapsto(x_0,g)$ restricts to a function on $T(\mathbb{A}_f)$ and determines an embedding $C_U\to S_U$.

If ϕ is a weight 0 Maass form on $\mathrm{GL}_2(\mathbb{A})$ with *parameter* t_v in the sense of [36, §4] at an archimedean place v then we set

$$B_v(a;\phi)=|a|_v^{1/2}\int_0^\infty e^{-\pi|a|_v(y+y^{-1})}y^{it_v}\ d^\times y.$$

Define $B_\infty(a;\phi) = \prod_{v|\infty} B_v(a;\phi)$ and define $\widehat{B}(\mathfrak{a};\phi)$ for $\mathfrak{a} = a\mathcal{O}_F$ by

$$B(a;\phi) = B_\infty(a;\phi)\cdot\widehat{B}(\mathfrak{a};\phi).$$

Let Π' be the automorphic representation of $G(\mathbb{A})$ whose Jacquet-Langlands lift is Π, and let $\phi^\chi_{\Pi'}$ be the toric newvector in Π' normalized by $\int_{S_U}|\phi^\chi_{\Pi'}| = 1$

Proposition 4.4.1. *Let $\phi^\#_\Pi$ be the orthogonal projection of the normalized newform $\phi_\Pi\in\Pi$ to the quasi-new line in Π. Then*

$$\begin{aligned}&2^{|S|}H_F\lambda_U^{-1}[\widehat{\mathcal{O}}^\times_E : U_T]\widehat{B}(\mathcal{O}_F;\phi^\#_\Pi)L(1/2,\Pi\times\Pi_\chi)\\ &= \quad |d|^{1/2}4^{[F:\mathbb{Q}]}||\phi^\#_\Pi||^2_{K_0(\mathfrak{d}\mathfrak{r})}\cdot\left|\int_{C_U} P_\chi(t)\phi^\chi_{\Pi'}(t)\ dt\ \right|^2\end{aligned}$$

in which S is the set of prime divisors of $\mathfrak{d}$.

Proof. Fix $a\in\mathbb{A}^\times$ and assume that $\mathfrak{a} = a\mathcal{O}_F$ is prime to $\mathfrak{c}$. We abbreviate $\Theta^*_\mathfrak{r} = \Theta^*_{\mathfrak{r},1/2}$. Suppose v is an infinite place of F. For each $a\in\mathbb{A}^\times$, $\gamma\in G(F_v)$, and η,ξ as in (3.1) define the multiplicity function

$$m^*_v(a,\gamma) = \begin{cases} 4e^{2\pi a_v(\xi-\eta)} & \text{if } \xi a_v\le 0 \text{ and } \eta a_v\ge 0\\ 0 & \text{otherwise.}\end{cases}$$

If $\gamma\in G(F_\infty)$ set $m^*_\infty(a,\gamma) = \prod_{v|\infty} m^*_v(a,\gamma_v)$. Exactly as in Proposition 4.3.1, using the formulas of §2.7 to supplement those of §2.4, we find

$$\frac{H_F}{\lambda_U}[\widehat{\mathcal{O}}^\times_E : U_T]\cdot B(a;\Theta^*_\mathfrak{r}) = |d|^{1/2}|a|\sum_{\gamma\in T(F)\backslash G(F)/T(F)}\langle P_{\chi,\mathfrak{a}}, P_\chi\rangle^\gamma_U\cdot m^*_\infty(a;\gamma). \tag{4.8}$$

The remainder of the proof is similar to that of Proposition 4.3.2; see [34, §4.4] for details. Briefly, for any Maass form ϕ on S_U the kernel

$$k_U(a;x,y) = \sum_{\gamma\in G(F)/(Z(F)\cap U)}\mathbf{1}_U(x_f^{-1}\gamma y_f)m^*_\infty(a;x_\infty^{-1}\gamma y_\infty)$$

satisfies

$$\begin{aligned}\int_{S_U} k_U(a;x,y)\phi(y)\ dy &= \int_X m^*_\infty(a;y_\infty)\phi(xy_\infty)\ dy_\infty\\ &= 4^{[F:\mathbb{Q}]}B_\infty(a;\phi)\cdot\phi(x).\end{aligned}$$

Exactly as in [34, Lemma 4.4.3] or [36, §16] this leads to a spectral decomposition of the kernel $k_U(a;x,y)$, and the proposition follows from (4.8), which is our analogue of [36, (16.1)], exactly as in [36, §16]. □

Theorem 4.4.2. *Let $\phi_\Pi\in\Pi$ be the normalized newvector (in the sense of §2.1) and let $\phi_{\Pi'}\in\Pi'$ be the newvector (in the sense of Definition 4.2.2) normalized by $\int_{S_V}|\phi_{\Pi'}|^2 = 1$. Then*

$$\frac{L(1/2,\Pi\times\Pi_\chi)}{||\phi_\Pi||^2_{K_0(\mathfrak{n})}} = \frac{4^{[F:\mathbb{Q}]}}{H_{F,\mathfrak{s}}\sqrt{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}\mathfrak{c}^2)}}\left|\int_{C_V} Q_\chi(t)\phi_{\Pi'}(t)\ dt\ \right|^2$$

where $H_{F,\mathfrak{s}}$ is the order of the ray class group of F of conductor $\mathfrak{s}$.

Proof. The proof is postponed until §4.6. □

4.5. **A particular family of Maass forms.** Fix a $\tau \in \mathbb{C}$ and if χ_0 is trivial assume that $\tau \neq 0, 1$. Let Π_τ denote the (irreducible) weight zero principal series representation

$$\Pi_\tau = \Pi(|\cdot|^{\tau-1/2}, \chi_0^{-1}|\cdot|^{1/2-\tau})$$

of $\mathrm{GL}_2(\mathbb{A})$ of conductor $\mathfrak{s}$ and central character χ_0^{-1}. We construct an Eisenstein series $\mathcal{E}_\tau \in \Pi_\tau$ as follows. Define a Schwartz function $\Omega = \prod_v \Omega_v$ on $\mathbb{A} \times \mathbb{A}$ by

$$\Omega_v(x, y) = \begin{cases} \mathbf{1}_{\mathcal{O}_{F,v}}(x)\mathbf{1}_{\mathcal{O}_{F,v}}(y) & \text{if } v \nmid \mathfrak{s}\infty \\ \chi_{0,v}^{-1}(y)\mathbf{1}_{\mathfrak{s}_v}(x)\mathbf{1}_{\mathcal{O}_{F,v}^\times}(y) & \text{if } v \mid \mathfrak{s} \\ e^{-\pi(x^2+y^2)} & \text{if } v \mid \infty. \end{cases}$$

The function

$$\mathcal{F}_\tau(g) = |\det(g)|^\tau \int_{\mathbb{A}^\times} \Omega\big([0, x] \cdot g\big)|x|^{2\tau}\chi_0(x)\ d^\times x$$

is a newvector in the induced representation $\mathcal{B}(|\cdot|^{\tau-1/2}, \chi_0^{-1}|\cdot|^{1/2-\tau})$ defined in [34, §2.2] and therefore the Eisenstein series (initially defined for $\mathrm{Re}(\tau) \gg 0$ and continued analytically)

$$\mathcal{E}_\tau(g) = \sum_{\gamma \in B(F)\backslash \mathrm{GL}_2(F)} \mathcal{F}_\tau(\gamma g)$$

is a newvector in Π_τ. The discrepancy between $\mathcal{E}_\tau$ and the normalized newvector is determined by the following

Lemma 4.5.1.

$$\int_{\mathbb{A}^\times} B(a; \mathcal{E}_\tau) \cdot |a|^{s-1/2}\ d^\times a = \frac{|\delta|^{\tau-1/2}\epsilon(1/2, \chi_0)}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{s})^{2\tau-1/2}} L(s, \Pi_\tau)$$

Proof. As in §2.2, using

$$B(a; \mathcal{E}_\tau) = \int_{\mathbb{A}} \mathcal{F}_\tau \begin{pmatrix} & 1 \\ -a\delta^{-1} & y \end{pmatrix} \psi(-y)\ dy$$

we see that $B(a; \mathcal{E}_\tau) = \prod_v B_v(a; \mathcal{E}_\tau)$ where

$$B_v(a; \mathcal{E}_\tau) = |\delta|_v^{\tau-1/2}|a|_v^\tau \chi_{0,v}(\delta) \int_{F_v} \psi_v^0(y) \int_{F_v^\times} \Omega_v(ax, xy)|x|_v^{2\tau}\chi_{0,v}(x)\ d^\times x dy.$$

If $v \nmid \mathfrak{s}\infty$ then a short calculation shows

$$\int_{F_v} \psi_v^0(y) \int_{F_v^\times} \Omega_v(ax, xy)|x|_v^{2\tau}\chi_{0,v}(x)\ d^\times x dy = |\delta|_v^{1/2} \sum_{k=0}^{\mathrm{ord}_v(a)} |\varpi^k|_v^{1-2\tau}\chi_{0,v}^{-1}(\varpi^k)$$

from which we deduce

$$\int_{F_v^\times} B_v(a; \mathcal{E}_\tau) \cdot |a|_v^{s-1/2}\ d^\times a = \chi_{0,v}(\delta)|\delta|_v^{\tau-1/2} L_v(s, \overline{\chi}_0|\cdot|^{1/2-\tau}) L_v(s, |\cdot|_v^{\tau-1/2}).$$

If $v \mid \mathfrak{s}$ then choose $\sigma \in F_v^\times$ with $\sigma\mathcal{O}_{F,v} = \mathfrak{s}_v$. We have

$$\begin{aligned}
&\int_{F_v} \psi_v^0(y) \int_{F_v^\times} \Omega_v(ax, xy)|x|_v^{2\tau}\chi_{0,v}(x)\ d^\times x\ dy \\
&= \int_{F_v^\times} \left[\int_{F_v} \psi_v^0(y)(yx)\mathbf{1}_{\mathcal{O}_{F,v}^\times}(yx)\ dy\right] \mathbf{1}_{\mathfrak{s}_v}(ax)|x|_v^{2\tau}\chi_{0,v}(x)\ d^\times x \\
&= |\delta\sigma|_v^{1/2}\epsilon_v(\chi_0, \psi_v^0) \int_{F_v^\times} \mathbf{1}_{\mathcal{O}_{F,v}^\times}(\sigma x^{-1})\mathbf{1}_{\mathfrak{s}_v}(ax)|x|_v^{2\tau-1}\ d^\times x \\
&= |\delta|_v^{1/2}|\sigma|_v^{2\tau-1/2}\epsilon_v(\chi_0, \psi_v^0)\mathbf{1}_{\mathcal{O}_{F,v}}(a).
\end{aligned}$$

Therefore

$$\int_{F_v^\times} B_v(a; \mathcal{E}_\tau) \cdot |a|_v^{s-1/2}\ d^\times a = \chi_{0,v}(\delta)|\delta|_v^{\tau-1/2}|\sigma|_v^{2\tau-1/2}\epsilon_v(\chi_0, \psi_v^0)L_v(s, |\cdot|^{\tau-1/2}).$$

If $v \mid \infty$ then

$$\begin{aligned}
&\int_{F_v} \psi_v^0(y) \int_{F_v^\times} \Omega_v(ax, xy)|x|_v^{2\tau}\chi_{0,v}(x)\ d^\times x dy \\
&= |\delta|_v^{1/2} \int_{-\infty}^{\infty} e^{2\pi i y} \int_{-\infty}^{\infty} e^{-\pi x^2(a^2+y^2)}|x|_v^{2\tau-1}\ d^{\mathrm{Leb}}x d^{\mathrm{Leb}}y.
\end{aligned}$$

We therefore have

$$\begin{aligned}
&\chi_{0,v}(\delta^{-1})|\delta|_v^{1/2-\tau} \int_{F_v^\times} B_v(a; \mathcal{E}_\tau) \cdot |a|_v^{s-1/2}\ d^\times a \\
&= \int_{\mathbb{R}^\times} \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^\times} |a|^{\tau+s-1/2} e^{-\pi x^2 a^2} d^\times a\right) e^{2\pi i y} e^{-\pi x^2 y^2}|x|^{2\tau}\ d^{\mathrm{Leb}}y\ d^\times x \\
&= G_1(s+\tau-1/2) \int_{\mathbb{R}^\times} \left(\int_{-\infty}^{\infty} e^{-2\pi i y x} e^{-\pi y^2}\ d^{\mathrm{Leb}}y\right) |x|^{s-\tau+1/2}\ d^\times x \\
&= G_1(s+\tau-1/2) \int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^{s-\tau+1/2}\ d^\times x \\
&= G_1(s+\tau-1/2)G_1(s-\tau+1/2).
\end{aligned}$$

Combining these calculations proves the lemma. □

We now assume that Π_τ satisfies Hypothesis 1.1.1, which is really just the condition that χ_v factors through $\mathrm{N} : E_v^\times \to F_v^\times$ for each $v \mid \mathfrak{s}$. Choosing $\Pi = \Pi_\tau$ in the introduction to §4, we wish to prove an analogue (Corollary 4.5.3) of Theorem 4.4.2 for the noncuspidal representation Π_τ by brute force. Note that now $\mathfrak{m} = \mathcal{O}_F$ and $\epsilon(1/2, \mathfrak{r}) = (-1)^{[F:\mathbb{Q}]}$. To put ourselves in the situation of §4.4, suppose B is a split quaternion algebra over F (so that (4.1) holds for all finite v) and as always fix an embedding $E \to B$. Let W be a two dimensional F-vector space on which B acts on the left, and fix an isomorphism of F-vector spaces $W \cong F \times F$. Writing elements of W as row vectors, there is an isomorphism $\rho : B \cong M_2(F)$ determined by $b \cdot [x, y] = [x, y] \cdot \rho(b)^t$, where the action on the left is the action of B on W, the action on the right is matrix multiplication, and the superscript t indicates transpose. The element $w_0 = [0, 1] \in W$ generates W as a left E-module, and we define

$$L = \mathcal{O}_{\mathfrak{c}\mathfrak{s}^{-1}} \cdot w_0 \qquad L' = \mathcal{O}_{\mathfrak{c}} \cdot w_0.$$

We may pick a $j \in \mathrm{GL}_2(\mathbb{A})$ having the following properties:

(a) if $v \mid \mathfrak{s}$ then j_v satisfies $[0,1]\cdot j_v^{-1} = w_0$ and

$$L_v = (\mathcal{O}_{F,v} \times \mathcal{O}_{F,v})\cdot j_v^{-1} \qquad L'_v = (\mathfrak{s}_v \times \mathcal{O}_{F,v})\cdot j_v^{-1},$$

(b) if $v \nmid \mathfrak{s}$ is a finite place of F then $j_f \cdot K_0(\mathfrak{m}) \cdot j_f^{-1} = \rho(V_v)$,

(c) if v is an archimedean place then $j_v \cdot \mathrm{SO}(F_v)\cdot j_v^{-1}$ is set of norm one elements of $\rho(T(F_v))$.

For every automorphic form ϕ on $\mathrm{GL}_2(\mathbb{A})$ we define an automorphic form ϕ' on $G(\mathbb{A})$ by $\phi'(g) = \phi(\rho(g)j)$. The space Π_τ of automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ thereby determines a space Π'_τ of automorphic forms on $G(\mathbb{A})$. Of course $G \cong \mathrm{GL}_2$ and $\Pi'_\tau \cong \Pi_\tau$, but it is useful to maintain these notational distinctions. Under the definition of §4.2 Π_τ is the Jacquet-Langlands lift of Π'_τ (a highly degenerate case). If $\phi \in \Pi_\tau$ is a newvector in the sense of §2.1 then $\phi' \in \Pi'_\tau$ is a newvector in the sense of §4.2.

Proposition 4.5.2. *Normalize the Haar measures on $T(\mathbb{A}_f)$ and $Z(\mathbb{A}_f)$ to give $\widehat{\mathcal{O}}_{\mathfrak{c}}^\times$ and $\widehat{\mathcal{O}}_F^\times$ each volume one, respectively, and give $T(F)\backslash T(\mathbb{A}_f)/Z(\mathbb{A}_f)$ the induced quotient measure. For every $\tau \in \mathbb{C}$*

$$\mathrm{N}_{F/\mathbb{Q}}\left(\mathfrak{d}\mathfrak{c}^2\mathfrak{s}^{-2}\right)^{\tau/2} \frac{1}{2^{[F:\mathbb{Q}]}} L(\tau,\chi) = \int_{T(F)\backslash T(\mathbb{A}_f)/Z(\mathbb{A}_f)} \chi(t)\mathcal{E}'_\tau(t)\ dt.$$

Proof. The restriction of $\mathcal{E}'_\tau$ to $T(\mathbb{A}_f)$ does not depend on the choice of embedding $E \to B$, and this embedding may be chosen so that

$$\rho(\alpha + \beta\sqrt{-\Delta}) = \begin{pmatrix} \alpha & \beta\Delta \\ -\beta & \alpha \end{pmatrix}$$

where $E = F[\sqrt{-\Delta}]$ with $\Delta \in F$ totally positive. As the embedding $\rho : T \to \mathrm{GL}_2$ identifies $Z(F)\backslash T(F)$ with $B(F)\backslash \mathrm{GL}_2(F)$ we have

$$\int_{T(F)\backslash T(\mathbb{A}_f)/Z(\mathbb{A}_f)} \chi(t)\mathcal{E}'_\tau(t)\ dt = \int_{T(\mathbb{A}_f)/Z(\mathbb{A}_f)} \chi(t)\mathcal{F}_\tau(\rho(t)j)\ dt.$$

Combining this with

$$\chi(t)\mathcal{F}_\tau(\rho(t)j) = |\det(j)|^\tau \int_{Z(\mathbb{A})} \Omega\big([0,1]\cdot\rho(tx)j\big)|\mathrm{N}(tx)|^\tau \chi(tx)\ dx$$

we find

$$\begin{aligned}\int_{T(F)\backslash T(\mathbb{A}_f)/Z(\mathbb{A}_f)} \chi(t)\mathcal{E}'_\tau(t)\ dt \quad &= \quad |\det(j)|^\tau \int_{T(\mathbb{A}_f)} \Omega\big([0,1]\cdot\rho(t)j\big)|\mathrm{N}(t)|^\tau\chi(t)\ dt \\ &\quad \cdot \prod_{v\mid\infty}\int_{F_v^\times} \Omega_v\big([0,1]\cdot x\big)|x|_v^{2\tau}\chi_{0,v}(x)\ d^\times x.\end{aligned}$$

We now compute the right hand side place-by-place. For an archimedean place v we may take $j_v = \begin{pmatrix} \sqrt{\Delta_v} & \\ & 1 \end{pmatrix}$ so that

$$\int_{F_v^\times} \Omega_v\big([0,1]\cdot xj\big)|x|_v^{2\tau}\chi_{0,v}(x)\ d^\times x = \int_{-\infty}^{\infty} e^{-\pi x^2}|x|^{2\tau-1}\ d^{\mathrm{Leb}}x.$$

The integral on the right is $2^{\tau-1}G_2(\tau) = 2^{\tau-1}L_v(\tau,\chi)$. If v is a finite place of F with $v \nmid \mathfrak{s}$ then

$$\begin{aligned}\int_{T(F_v)} \Omega_v([0,1]\cdot\rho(t)j)|\mathrm{N}(t)|_v^\tau \chi_v(t)\ dt &= \int_{T(F_v)} \mathbf{1}_{L_v}(\bar{t}\cdot w_0)|\mathrm{N}(t)|_v^\tau\chi_v(t)\ dt \\ &= \int_{T(F_v)} \mathbf{1}_{\mathcal{O}_{\mathfrak{c},v}}(\bar{t})|\mathrm{N}(t)|_v^\tau\chi_v(t)\ dt \\ &= \mathrm{Vol}(\mathcal{O}_{\mathfrak{c},v}^\times)\cdot L_v(\tau,\chi),\end{aligned}$$

the final equality by the argument of [36, p. 238]. Finally suppose that $v \mid \mathfrak{s}$. For any $t \in E_v^\times$ the value of $\Omega_v([0,1]\cdot\rho(t)j)$ is nonzero if and only if $[0,1]\rho(t)j$ generates the $\mathcal{O}_{F,v}$-module $(\mathfrak{s}_v \times \mathcal{O}_{F,v})/(\mathfrak{s}_v\times\mathfrak{s}_v)$, and when this is the case $\Omega_v([0,1]\cdot\rho(t)j) = \chi_{0,v}^{-1}(y)$ where $y \in \mathcal{O}_{F,v}^\times$ satisfies $[0,1]\rho(t)j \in [0,y]+\mathfrak{s}_v^2$. This condition is equivalent to tw_0 being an $\mathcal{O}_{F,w}$-generator of $L_v'/\mathfrak{s}_vL_v$, in which case the $y \in \mathcal{O}_{F,v}^\times$ above satisfies $\bar{t}w_0 \in yw_0 + \mathfrak{s}_vL_v$. Thus $y \equiv \vartheta_v(\bar{t}) \equiv \vartheta_v(t) \pmod{\mathfrak{s}_v}$ in the notation of §4.1. By Lemma 4.1.2 $\chi_{0,v}^{-1}(y) = \chi_v^{-1}(t)$. As the generators of $\mathcal{O}_{\mathfrak{c},v}/\mathfrak{s}_v\mathcal{O}_{\mathfrak{c}\mathfrak{s}^{-1},v}$ are exactly the units of $\mathcal{O}_{\mathfrak{c},v}$ we find

$$\begin{aligned}\int_{T(F_v)} \Omega_v([0,1]\cdot\rho(t)j)\cdot|\mathrm{N}(t)|_v^\tau\chi_v(t)\ dt &= \int_{\mathcal{O}_{\mathfrak{c},v}^\times} \chi_v^{-1}(t)\cdot|\mathrm{N}(t)|_v^\tau\chi_v(t)\ dt \\ &= \mathrm{Vol}(\mathcal{O}_{\mathfrak{c},v}^\times).\end{aligned}$$

It only remains to compute $\det(j)$. From the relation

$$[(\mathcal{O}_F + \mathcal{O}_F\sqrt{-\Delta})\cdot w_0]\cdot j^{-1} = \mathcal{O}_{\mathfrak{c}\mathfrak{s}^{-1}}\cdot w_0$$

we find

$$4\Delta\det(j)^{-2}\mathcal{O}_F = \mathrm{disc}(\mathcal{O}_F+\mathcal{O}_F\sqrt{-\Delta})\cdot\det(j)^{-2}\mathcal{O}_F = \mathrm{disc}(\mathcal{O}_{\mathfrak{c}\mathfrak{s}^{-1}}) = \mathfrak{d}(\mathfrak{c}/\mathfrak{s})^2.$$

Using $|\det(j)|_v^2 = \Delta_v$ for $v \mid \infty$ we obtain $2^{[F:\mathbb{Q}]}|\det(j)| = \sqrt{\mathrm{N}(\mathfrak{d}\mathfrak{c}^2\mathfrak{s}^{-2})}$. The proposition follows by combining these calculations. □

Corollary 4.5.3. *Suppose* $\mathrm{Re}(\tau) = 1/2$ *and let* $\phi_\tau \in \Pi_\tau$ *be the normalized newvector. Then*

$$L(1/2,\Pi_\tau\times\Pi_\chi) = \frac{4^{[F:\mathbb{Q}]}}{\sqrt{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}\mathfrak{c}^2)}}\left|\frac{1}{H_{F,\mathfrak{s}}}\int_{C_V} Q_\chi(g)\phi_\tau'(g)\ dg\right|^2$$

where S_V *is the measure space of §4.4 defined with* V *in place of* U*.*

Proof. Using $\widehat{\mathcal{O}}_{\mathfrak{c}}^\times/V_T \cong (\mathcal{O}_F/\mathfrak{s})^\times$, the measures on $T(F)\backslash T(\mathbb{A}_f)/Z(\mathbb{A}_f)$ and C_V are related by

$$\int_{C_V} Q_\chi(t)\phi_\tau'(t)\ dg = H_{F,\mathfrak{s}}\int_{T(F)\backslash T(\mathbb{A}_f)/Z(\mathbb{A}_f)} Q_\chi(t)\phi_\tau'(t)\ dg$$

while Lemma 4.5.1 implies

$$\epsilon(1/2,\chi_0)\cdot\phi_\tau = \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{s})^{2\tau-1/2}\cdot\mathcal{E}_\tau.$$

The corollary now follows immediately from $|L(\tau,\chi)|^2 = L(1/2,\Pi_\tau\times\Pi_\chi)$, Proposition 4.5.2, and the fact that the restriction of Q_χ to $T(\mathbb{A}_f)$ is simply χ. □

4.6. **Descent to low level.** Assume that either Π_v is a weight 2 discrete series at each archimedean v or that Π_v is a weight 0 principal series at each archimedean v. In the weight 2 case we assume that $\epsilon(1/2,\mathfrak{r}) = 1$ and B is totally definite, as in §4.3, and in the weight 0 case we assume that $\epsilon(1/2,\mathfrak{r}) = (-1)^{[F:\mathbb{Q}]}$ and B is totally indefinite, as in §4.4. For each $v \mid \mathfrak{d}\mathfrak{c}$ the representation Π_v is isomorphic to a principal series $\Pi(\mu_v, \chi_{0,v}^{-1}\mu_v^{-1})$ with μ_v unramified, and we set $\alpha_v = \mu_v(\varpi)$ for any uniformizer ϖ of F_v. By the argument of [36, §17] for each $v \mid \mathfrak{d}\mathfrak{c}$ there are rational functions $\mathbf{a}_v$, $\mathbf{b}_v$, $\mathbf{c}_v$ which, crucially, depend only on the data (F_v, E_v, χ_v) and not on the representation Π, such that

$$\widehat{B}(\mathcal{O}_F;\phi_\Pi^{\#}) = \widehat{B}(\mathcal{O}_F;\phi_\Pi)\cdot\prod_{v\mid\mathfrak{d}\mathfrak{c}}\mathbf{a}_v(\alpha_v)$$

and

$$||\phi_\Pi^{\#}||_{K_0(\mathfrak{d}\mathfrak{r})} = ||\phi_\Pi||^2_{K_0(\mathfrak{n})}\cdot\prod_{v\mid\mathfrak{d}\mathfrak{c}}\mathbf{b}_v(\alpha_v)$$

where $\phi_\Pi\in\Pi$ is the normalized newvector and $\phi_\Pi^{\#}\in\Pi$ is the projection of ϕ_Π to the quasi-new line. Using (4.2) in place of [36, Lemma 17.2], the rational function $\mathbf{c}_v$ is defined by the relation

$$\frac{1}{||\phi_{\Pi'}||^2}\left|\int_{C_U} Q_\chi(g)\phi_{\Pi'}(g)\ dg\right|^2 = \frac{1}{||\phi^\chi_{\Pi'}||^2}\left|\int_{C_U} P_\chi(g)\phi^\chi_{\Pi'}(g)\ dg\right|^2\cdot\prod_{v\mid\mathfrak{d}\mathfrak{c}}\mathbf{c}_v(\alpha_v)$$

where Π' is the automorphic representation of $G(\mathbb{A})$ whose Jacquet-Langlands lift is Π, $\phi^\chi_{\Pi'}$ is a toric newvector in Π' in the sense of Definition 4.2.4, $\phi_{\Pi'}\in\Pi'$ is a newvector in the sense of Definition 4.2.2, and $||\cdot||$ is any $G(\mathbb{A})$-invariant norm on Π' (e.g. $||\cdot||^2=\int_{S_U}|\cdot|^2$). If $v\nmid\mathfrak{s}$ then $\chi_{0,v}$ is unramfied and we must have $\mathbf{a}_v(\alpha_v) = \mathbf{a}_v(\alpha_v^{-1}\chi_{0,v}^{-1}(\varpi))$ due to the the isomorphism $\Pi(\mu_v,\chi_{0,v}^{-1}\mu_v^{-1})\cong\Pi(\chi_{0,v}^{-1}\mu_v^{-1},\mu_v)$, and similarly for $\mathbf{b}_v$ and $\mathbf{c}_v$. Set $\mathbf{a}_\Pi=\prod_{v\mid\mathfrak{d}\mathfrak{c}}\mathbf{a}_v(\alpha_v)$ and define $\mathbf{b}_\Pi$ and $\mathbf{c}_\Pi$ similarly. Proposition 4.3.2 (for the weight 2 case) and Proposition 4.4.1 (for the weight 0 case) give

$$\begin{aligned}&2^{|S|}H_F\lambda_U^{-1}[\widehat{\mathcal{O}}_E^\times:U_T]\widehat{B}(\mathcal{O}_F;\phi_\Pi^{\#})L(1/2,\Pi\times\Pi_\chi)\\ &\quad= |d|^{1/2}2^{f\cdot[F:\mathbb{Q}]}||\phi_\Pi^{\#}||^2_{K_0(\mathfrak{d}\mathfrak{r})}\cdot\frac{\left|\int_{C_U}P_\chi(g)\phi^\chi_{\Pi'}(g)\ dg\right|^2}{\int_{S_U}|\phi^\chi_{\Pi'}(g)|^2\ dg}.\end{aligned}$$

where $f=1$ in the weight 2 case and $f=2$ in the weight 0 case. As $\widehat{B}(\mathcal{O}_F,\phi_\Pi)=1$ we find, using $\lambda_V H_{F,\mathfrak{s}} = H_F[\widehat{\mathcal{O}}_\mathfrak{c}^\times:V_T]$ and (3.3) (which holds also with C_U and C_V replaced by S_U and S_V), that

$$\kappa\cdot\mathbf{a}_\Pi\mathbf{c}_\Pi\cdot\frac{L(1/2,\Pi\times\Pi_\chi)}{||\phi_\Pi||^2_{K_0(\mathfrak{n})}} = \frac{\mathbf{b}_\Pi\cdot 2^{f\cdot[F:\mathbb{Q}]}}{H_{F,\mathfrak{s}}\sqrt{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}\mathfrak{c}^2)}}\cdot\frac{\left|\int_{C_V}Q_\chi(t)\phi_{\Pi'}(t)\ dt\right|^2}{\int_{S_V}|\phi_{\Pi'}(g)|^2\ dg}. \tag{4.9}$$

Here $\kappa=\prod_{v\mid\mathfrak{d}\mathfrak{c}}\kappa_v$ with

$$\kappa_v = \frac{[\mathcal{O}^\times_{E,v}:U_{T,v}]}{[\widehat{\mathcal{O}}^\times_{\mathfrak{c},v}:V_{T,v}]}\cdot\frac{|c|_v}{[V_v:U_v]}\begin{cases}2 & \text{if } v\mid\mathfrak{d}\\ 1 & \text{if } v\mid\mathfrak{c}\end{cases}$$

where $c\in\mathbb{A}^\times$ satisfies $c\mathcal{O}_F=\mathfrak{c}$.

Proof of Theorems 4.3.3 and 4.4.2. It follows from the definition of the quasi-new line that $\phi_\Pi^\# \neq 0$ (in the notation of §2.8 we have $\Lambda_v(\phi_{\Pi,v}) \neq 0$ for each $v \mid \mathfrak{d}\mathfrak{r}$, and so $\phi_{\Pi,v}$ has nontrivial projection to the quasi-new line in Π_v), and hence $\mathbf{b}_\Pi \neq 0$. It therefore suffices by (4.9) to prove that $\kappa \cdot \mathbf{a}_\Pi \mathbf{c}_\Pi = \mathbf{b}_\Pi$. Let us suppose for the moment that Π is of parallel weight 0 and that $\mathfrak{m} = \mathcal{O}_F$. Thus $\epsilon(1/2, \mathfrak{r}) = (-1)^{[F:\mathbb{Q}]}$ and we are in the situation of §4.4. The quaternion algebra B is split, and we let $\rho : G \cong \mathrm{GL}_2$ and $j \in \mathrm{GL}_2(\mathbb{A})$ be as in §4.5. Set $\Pi' = \Pi$ and for each $\phi \in \Pi$ set $\phi'(g) = \phi(\rho(g)j)$. Fix a Haar measure on $\mathrm{GL}_2(\mathbb{A}_f)$ and, as always, normalize the Haar measure on $Z(\mathbb{A}_f)$ to give $\widehat{\mathcal{O}}_F^\times$ volume 1. Define a Haar measure on $G(\mathbb{A}_f)$ by demanding that ρ be an isomorphism of measure spaces. For any $\phi \in \Pi$ we now have, tediously keeping track of the normalizations of measures,

$$\begin{aligned}
\int_{S_V} |\phi'|^2 &= \mathrm{Vol}(V)^{-1} \int_{G(F)\backslash X \times G(\mathbb{A}_f)/V} |\phi'|^2 \\
&= \mathrm{Vol}(V)^{-1} \frac{1}{[Z(F) \cap \widehat{\mathcal{O}}_F^\times : Z(F) \cap V]} \int_{G(F)\backslash X \times G(\mathbb{A}_f)/\widehat{\mathcal{O}}_F^\times} |\phi'|^2 \\
&= \mathrm{Vol}(V)^{-1} \frac{[Z(\mathbb{A}_f) : Z(F)\widehat{\mathcal{O}}_F^\times]}{[Z(F) \cap \widehat{\mathcal{O}}_F^\times : Z(F) \cap V]} \int_{G(F)\backslash X \times G(\mathbb{A}_f)/Z(\mathbb{A}_f)} |\phi'|^2.
\end{aligned}$$

Using $jKj^{-1} = \rho(V)$ and $V_Z = \{x \in \widehat{\mathcal{O}}_F^\times \mid x \in 1 + \widehat{s}\}$ we find that

$$\int_{S_V} |\phi'|^2 = H_F \lambda_V^{-1} ||\phi||_K^2 = H_{F,\mathfrak{s}} ||\phi||_{K_0(\mathfrak{n})}^2.$$

We may now write (4.9) as

$$\kappa \cdot \mathbf{a}_\Pi \mathbf{c}_\Pi \cdot L(1/2, \Pi \times \Pi_\chi) = \frac{\mathbf{b}_\Pi \cdot 2^{f \cdot [F:\mathbb{Q}]}}{\sqrt{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}\mathfrak{c}^2)}} \cdot \left| \frac{1}{H_{F,\mathfrak{s}}} \int_{C_V} Q_\chi(t) \phi'_\Pi(t)\, dt \right|^2. \tag{4.10}$$

The point is that in this formulation no L^2 norms appear, and the statement of the formula makes sense even if Π is noncuspidal. The argument of [36, §18] shows that the equality (4.10) can be extended to the principal series representation Π_τ of §4.5 for any $\tau \in \mathbb{C}$ with $\mathrm{Re}(\tau) = 1/2$ (so that Π_τ is unitary), provided that χ does not factor through the norm map $\mathbb{A}_E^\times \to \mathbb{A}^\times$ (so that Π_χ is cuspidal by Lemma 2.3.3 and (2.11) still holds).

If for each $v \mid \mathfrak{d}\mathfrak{c}$ we let q_v denote the cardinality of the residue field of v, then taking $\Pi = \Pi_\tau$ and $\phi_\Pi = \phi_\tau$ in (4.10) and comparing with Lemma 4.5.3 (and still assuming that Π_χ is cuspidal) gives

$$\prod_{v \mid \mathfrak{d}\mathfrak{c}} \kappa_v \mathbf{a}_v(q_v^{1/2-\tau}) \mathbf{c}_v(q_v^{1/2-\tau}) = \prod_{v \mid \mathfrak{d}\mathfrak{c}} \mathbf{b}_v(q_v^{1/2-\tau}).$$

As in the proof of [36, Proposition 19.2], letting τ vary and letting χ vary over characters which do not factor through the norm while holding the components χ_v for $v \mid \mathfrak{d}\mathfrak{c}$ fixed, we find the equality of rational functions $\kappa \prod \mathbf{a}_v \mathbf{c}_v = \prod \mathbf{b}_v$ where each product is over all $v \mid \mathfrak{d}\mathfrak{c}$. □

5. Central derivatives

In this section we relate the Néron-Tate heights of certain CM points on Shimura curves to derivatives of automorphic L-functions. As in [34] the method is to compute the arithmetic intersection pairings of various CM-divisors and compare these intersection multiplicities to the Whittaker coefficients of the automorphic form $\Phi_{\mathfrak{r}}$ of §2.6. These intersection multiplicities decompose as a sum of local intersection multiplicities, and the calculations of §5 and §6 of [34] show that the calculation of local multiplicities can be reduced to the calculation of linking numbers of CM-cycles on totally definite quaternion algebras. Fortunately for us, this reduction step is done in [34] in a very general context, and includes not only on Shimura curves with arbitrary level structure but also Shimura curves associated to the algebraic group G below (as opposed to the group G/Z). Thus we may cite from Zhang the crucial Propositions 5.3.1 and 5.4.1 below, which reduce the local intersection theory at nonsplit primes to the calculations we have done in §3.

Throughout §5 we assume that the representation Π of §1.1 satisfies Hypothesis 1.1.1 and that Π_v lies in the discrete series of weight 2 for every archimedean v. Set $\mathfrak{r} = \mathfrak{m}\mathfrak{c}^2$ and assume that $\omega(\mathfrak{m}) = (-1)^{[F:\mathbb{Q}]-1}$. The epsilon factor of §2.4 then satisifies $\epsilon(1/2, \mathfrak{r}) = -1$ and so $L(1/2, \Pi \times \Pi_\chi) = 0$ by the functional equation (2.6) and the Rankin-Selberg integral representation (2.8) with $b = 1$. Fix an archimedean place w_∞ of F and let B be the quaternion algebra over F characterized by

$$B_v \text{ is split} \iff \epsilon_v(1/2, \mathfrak{r}, \psi) = 1 \text{ or } v = w_\infty$$

for every place v. Thus B is indefinite at w_∞ and definite at all other archimedean places. The reduced discriminant of B divides $\mathfrak{m}$ and, as E_v is a field whenever B_v is nonsplit, there is an embedding $E \to B$ which we fix. Let G, T, and Z be the algebraic groups over F defined at the beginning of §3. For any ideal $\mathfrak{b} \subset \mathcal{O}_F$ let $\mathcal{O}_{\mathfrak{b}} = \mathcal{O}_F + \mathfrak{b}\mathcal{O}_E$ denote the order of $\mathcal{O}_E$ of conductor $\mathfrak{b}$. Fix an algebraic closure F^{alg} of F containing E and an embedding $F^{\mathrm{alg}} \hookrightarrow \mathbb{C}$ lying above w_∞.

General references for Shimura curves include [3, 23, 24, 26, 35, 34].

5.1. **Shimura curves.** Throughout §5.1 we let U be an arbitrary compact open subgroup of $G(\mathbb{A}_f)$. The chosen embedding $E \to \mathbb{C}$ determines an isomorphism of real algebraic groups $\mathbb{S} \cong T \times_F \mathbb{R}$, where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. The embedding $T \to G$ therefore determines an embedding of real algebraic groups

$$x_0 : \mathbb{S} \to G \times_F \mathbb{R} \to (\mathrm{Res}_{F/\mathbb{Q}} G) \times_{\mathbb{Q}} \mathbb{R}.$$

Let X be the $G(\mathbb{R})$-conjugacy class of x_0 in the set of all such embeddings. If $F \neq \mathbb{Q}$ or if $B \not\cong M_2(F)$ we define a compact Riemann surface

$$X_U(\mathbb{C}) = G(F)\backslash X \times G(\mathbb{A}_f)/U. \tag{5.1}$$

For $x \in X$ and $g \in G(\mathbb{A}_f)$ let $[x, g]$ denote the image of (x, g) in $X_U(\mathbb{C})$. If $F = \mathbb{Q}$ and B is split then the right hand side of (5.1) is noncompact, and $X_U(\mathbb{C})$ is defined as the usual compactification of the right hand side obtained by adjoining finitely many cusps. The connected components of $X_U(\mathbb{C})$ are indexed by the set

$$Z_U(\mathbb{C}) = Z(F)^+\backslash Z(\mathbb{A}_f)/\mathrm{N}(U)$$

where $Z(F)^+ \subset Z(F) \cong F^\times$ is the subgroup of totally positive elements and $\mathrm{N}(U)$ is the image of U under the reduced norm $G(\mathbb{A}_f) \to Z(\mathbb{A}_f)$. The canonical map $X_U(\mathbb{C}) \to Z_U(\mathbb{C})$ is given by $[x, g] \mapsto \mathrm{N}(g)$.

Let X_U denote Shimura's canonical model of $X_U(\mathbb{C})$ over $\mathrm{Spec}(F)$. Let F_U/F be the abelian extension of F which, under the reciprocity map of class field theory, has $\mathrm{Gal}(F_U/F) \cong Z_U(\mathbb{C})$. The component map $X_U(\mathbb{C}) \to Z_U(\mathbb{C})$ arises from a morphism of F-schemes $X_U \to Z_U$ where Z_U is (noncanonically) isomorphic to $\mathrm{Spec}(F_U)$. For each geometric point $\alpha : \mathrm{Spec}(F^{\mathrm{alg}}) \to Z_U$ define a smooth connected projective curve over F^{alg}

$$X_U^\alpha = X_U \times_{Z_U} \mathrm{Spec}(F^{\mathrm{alg}}).$$

The *Jacobian* J_U of X_U is the abelian variety over F defined by

$$J_U = \mathrm{Res}_{Z_U/F}(\mathrm{Pic}^0_{X_U/Z_U})$$

so that the geometric fiber of J_U decomposes as

$$J_U \times_F F^{\mathrm{alg}} \cong \prod_{\alpha \in Z_U(F^{\mathrm{alg}})} J_U^\alpha$$

where J_U^α is the Jacobian of X_U^α. There is a $\mathrm{Gal}(F^{\mathrm{alg}}/F)$ invariant function

$$\mathrm{Hg} : X_U(F^{\mathrm{alg}}) \to J_U(F^{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

the *Hodge embedding*, described in detail in [6, §3.5]. Briefly, Zhang [34, §6.2] constructs the *Hodge class* $\mathcal{L} \in \mathrm{Pic}(X_U) \otimes_{\mathbb{Z}} \mathbb{Q}$ having degree 1 on every geometric component. Each $P \in X_U(F^{\mathrm{alg}})$ determines a geometric point $\alpha \in Z_U(F^{\mathrm{alg}})$, and we let $\mathcal{L}_P$ denote the restriction of $\mathcal{L}$ to X_U^α. Letting $\mathcal{O}(P) \in \mathrm{Pic}(X_U \times_F F^{\mathrm{alg}})$ denote the class of P we define

$$\mathrm{Hg}(P) = \mathcal{O}(P) \otimes \mathcal{L}_P^{-1} \in J_U^\alpha(F^{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For any finite extension L/F the Néron-Tate height on $J_U(L)$ is denoted by $\langle \cdot, \cdot \rangle_{U,L}^{\mathrm{NT}}$. The *normalized* Néron-Tate height on $J_U(F^{\mathrm{alg}})$ is defined by

$$\langle x, y \rangle_U^{\mathrm{NT}} = \frac{1}{[L : F]} \langle x, y \rangle_{U,L}^{\mathrm{NT}}$$

where L is any finite extension of F large enough that x and y are defined over L. Fix two points $P, Q \in X_U(F^{\mathrm{alg}})$ and choose a finite Galois extension L/F large enough that P and Q are both defined over L. To compute the Néron-Tate pairing of $\mathrm{Hg}(P)$ and $\mathrm{Hg}(Q)$ we use the arithmetic intersection theory of Gillet-Soulé [9, 29] as in §5.3 and §6.1 of [34]. Suppose that U is small enough that X_U admits a canonical regular model $\underline{X}_U$, proper and flat over $\mathcal{O}_F$, as in [35, §1.2.5]. Let $\underline{Z}_U$ be the normalization of $\mathrm{Spec}(\mathcal{O}_F)$ in Z_U, so that $\underline{Z}_U \cong \mathrm{Spec}(\mathcal{O}_{F_U})$ (noncanonically) and the component map $X_U \to Z_U$ extends to a map of $\mathcal{O}_F$-schemes $\underline{X}_U \to \underline{Z}_U$. As $Z_U(L) \neq \emptyset$ there are $[F_U : F]$ distinct embeddings $F_U \to L$, and so $[F_U : F]$ distinct morphisms $\mathrm{Spec}(\mathcal{O}_L) \to \underline{Z}_U$. Let $\mathcal{Z}_U$ denote the disjoint union of $[F_U : F]$ copies of $\mathrm{Spec}(\mathcal{O}_L)$ so that $\mathcal{Z}_U$ is naturally an $\mathcal{O}_L$-scheme which admits an $\mathcal{O}_F$-morphism $\mathcal{Z}_U \to \underline{Z}_U$. Let $\mathcal{X}_U$ be the minimal resolution of singularities of the $\mathcal{O}_L$-scheme $\underline{X}_U \times_{\underline{Z}_U} \mathcal{Z}_U$. The scheme $\mathcal{X}_U$ has generic fiber $X_U \times_F L$ and is a disjoint union of $[F_U : F]$ proper and flat curves over $\mathcal{O}_L$ indexed by $Z_U(F^{\mathrm{alg}})$, each with geometrically connected generic fiber. The Hodge class $\mathcal{L}$ on X_U admits a natural extension to $\underline{X}_U$ [35, §4.1.4] which we pull back to a class $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}_U) \otimes_{\mathbb{Z}} \mathbb{Q}$. For each embedding $i : L \to \mathbb{C}$ the Riemann surface $(\mathcal{X}_U \times_{\mathcal{O}_L} \mathbb{C})(\mathbb{C})$ has a canonical volume form μ which on each connected component has total volume 1 and whose pull back to the upper half-plane (under any such parametrization) is a multiple of the hyperbolic volume form $y^{-2}dxdy$. By [20, Theorem I.4.2] there is a Hermitian metric ρ_i, unique up to scaling, on the pull-back of $\mathcal{L}$ to $\mathcal{X}_U \times_{\mathcal{O}_L} \mathbb{C}$ whose Chern form is μ. Letting ρ denote the tuple (ρ_i) indexed by embeddings i as above, the pair $\widehat{\mathcal{L}} = (\mathcal{L}, \rho)$ is then an element of $\widehat{\mathrm{Pic}}(\mathcal{X}_U)$ as in [34, §6.1].

Going back to the point $P \in X_U(L)$, let $\mathcal{X}_U^\alpha$ be the connected component of $\mathcal{X}_U$ containing P. The *arithmetic closure* (as in [34, §6.1] or [36, §9]) $\widehat{P} \in \widehat{\mathrm{Div}}(\mathcal{X}_U)$ of P with respect to $\widehat{\mathcal{L}}$ is a pair $\widehat{P} = (\mathcal{P} + D_P, g_P)$ where $\mathcal{P}$ is the Zariski closure of P on $\mathcal{X}_U$ and $g_P = (g_{P,i})$ is a tuple indexed by embeddings $i : L \to \mathbb{C}$ with $g_{P,i}$ a smooth function on the complement of P in $(\mathcal{X}_U \times_{\mathcal{O}_L} \mathbb{C})(\mathbb{C})$ such that $2 \cdot g_{P,i}$ is a Green's function for P with respect to μ (in the sense of [20, §II.1]) on the component indexed by α, and is identically 0 on the other components. Lang and Zhang use different normalizations for Green's functions, hence the factor of 2; our g_P is Zhang's $g(P, \cdot)$. Finally D_P is a vertical divisor on $\mathcal{X}_U^\alpha$ chosen so that $\mathcal{P} + D_P$ has trivial intersection multiplicity with every vertical divisor, and so that for any finite place w of L the restriction of $\mathcal{L}$ to the sum of the components of D_P above w has degree 0. One defines $\widehat{Q} = (\mathcal{Q} + D_Q, g_Q)$ in the same way. The Hodge index theorem now tells us that

$$\langle \mathrm{Hg}(P), \mathrm{Hg}(Q) \rangle_U^{\mathrm{NT}} = \frac{-1}{[L:F]} \langle \widehat{P} - \widehat{\mathcal{L}}_P, \widehat{Q} - \widehat{\mathcal{L}}_Q \rangle_{\mathcal{X}_U}^{\mathrm{Ar}}$$

where $\widehat{\mathcal{L}}_P$ is the restriction of $\widehat{\mathcal{L}}$ to the component of $\mathcal{X}_U$ containing P (and similarly with P replaced by Q) and the pairing on the right is the Gillet-Soule arithmetic intersection pairing on $\widehat{\mathrm{Pic}}(\mathcal{X}_U)$ defined by [36, (9.3)].

For each place w of F fix an extension w^{alg} to F^{alg}. As we assume that $P \neq Q$ there is a decomposition of the arithmetic intersection pairing as a sum of local Green's functions

$$\langle \widehat{P}, \widehat{Q} \rangle_{\mathcal{X}_U}^{\mathrm{Ar}} = \sum_w \sum_{\sigma \in \mathrm{Gal}(L/F)} d_w \cdot g(P^\sigma, Q^\sigma)_{U, w^{\mathrm{alg}}}$$

where the sum is over all places of F and terms on the right are as follows. If $w \mid \infty$ then $d_w = 1$ and $g(P, Q)_{U, w^{\mathrm{alg}}} = g_{P,i}(Q)$ where $i : L \to \mathbb{C}$ is the embedding determined by w^{alg}. If w is nonarchimedean then $d_w = \log q_w$ where q_w is the size of the residue field of w, and

$$g(P, Q)_{U, w^{\mathrm{alg}}} = e(L_{w^{\mathrm{alg}}}/F_w)^{-1} i_{w^{\mathrm{alg}}}(\mathcal{P} + D_P, \mathcal{Q} + D_Q)_{\mathcal{X}_U}$$

where $e(L_{w^{\mathrm{alg}}}/F_w)$ is the ramification index and $i_{w^{\mathrm{alg}}}(\cdot, \cdot)_{\mathcal{X}_U}$ is the intersection pairing on $\mathcal{X}_U \times_{\mathcal{O}_L} \mathcal{O}_{L, w^{\mathrm{alg}}}$ defined in [20, III.2] for divisors with no common components and extended in [20, III.3] to divisors with common vertical components. The Green's function $g(P, Q)_{U, w^{\mathrm{alg}}}$ does not depend on the choice of L and extends bi-additively to a Hermitian pairing on divisors with complex coefficients on $X_U \times_F F^{\mathrm{alg}}$ having disjoint support.

If U is not sufficiently small in the sense of [35, §1.2.5] then choose $U' \subset U$ which is sufficiently small and define

$$g(P, Q)_{U, w^{\mathrm{alg}}} = \frac{1}{\deg(\pi)} g(\pi^* P, \pi^* Q)_{U', w^{\mathrm{alg}}}$$

where $\pi : X_{U'} \to X_U$ is the degeneracy map with $\deg(\pi) = [F^\times U : F^\times U']$. This does not depend on the choice of sufficiently small U'.

5.2. **Special cycles and Hecke correspondences.** For the remainder of §5 we let U and V denote the compact open subgroups of $G(\mathbb{A}_f)$ constructed in §4.1 and recall that we constructed there CM cycles P_χ and $P_{\chi, \mathfrak{a}}$ of level U (for $\mathfrak{a}$ any ideal of $\mathcal{O}_F$ prime to $\mathfrak{c}$) and a CM cycle Q_χ of level V. Let $\epsilon_v \in B_v$ be the element of Lemma 4.1.3 used in the construction of U, and note that U_v is a maximal compact open subgroup of $G(F_v)$ for $v \nmid \mathfrak{d}\mathfrak{r}\infty$. For $\mathfrak{a}$ prime to $\mathfrak{d}\mathfrak{r}$ there are algebraic Hecke correspondence $T_{\mathfrak{a}}^{\mathrm{Pic}}$ and $T_{\mathfrak{a}}^{\mathrm{Alb}}$ on X_U characterized by their action on points of $X_U(\mathbb{C})$

$$T_{\mathfrak{a}}^{\mathrm{Pic}}[x,g] = \sum_{h \in U \backslash H(\mathfrak{a})} [x, gh^{-1}] \qquad T_{\mathfrak{a}}^{\mathrm{Alb}}[x,g] = \sum_{h \in H(\mathfrak{a})/U} [x, gh],$$

where $H(\mathfrak{a})$ was defined in §4.1. We also have diamond automorphisms of X_U defined by

$$\langle \mathfrak{a} \rangle^{\mathrm{Pic}}[x,g] = [x, ga^{-1}] \qquad \langle \mathfrak{a} \rangle^{\mathrm{Alb}}[x,g] = [x, ga]$$

where $a \in \mathbb{A}^\times$ satisfies $a\mathcal{O}_F = \mathfrak{a}$ and $a_v = 1$ for $v \mid \infty$. Restricting $T_{\mathfrak{a}}^{\mathrm{Pic}}$, $T_{\mathfrak{a}}^{\mathrm{Alb}}$ and the diamond automorphisms to divisors on X_U which have degree zero on every geometric component we obtain endomorphisms, denoted the same way, of J_U.

We view the set of CM points of level U on G as a subset of $X_U(\mathbb{C})$ using the injection $C_U \to X_U(\mathbb{C})$ defined by $T(F)gU \mapsto [x_0, g]$. By Shimura's reciprocity law [24, §12] all points of C_U are defined over the maximal abelian extension of E in $\mathbb{C}$ and satisfy

$$[x_0, g]^\sigma = [x_0, t^{-1}g]$$

where $\sigma = [t, E]$ is the arithmetic Artin symbol of t as in [28, §5.2]. Any CM-cycle P of level U can be written as a sum of characteristic functions of CM points, and so can be viewed as a divisor (with complex coefficients) on $X_U \times_F F^{\mathrm{alg}}$ in an obvious way. Setting $P = [x_0, 1]$ we then have

$$P_\chi = \sum_{t \in T(F) \backslash T(\mathbb{A}_f)/U_T} \overline{\chi(t)} \cdot P^{[t,E]}.$$

This divisor is rational over the abelian extension E_χ/E cut out by χ. As divisors on $X_U \times_F E_\chi$ we have $T_{\mathfrak{a}}^{\mathrm{Pic}} P_\chi = P_{\chi,\mathfrak{a}}$ and $\langle \mathfrak{a} \rangle^{\mathrm{Pic}} P_\chi = \chi_0(\mathfrak{a}) P_\chi$.

For $\mathfrak{a}$ prime to $\mathfrak{d}\mathfrak{r}$ let $P^0_{\chi,\mathfrak{a}}$ denote the restriction of $P_{\chi,\mathfrak{a}}$ to the complement of the image of $T(\mathbb{A}_f) \to C_U$. In particular $P^0_{\chi,\mathfrak{a}}$ and P_χ have disjoint support. Fix $a \in \mathbb{A}^\times$ with $a\mathcal{O}_F = \mathfrak{a}$ and define

$$r_\chi(\mathfrak{a}) = \prod_{v \nmid \infty} |a|_v^{-1/2} B_v(a; \theta).$$

We note that r_χ is a derivation of $\Pi_\chi \otimes |\cdot|^{1/2}$ in the sense of [34, Definition 3.5.3]. Exactly as in [34, Lemma 6.2.1], (using our Corollaries 3.3.9 and 3.4.5 to evaluate $P_{\chi,\mathfrak{a}}(1)$ instead of [34, Lemma 4.2.1]) we have

$$P_{\chi,\mathfrak{a}} = P^0_{\chi,\mathfrak{a}} + r_\chi(\mathfrak{a}) \cdot P_\chi. \tag{5.2}$$

5.3. **Intersections at nonsplit primes away from $\mathfrak{d}\mathfrak{r}$.** Suppose $w \nmid \mathfrak{d}\mathfrak{r}$ is a finite place of F which is inert in E and fix a place w^{alg} of F^{alg} above w. Note that the quaternion algebra B_w is split and, as $R_w = \mathcal{O}_{E,w} + \mathcal{O}_{E,w}\epsilon_w$ is a maximal order of B_w, $U_w = R_w^\times$ is a maximal compact open subgroup $G(F_w)$. We wish to compute $g(P_\chi, P^0_{\chi,\mathfrak{a}})_{U,w^{\mathrm{alg}}}$. Let $\tilde{B}$ be the totally definite quaternion algebra obtained from B by interchanging invariants at w_∞ and w. That is, $\tilde{B}$ is defined by $\{\text{places } v \text{ of } F \mid \tilde{B}_v \not\cong B_v\} = \{w, w_\infty\}$. As E_v is a field for every place v at which $\tilde{B}$ is nonsplit, we may fix an embedding $E \to \tilde{B}$. Denote by $\tilde{G}$ the algebraic group over F defined by $\tilde{G}(A) = (\tilde{B} \otimes_F A)^\times$.

For each finite place $v \neq w$ fix an isomorphism $\sigma_v : G(F_v) \cong \tilde{G}(F_v)$ compatible with the embeddings of $T(F_v)$ into $G(F_v)$ and $\tilde{G}(F_v)$ and define

$$\tilde{\epsilon}_v = \sigma_v(\epsilon_v) \qquad \tilde{U}_v = \sigma_v(U_v).$$

Pick $\tilde{\epsilon}_w \in \tilde{B}_w$ so that $E_w \tilde{\epsilon}_w = \tilde{B}_w^-$ and $\mathrm{ord}_w(\tilde{\mathrm{N}}(\tilde{\epsilon}_w)) = 1$, where $\tilde{\mathrm{N}}$ is the reduced norm on $\tilde{B}_w$. Then $\tilde{R}_w = \mathcal{O}_{E,w} + \mathcal{O}_{E,w}\tilde{\epsilon}_w$ is the unique maximal order in $\tilde{B}_w$, and we define $\tilde{U}_w = \tilde{R}_w^\times$. Define a function $\sigma_w : G(F_v) \to \tilde{G}(F_w)/\tilde{U}_w$ by $\sigma_w(g) = \tilde{g}\tilde{U}_w$ for any $\tilde{g} \in \tilde{G}(F_w)$ satisfying $\mathrm{ord}_w(\mathrm{N}(g)) = \mathrm{ord}_w(\tilde{\mathrm{N}}(\tilde{g}))$. Set $\tilde{U} = \prod_v \tilde{U}_v$, a compact open subgroup of $\tilde{G}(\mathbb{A}_f)$. Taking the product of the σ_v we obtain a map of left $T(\mathbb{A}_f)$-sets $\sigma : G(\mathbb{A}_f)/U \to \tilde{G}(\mathbb{A}_f)/\tilde{U}$ and a push-forward map $f \mapsto \sigma_* f$ from finitely supported functions on $G(\mathbb{A}_f)/U$ to finitely supported functions on $\tilde{G}(\mathbb{A}_f)/\tilde{U}$ defined by

$$(\sigma_* f)(x) = \sum_{\sigma(y)=x} f(y).$$

As the natural projection $G(\mathbb{A}_f)/U \to C_U$ has finite fibers, any CM-cycle of level U may be viewed as a finitely supported function on $G(\mathbb{A}_f)/U$. The push-forward is then a left $T(F)$-invariant function on $\tilde{G}(\mathbb{A}_f)/\tilde{U}$, and so there is an induced push-forward σ_* from CM-cycles on G of level U to CM-cycles on $\tilde{G}$ of level $\tilde{U}$.

Fix a uniformizer ϖ of F_w and for each $k \geq 0$ let $A_k = \mathcal{O}_{F,w} + \varpi^k \mathcal{O}_{E,w}$. For each $x \in C_U$ define the *w-conductor* of $x = T(F)gU$ to be the integer k determined by

$$A_k^\times = g_w U_w g_w^{-1} \cap T(F_w).$$

Proposition 5.3.1. *Suppose that P and Q are disjoint CM-cycles of level U with P supported on points of w-conductor k and Q supported on points of w-conductor 0. Then*

$$g(P,Q)_{U,w^{\mathrm{alg}}} = \sum_{\gamma \in T(F)\backslash \tilde{G}(F)/T(F)} \langle \sigma_* P, \sigma_* Q\rangle_{\tilde{U}}^{\gamma} \cdot M_k(\gamma)$$

where

$$M_k(\gamma) = \begin{cases} \frac{\mathrm{ord}_w(\xi\varpi)}{2} & \text{if } k = 0 \text{ and } \xi \neq 0 \\ 0 & \text{if } k = 0 \text{ and } \xi = 0 \\ [\mathcal{O}_{E,w}^\times : A_k^\times]^{-1} & \text{if } k > 0. \end{cases}$$

Proof. See Lemmas 5.5.2 and 6.3.5 of [34]. □

Suppose $\mathfrak{a}$ is an ideal of $\mathcal{O}_F$ prime to $\mathfrak{d}\mathfrak{r}$. For any finite place v we may replace B_v by $\tilde{B}_v$ and ϵ_v by $\tilde{\epsilon}_v$ everywhere in §3.3 and §3.4, giving a function $\tilde{P}_{\chi,\mathfrak{a},v}$ on $\tilde{G}(F_v)/\tilde{U}_v$. Taking the product over all finite v gives a CM-cycle $\tilde{P}_{\chi,\mathfrak{a}}$ of level $\tilde{U}$ on $\tilde{G}$. When $\mathfrak{a} = \mathcal{O}_F$ we omit it from the notation. Define an ideal $\mathfrak{c}$ of $\mathcal{O}_F$ by $\mathrm{ord}_v(\mathfrak{c}) = \mathrm{ord}_v(\mathrm{N}(\tilde{\epsilon}_v))$ for all finite places v, so that

$$\mathrm{ord}_v(\mathfrak{c}) = \mathrm{ord}_v(\mathfrak{r}) + \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Proposition 5.3.2. *Suppose $\mathfrak{a}$ is prime to $\mathfrak{c}$. There is a constant κ, independent of $\mathfrak{a}$, such that*

$$g(P_{\chi,\mathfrak{a}}^0, P_\chi)_{U,w^{\mathrm{alg}}} = \kappa \cdot r_\chi(\mathfrak{a}) + \sum_{\gamma \in T(F)\backslash \tilde{G}(F)/T(F)} \langle \tilde{P}_{\chi,\mathfrak{a}}, \tilde{P}_\chi\rangle_{\tilde{U}}^{\gamma} \cdot m_{\mathfrak{a}}(\gamma)$$

where

$$m_{\mathfrak{a}}(\gamma) = \frac{1}{2}\begin{cases} \mathrm{ord}_w(\xi\mathfrak{a})+1 & \text{if } \xi \neq 0 \text{ and } \mathrm{ord}_w(\xi\mathfrak{a}) \text{ is odd and nonnegative} \\ \mathrm{ord}_w(\mathfrak{a}) & \text{if } \xi = 0 \text{ and } \mathrm{ord}_w(\mathfrak{a}) \text{ is even and nonnegative} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is our analogue of [34, Lemma 6.3.5]. Decompose

$$P^0_{\chi,\mathfrak{a}} = \sum_{k=0}^{\infty} \mathfrak{P}^0_k \qquad P_{\chi,\mathfrak{a}} = \sum_{k=0}^{\infty} \mathfrak{P}_k$$

where $\mathfrak{P}^0_k$ is the restriction of $P^0_{\chi,\mathfrak{a}}$ to points of w-conductor k, and similarly for $\mathfrak{P}_k$. By (5.2)

$$\mathfrak{P}_k = \mathfrak{P}^0_k + \begin{cases} r_\chi(\mathfrak{a})P_\chi & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$$

and Proposition 5.3.1 gives

$$\begin{aligned} g(P^0_{\chi,\mathfrak{a}}, P_\chi)_{U,w^{\mathrm{alg}}} &= \sum_{\gamma\in T(F)\backslash\tilde{G}(F)/T(F)} \sum_{k=0}^{\infty} \langle \sigma_*\mathfrak{P}_k, \sigma_* P_\chi\rangle^{\gamma}_{\tilde{U}} \cdot M_k(\gamma) \\ &\quad - r_\chi(\mathfrak{a}) \sum_{\gamma\in T(F)\backslash\tilde{G}(F)/T(F)} \langle \sigma_* P_\chi, \sigma_* P_\chi\rangle^{\gamma}_{\tilde{U}} \cdot M_0(\gamma). \end{aligned}$$

The next claim is that $\sigma_*\mathfrak{P}_k = c_k \tilde{P}_{\chi,\mathfrak{a}}$ where

$$c_k = \begin{cases} [\mathcal{O}^\times_{E,w} : A^\times_k] & \text{if } \mathrm{ord}_w(\mathfrak{a}) - k \text{ is even and nonnegative} \\ 0 & \text{otherwise.} \end{cases}$$

To prove this define

$$\begin{aligned} H^k_w(\mathfrak{a}) &= \{h \in H_w(\mathfrak{a}) \mid hU_wh^{-1} \cap T(F_w) = A^\times_k\} \\ H^k(\mathfrak{a}) &= \{h \in H(\mathfrak{a}) \mid h_w \in H^k_w(\mathfrak{a})\} \\ \tilde{H}(\mathfrak{a}) &= \tilde{H}_w(\mathfrak{a}) \cdot \prod_{v\neq w} \sigma_v(H_v(\mathfrak{a})) \end{aligned}$$

where $\tilde{H}_w(\mathfrak{a}) = \{h \in \tilde{R}_w \mid \tilde{\mathrm{N}}(h)\mathcal{O}_F = \mathfrak{a}_v\}$. The CM-cycles in question are now given by

$$\begin{aligned} \mathfrak{P}_k(g) &= \chi_0(\mathfrak{a}) \sum_{t\in T(\mathbb{A}_f)/U_T} \chi(t)\mathbf{1}_{H^k(\mathfrak{a})}(t^{-1}g) \\ \tilde{P}_{\chi,\mathfrak{a}}(g) &= \chi_0(\mathfrak{a}) \sum_{t\in T(\mathbb{A}_f)/U_T} \chi(t)\mathbf{1}_{\tilde{H}(\mathfrak{a})}(t^{-1}g). \end{aligned}$$

As in the proof of [34, Lemma 6.3.5] there is a decomposition

$$G(F_w) = \bigsqcup_{k=0}^{\infty} T(F_w)h_kU_w$$

where each $h_k \in R_w$ satisfies $\mathrm{ord}_w(\mathrm{N}(h_k)) = k$ and $h_kU_wh_k^{-1} \cap T(F_w) = A^\times_k$. Fixing a uniformizer $\varpi \in F^\times_w$ we therefore find

$$H^k_w(\mathfrak{a}) = \begin{cases} \varpi^{\frac{\mathrm{ord}_w(\mathfrak{a})-k}{2}} \mathcal{O}^\times_{E,w} h_k U_w & \text{if } \mathrm{ord}_w(\mathfrak{a}) - k \text{ is even and nonnegative} \\ \emptyset & \text{otherwise.} \end{cases}$$

From this it follows that $\#(H^k_w(\mathfrak{a})/U_w) = c_k$. Write $H^k_w(\mathfrak{a}) = \sqcup_{i=1}^{c_k} s_i U_w$. For any $t \in T(\mathbb{A}_f)$ we have $\sigma_w(ts_i) = t\tilde{H}_w(\mathfrak{a})$, and hence $\sigma_* \mathbf{1}_{tH^k(\mathfrak{a})} = c_k \cdot \mathbf{1}_{t\tilde{H}(\mathfrak{a})}$ from which $\sigma_* \mathfrak{P}_k = c_k \tilde{P}_{\chi,\mathfrak{a}}$ follows immediately.

It follows from the above that

$$\sum_{k=0}^{\infty} \langle \sigma_* \mathfrak{P}_k, \sigma_* P_\chi \rangle^\gamma_{\tilde{U}} \cdot M_k(\gamma) = \langle \tilde{P}_{\chi,\mathfrak{a}}, \tilde{P}_\chi \rangle^\gamma_{\tilde{U}} \cdot \sum_{k=0}^{\infty} c_k \cdot M_k(\gamma).$$

Assume $\langle \tilde{P}_{\chi,\mathfrak{a}}, \tilde{P}_\chi \rangle^\gamma_{\tilde{U}} \neq 0$. Suppose first that γ is nondegenerate. In particular $O^\gamma_{\tilde{U}}(\tilde{P}_{\chi,\mathfrak{a},w}) \neq 0$ by (3.7), and so Proposition 3.3.1 implies that $\mathrm{ord}_w(\eta\mathfrak{a})$ and $\mathrm{ord}_w(\xi\mathfrak{a}) - 1$ are both even and nonnegative. If $\mathrm{ord}_w(\mathfrak{a})$ is odd then $\mathrm{ord}_w(\eta)$ is odd, and as $\eta + \xi = 1$ we must have $\mathrm{ord}_w(\xi) = 0$. Thus

$$\sum_{k=0}^{\infty} c_k \cdot M_k(\gamma) = \#\{k \mid 1 \leq k \leq \mathrm{ord}_w(\mathfrak{a}), k \text{ odd}\} = m_\mathfrak{a}(\gamma). \tag{5.4}$$

If $\mathrm{ord}_w(\mathfrak{a})$ is even then

$$\sum_{k=0}^{\infty} c_k \cdot M_k(\gamma) = \frac{\mathrm{ord}_w(\xi) + 1}{2} + \#\{k \mid 1 \leq k \leq \mathrm{ord}_w(\mathfrak{a}), k \text{ even}\} = m_\mathfrak{a}(\gamma).$$

Now suppose γ is degenerate, so that $\tilde{P}_{\chi,\mathfrak{a}}(\gamma) \neq 0$ by Lemma 3.2.1. If $\xi = 0$ then we may assume $\gamma = 1$ so that Lemma 3.3.6 implies $\mathrm{ord}_w(\mathfrak{a})$ is even. Thus

$$\sum c_k \cdot M_k(\gamma) = \#\{k \mid 1 \leq k \leq \mathrm{ord}_w(\mathfrak{a}), k \text{ even}\} = m_\mathfrak{a}(\gamma).$$

If $\xi = 1$ then similarly $\mathrm{ord}_w(\mathfrak{a}\mathfrak{e}^{-1}) = \mathrm{ord}_w(\mathfrak{a}) - 1$ is even and so again (5.4) holds. □

Corollary 5.3.3. *Suppose* $\mathfrak{a}$ *is prime to* $\mathfrak{d}\mathfrak{r}$. *Then*

$$2^{[F:\mathbb{Q}]+1} \log|\varpi|_w \cdot g(P_\chi, P^0_{\chi,\mathfrak{a}})_{U,w^{\mathrm{alg}}} = [\widehat{\mathcal{O}}^\times_E : U_T] H_F \lambda_U^{-1} \cdot \mathrm{N}(\mathfrak{a}) \widehat{B}^w(\mathfrak{a}; \Phi_\mathfrak{r}) + A(\mathfrak{a})$$

where $A(\mathfrak{a})$ *is a derivation of* $\Pi_{\overline{\chi}} \otimes |\cdot|^{1/2}$ *in the sense of* [34, Definition 3.5.3].

Proof. Fix a nondegenerate $\gamma \in \tilde{G}(F)$ and an $a \in \mathbb{A}^\times$ with $a\mathcal{O}_F = \mathfrak{a}$. For any place v of F, Lemma 3.1.1 and the definition of $\tilde{B}$ give

$$\omega_v(-\eta\xi) = \epsilon_v(1/2, \mathfrak{r}) \cdot \begin{cases} -1 & \text{if } v = w \\ 1 & \text{if } v \neq w. \end{cases}$$

Thus $\mathrm{Diff}_\mathfrak{r}(\eta, \xi) = \{w\}$, and conversely a pair $\eta, \xi \in F^\times$ with $\eta + \xi = 1$ arises from some choice of nondegenerate $\gamma \in \tilde{G}(F)$ if and only if $\mathrm{Diff}_\mathfrak{r}(\eta, \xi) = \{w\}$. Comparing Propositions 2.6.1 and 3.3.1, and recalling (5.3), we find

$$B_w(a, \eta, \xi; \Theta'_\mathfrak{r}) = |a|_w \tau_w(\gamma) \cdot O^\gamma_{\tilde{U}}(\tilde{P}_{\chi,\mathfrak{a},w}) \cdot m_\mathfrak{a}(\gamma) \log|\varpi^2|_w.$$

On the other hand for any finite place $v \neq w$ we have, using (5.3) and Corollaries 3.3.5 and 3.4.3,

$$[\mathcal{O}^\times_{E,v} : \mathcal{O}^\times_{F,v} U_{T,v}] B_v(a, \eta, \xi; \Theta_\mathfrak{r}) = |a|_v \tau_v(\gamma) \cdot O^\gamma_{\tilde{U}}(\tilde{P}_{\chi,\mathfrak{a},v}).$$

Using (2.9), Lemma 3.1.2, and (3.7) we find

$$[\widehat{\mathcal{O}}^\times_E : U_T] H_F \lambda_U^{-1} \cdot \mathrm{N}(\mathfrak{a}) \widehat{B}^w(\mathfrak{a}, \Phi_\mathfrak{r}) = 2^{[F:\mathbb{Q}]+1} \log|\varpi|_w \sum \langle \tilde{P}_\chi, \tilde{P}_{\chi,\mathfrak{a}} \rangle^\gamma_{\tilde{U}} \cdot m_\mathfrak{a}(\gamma)$$

where the sum is over all nondegenerate $\gamma \in T(F)\backslash\tilde{G}(F)/T(F)$. If γ is degenerate then $\langle \tilde{P}_\chi, \tilde{P}_{\chi,\mathfrak{a}}\rangle^\gamma_{\tilde{U}} \cdot m_\mathfrak{a}(\gamma)$ is a derivation of $\Pi_{\overline{\chi}} \otimes |\cdot|^{1/2}$ (using Lemma 3.2.1 and Corollaries 3.3.9 and 3.4.5). Thus the claim follows from Proposition 5.3.2. □

5.4. **Intersections at nonsplit primes dividing $\mathfrak{d}\mathfrak{r}$.** Suppose that w is a place of F which is nonsplit in E with $w \mid \mathfrak{d}\mathfrak{r}$ and fix a place w^{alg} of F^{alg} above w. Again let $\tilde{B}$ be the quaternion algebra over F obtained from B by interchanging invariants at w and w_∞, so that $\{\text{places } v \text{ of } F \mid \tilde{B}_v \not\cong B_v\} = \{w, w_\infty\}$. Fix an embedding $E \to \tilde{B}$ and for each finite place $v \neq w$ let σ_v and $\tilde{\epsilon}_v$ be as in §5.3. Choose $\tilde{\epsilon}_w$ so that $\tilde{B}^-_w = E_w\tilde{\epsilon}_w$ and

$$\mathrm{ord}_w(\mathrm{N}(\tilde{\epsilon}_w)) = \mathrm{ord}_w(\mathfrak{r}) + \begin{cases} 1 & \text{if } w \nmid \mathfrak{d} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{a}$ be prime to $\mathfrak{d}\mathfrak{r}$. As in §5.3, for any finite place v we may repeat the constructions of §3.3 and §3.4 with B replaced by $\tilde{B}$ and ϵ_v replaced by $\tilde{\epsilon}_v$, giving a compact open subgroup $\tilde{U}_v \subset \tilde{G}(\mathbb{A}_f)$ and a function $\tilde{P}_{\chi,\mathfrak{a},v}$ on $\tilde{G}(F_v)/\tilde{U}_v$ for each v. Taking the product over all finite v gives a CM-cycle $\tilde{P}_{\chi,\mathfrak{a}}$ of level $\tilde{U}$.

Define the *w-special* CM points of level U, denoted C^0_U, to be the image of

$$T(F_w) \times G(\mathbb{A}^w_f) \to C_U$$

where $\mathbb{A}^w_f = \{x \in \mathbb{A}_f \mid x_w = 0\}$. By a w-special CM cycle we mean a CM cycle supported on w-special points. Define $C^0_{\tilde{U}}$ similarly, and note that there are bijections

$$C^0_U \cong T^0(F)\backslash G(\mathbb{A}^w_f)/U^w \cong T^0(F)\backslash\tilde{G}(\mathbb{A}^w_f)/\tilde{U}^w \cong C^0_{\tilde{U}}$$

where $U^w = \prod_{v\neq w} U_v$ and similarly for $\tilde{U}^w$, and $T^0(F)$ is defined as

$$T(F) \cap U_w = T(F) \cap (1 + \mathfrak{c}\mathcal{O}_{E,w})^\times = T(F) \cap \tilde{U}_w.$$

Thus we may identify w-special cycles of level U with w-special cycles of level $\tilde{U}$, and we denote this bijection by $P \mapsto \sigma_* P$. As $\mathfrak{a}$ is prime to $\mathfrak{d}\mathfrak{r}$, $\mathrm{ord}_w(\mathfrak{a}) = 0$ and it follows from the construction that $P_{\chi,\mathfrak{a}}$ is w-special. It is easy to see that $\sigma_* P_{\chi,\mathfrak{a}} = \tilde{P}_{\chi,\mathfrak{a}}$ (as one only needs to check equality locally at $v \neq w$).

Proposition 5.4.1. *Suppose P and Q are w-special CM cycles of level U with disjoint support. There is a locally constant function (independent of P and Q) $K(x,y)$ on $\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_f)$ such that*

$$\begin{aligned} g(P,Q)_{U,w^{\mathrm{alg}}} &= \sum_{\gamma \in T(F)\backslash\tilde{G}(F)/T(F)} \langle \sigma_* P, \sigma_* Q\rangle^\gamma_{\tilde{U}} \cdot M(\gamma) \\ &\quad + \int_{[T(F)\backslash\tilde{G}(\mathbb{A}_f)]^2} (\sigma_* P)(x) K(x,y) \overline{(\sigma_* Q)(y)}\, dx\, dy \end{aligned}$$

where

$$M(\gamma) = \begin{cases} \frac{\mathrm{ord}_w(\xi)}{2} & \text{if } \xi \neq 0 \text{ and } \mathrm{ord}_w(\xi) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See Lemmas 6.3.7 and 6.3.8 of [34]. □

Proposition 5.4.2. *If $\mathfrak{a}$ is prime to $\mathfrak{d}\mathfrak{r}$ then*

$$\begin{aligned} g(P^0_{\chi,\mathfrak{a}}, P_\chi)_{U,w^{\mathrm{alg}}} &= \kappa\cdot r_\chi(\mathfrak{a}) + \sum_{\gamma\in T(F)\backslash\tilde{G}(F)/T(F)} \langle \tilde{P}_{\chi,\mathfrak{a}}, \tilde{P}_\chi\rangle^\gamma_{\tilde{U}}\cdot m(\gamma) \\ &\quad + \int_{[T(F)\backslash\tilde{G}(\mathbb{A}_f)]^2} \tilde{P}_{\chi,\mathfrak{a}}(x)K(x,y)\overline{\tilde{P}_\chi(y)}\;dx\;dy \end{aligned}$$

where $K(x,y)$ is a locally constant function on $[\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_f)]^2$ and

$$m(\gamma) = \frac{1}{2}\begin{cases} \mathrm{ord}_w(\xi\mathfrak{r}^{-1})+1 & \text{if } \xi\neq 0, \mathrm{ord}_w(\xi)\geq 0, \text{ and } w\mid\mathfrak{r} \\ \mathrm{ord}_w(\xi\mathfrak{d}) & \text{if } \xi\neq 0, \mathrm{ord}_w(\xi)\geq 0, \text{ and } w\mid\mathfrak{d} \\ 0 & \text{otherwise.}\end{cases}$$

Proof. It follows from (5.2) and Proposition 5.4.1 that the claim holds if one replaces $m(\gamma)$ with $M(\gamma)$. Thus if we set $m' = m - M$ it suffices to show that

$$\sum_{\gamma\in T(F)\backslash\tilde{G}(F)/T(F)} \langle \tilde{P}_{\chi,\mathfrak{a}}, \tilde{P}_\chi\rangle^\gamma_{\tilde{U}}\cdot m'(\gamma) = \int_{T(F)\backslash\tilde{G}(\mathbb{A}_f)} \tilde{P}_{\chi,\mathfrak{a}}(x)k(x,y)\overline{\tilde{P}_\chi(y)}\;dx\;dy$$

for k some locally constant function on $\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_f)$. Note that m' is locally constant for the topology on $G(F)$ induced from $G(F_w)$ (i.e. m and M have the same singularity near $\xi=0$) and let $\tilde{U}'_w\subset\tilde{U}_w$ be small enough that m' is a constant, μ, on $\tilde{U}'_w$. Let $\tilde{U}'$ be the subgroup obtained by shrinking the w-component of $\tilde{U}$ from $\tilde{U}_w$ to $\tilde{U}'_w$. The crucial point is that on the image of $\{1\}\times\tilde{G}(\mathbb{A}^w)\to C_{\tilde{U}'}$ we have

$$k^{m'}_{\tilde{U}'}(x,y) = k^\mu_{\tilde{U}'}(x,y)$$

where $k^\mu_{\tilde{U}'}$ is the kernel (3.4) constructed with constant multiplicity function μ. The w-special CM-cycles $\tilde{P}_{\chi,\mathfrak{a}}$ and $\tilde{P}_\chi$ are supported on the image of $T(F_w)\times\tilde{G}(\mathbb{A}^w)$ in $C_{\tilde{U}'}$, which equals the image of $\{1\}\times\tilde{G}(\mathbb{A}^w)$ as $T(F_w)\subset T(F)\tilde{U}'_w$. Therefore the pairings (3.5) satisfy

$$\langle \tilde{P}_{\chi,\mathfrak{a}}, \tilde{P}_\chi\rangle^{m'}_{\tilde{U}'} = \langle \tilde{P}_{\chi,\mathfrak{a}}, \tilde{P}_\chi\rangle^{\mu}_{\tilde{U}'},$$

and it follows that $\langle \tilde{P}_{\chi,\mathfrak{a}}, \tilde{P}_\chi\rangle^{m'}_{\tilde{U}} = \langle \tilde{P}_{\chi,\mathfrak{a}}, \tilde{P}_\chi\rangle^{\mu}_{\tilde{U}}$ (replacing $\tilde{U}'$ by $\tilde{U}$ changes each pairing by a constant depending on the normalizations of measures in §3.2 but not on the multiplicity function). As the multiplicity function μ is constant the kernel $k^\mu_{\tilde{U}}$ is right $\tilde{G}(F)$-invariant, and we take $k = k^\mu_{\tilde{U}}$. □

Corollary 5.4.3. *Define a function $\mathcal{P}_{\overline{\chi}}$ on $S_{\tilde{U}} = \tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_f)/\tilde{U}$ by*

$$\mathcal{P}_{\overline{\chi}}(g) = \sum_{\gamma\in T(F)\backslash G(F)} \overline{\tilde{P}_\chi(\gamma g)}.$$

For any $\mathfrak{a}$ prime to $\mathfrak{d}\mathfrak{r}$

$$\begin{aligned} &2^{[F:\mathbb{Q}]+1}|d|^{1/2}\log|\varpi|_w\cdot g(P_\chi, P^0_{\chi,\mathfrak{a}})_{U,w^{\mathrm{alg}}} \\ &= [\widehat{\mathcal{O}}^\times_E : U_T]H_F\lambda_U^{-1}\cdot\mathrm{N}(\mathfrak{a})\widehat{B}^w(\mathfrak{a};\Phi_\mathfrak{r}) + A(\mathfrak{a}) + \int_{\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_f)} (T_\mathfrak{a}\mathcal{P}_{\overline{\chi}})(x)\cdot g(x)\;dx \end{aligned}$$

where $A(\mathfrak{a})$ is a derivation of $\Pi_{\overline{\chi}}\otimes|\cdot|^{1/2}$, $g(x)$ is a locally constant function on $\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_f)$, and $T_\mathfrak{a}$ is the Hecke operator on $L^2(S_{\tilde{U}})$ defined in §4.3.

Proof. This is deduced from Proposition 5.4.2 exactly as in the proof of Corollary 5.3.3, taking

$$g(x) = \int_{T(F)\backslash\tilde{G}(\mathbb{A}_f)} \overline{K(x,y)}\tilde{P}_\chi(y)\ dy.$$

□

5.5. **Archimedean intersections.** Let w be an archimedean place of F and choose a place w^{alg} of F^{alg} above w. If $w = w_\infty$ is the archimedean place at which B is split then set $\tilde{B} = B$. If $w \neq w_\infty$ then let $\tilde{B}$ be the quaternion algebra obtained from B be interchanging invariants at w and w_∞ as in §5.3. As in §5.3 fix an embedding $E \to \tilde{B}$ and, for every finite place v of F, choose $\sigma_v : B_v \cong \tilde{B}_v$ compatible with the embeddings of E_v into B_v and $\tilde{B}_v$. Define $\tilde{\epsilon}_v = \sigma_v(\epsilon_v)$, set $\tilde{U}_v = \sigma_v(U_v)$, and let σ_* denote the induced isomorphism from CM cycles of level U on G to CM cycles of level $\tilde{U}$ on $\tilde{G}$.

For $\gamma \in \tilde{G}(F)$ view $\xi \in F$ as a real number using the embedding $F \to \mathbb{R}$ determined by w and define

$$m_s(\gamma) = \begin{cases} Q_s(1-2\xi) & \text{if } \xi < 0 \\ 0 & \text{otherwise,} \end{cases}$$

where Q_s is defined by [34, (6.3.3)], and a function on $\tilde{G}(\mathbb{A}_f) \times \tilde{G}(\mathbb{A}_f)$

$$k^s_{\tilde{U}}(x,y) = \sum_{\gamma \in \tilde{G}(F)/(Z(F)\cap\tilde{U})} \mathbf{1}_{\tilde{U}}(x^{-1}\gamma y) m_s(\gamma).$$

We now recall the statement of [34, Lemma 6.3.1]. For any distinct points $P, Q \in C_U$ the sum defining $k^s_{\tilde{U}}(\sigma_*P, \sigma_*Q)$ is convergent for $\mathrm{Re}(s) > 0$ and extends to a meromorphic function in a neighborhood of $s = 0$ with a simple pole at $s = 0$. Thus for any CM-cycles P and Q of level U the pairing $\langle \sigma_*P, \sigma_*Q\rangle^{m_s}_{\tilde{U}}$ of (3.5) has meromorphic continuation with a pole of order at most 1 at $s = 0$, and moreover

$$g(P,Q)_{U,w^{\mathrm{alg}}} = \mathrm{const}_{s\to 0}\langle \sigma_*P, \sigma_*Q\rangle^{m_s}_{\tilde{U}}.$$

In particular, if $\mathfrak{a}$ is prime to $\mathfrak{d}\mathfrak{r}$ then

$$g(P^0_{\chi,\mathfrak{a}}, P_\chi)_{U,w^{\mathrm{alg}}} = \mathrm{const}_{s\to 0} \sum_{\gamma\in T(F)\backslash\tilde{G}(F)/T(F)} \langle \tilde{P}^0_{\chi,\mathfrak{a}}, \tilde{P}_\chi\rangle^{\gamma}_{\tilde{U}} \cdot m_s(\gamma) \tag{5.5}$$

where $\tilde{P}^0_{\chi,\mathfrak{a}} = \sigma_* P^0_{\chi,\mathfrak{a}}$ is the cycle defined by replacing U by $\tilde{U}$ and B by $\tilde{B}$ in the definition of $P^0_{\chi,\mathfrak{a}}$, and similarly for $\tilde{P}_\chi$.

Corollary 5.5.1. *For any* $\mathfrak{a}$ *prime to* $\mathfrak{d}\mathfrak{r}$

$$-2^{[F:\mathbb{Q}]+1}|d|^{1/2} g(P_\chi, P^0_{\chi,\mathfrak{a}})_{U,w^{\mathrm{alg}}} = [\widehat{\mathcal{O}}^\times_E : U_T] H_F \lambda_U^{-1} \mathrm{N}(\mathfrak{a}) \cdot \mathrm{const}_{s\to 0}\widehat{B}^w(s,\mathfrak{a};\Phi_{\mathfrak{r}})$$

up to a derivation of $\Pi_{\overline{\chi}} \otimes |\cdot|^{1/2}$.

Proof. Suppose $\mathrm{Re}(\sigma) > 0$ and, for any $\gamma \in \tilde{G}(F)$, write $M_s(\gamma) = M_s(\xi_w)$ where the M_σ on the right is the function on $\mathbb{R}$ defined in §2.6. Combining (2.10) with Corollaries 3.3.5 and 3.4.3, and arguing as in the proof of Corollary 5.3.3, we find

$$\begin{aligned}&[\widehat{\mathcal{O}}^\times_E : U_T\widehat{\mathcal{O}}^\times_F]\mathrm{N}(\mathfrak{a})\widehat{B}^w(s,\mathfrak{a};\Phi_{\mathfrak{r}}) \\ &\qquad = (-2i)^{[F:\mathbb{Q}]}\omega_\infty(\delta)|d|^{1/2} \sum |\eta\xi|^{1/2}_\infty M_s(\gamma) \prod_{v\nmid\infty} \overline{\tau_v(\gamma)} \cdot \overline{O^\gamma_v(\tilde{P}_{\chi,\mathfrak{a},v})}\end{aligned}$$

where the sum is over all nondegenerate $\gamma \in T(F)\backslash\tilde{G}(F)/T(F)$. By Lemma 3.1.2 we have

$$\prod_{v\nmid\infty} \tau_v(\gamma) = \omega_\infty(\delta)(-i)^{[F:\mathbb{Q}]}|\eta\xi|_\infty^{-1/2},$$

and combining this with (3.7) gives

$$[\widehat{\mathcal{O}}_E^\times : U_T]H_F\lambda_U^{-1}\mathrm{N}(\mathfrak{a})\widehat{B}^w(s,\mathfrak{a};\Phi_\mathfrak{r}) = 2^{[F:\mathbb{Q}]}|d|^{1/2}\sum\langle\tilde{P}_\chi,\tilde{P}_{\chi,\mathfrak{a}}\rangle_{\tilde{U}}^\gamma\cdot M_s(\gamma)$$

where the sum is again over all nondegenerate γ as above. By the argument in the proof of [34, Lemma 6.4.1] the constant term as $s\to 0$ is unchanged if we replace $M_s(\gamma)$ by $-2m_s(\gamma)$. Adding in the terms corresponding to the two degenerate choices of γ add derivations of $\Pi_{\overline{\chi}}\otimes|\cdot|^{1/2}$, as in the proof of Corollary 5.3.3, and replacing $\tilde{P}_{\chi,\mathfrak{a}}$ by $\tilde{P}^0_{\chi,\mathfrak{a}}$ also adds a derivation of $\Pi_{\overline{\chi}}\otimes|\cdot|^{1/2}$, by (5.2) with P replaced by $\tilde{P}$. Thus the claim follows from (5.5). □

5.6. **The twisted Gross-Zagier theorem.** Let $\mathbb{T}$ denote the $\mathbb{Z}$-algebra generated by the Hecke operators $T_\mathfrak{a}$ and the nebentype operators $(\langle\mathfrak{a}\rangle\phi)(g) = \phi(ga)$, where $a\mathcal{O}_F = \mathfrak{a}$ and $a_v = 1$ for $v \mid \infty$, acting on holomorphic automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ of parallel weight 2 and level $K_1(\mathfrak{d}\mathfrak{r})$. Let ϕ_Π denote the normalized newform in Π. The $\mathbb{C}$-algebra $\mathbb{T}_\mathbb{C} = \mathbb{T}\otimes_\mathbb{Z}\mathbb{C}$ is semi-simple, and we let $\mathbb{T}_\Pi$ be the maximal summand of $\mathbb{T}_\mathbb{C}$ in which

$$T_\mathfrak{a} = \widehat{B}(\mathcal{O}_F;T_\mathfrak{a}\phi_\Pi) \qquad \langle\mathfrak{a}\rangle = \chi_0^{-1}(\mathfrak{a}).$$

Let e_Π be the idempotent in $\mathbb{T}_\mathbb{C}$ satisfying $e_\Pi\mathbb{T}_\mathbb{C} = \mathbb{T}_\Pi$. It follows from the Jacquet-Langlands correspondence and the Eichler-Shimura theory that there is a ring homomorphism $\mathbb{T}\to\mathrm{End}(J_U)$ taking $T_\mathfrak{a}\mapsto T_\mathfrak{a}^{\mathrm{Alb}}$ and $\langle\mathfrak{a}\rangle\mapsto\langle\mathfrak{a}\rangle^{\mathrm{Alb}}$, and so $\mathbb{T}_\mathbb{C}$ acts on $J_U(E_\chi)\otimes_\mathbb{Z}\mathbb{C}$.

Proposition 5.6.1. *Abbreviating* $P_{\chi,\Pi} = e_\Pi\cdot\mathrm{Hg}(P_\chi)$,

$$\frac{2^{|S|}H_F[\widehat{\mathcal{O}}_E^\times:U_T]}{\lambda_U||\phi_\Pi^\#||^2_{K_0(\mathfrak{d}\mathfrak{r})}}\widehat{B}(\mathcal{O}_F,\phi_\Pi^\#)L'(1/2,\Pi\times\Pi_\chi) = 2^{[F:\mathbb{Q}]+1}|d|^{1/2}\langle P_{\chi,\Pi},P_{\chi,\Pi}\rangle_U^{\mathrm{NT}}.$$

Proof. This follows easily from the formulae of the previous subsections, exactly as in [34, §6.4], "Conclusion of the Proof of Theorem 1.3.2". We quickly sketch the argument.

Suppose $\mathfrak{a}$ is prime to $\mathfrak{d}\mathfrak{r}$. Using the argument of [34, Lemma 6.2.2], up to sums of derivations of principal series and $\Pi_{\overline{\chi}}\otimes|\cdot|^{1/2}$ we have

$$\begin{aligned}\langle T_\mathfrak{a}^{\mathrm{Alb}}\mathrm{Hg}(P_\chi),\mathrm{Hg}(P_\chi)\rangle_U^{\mathrm{NT}} &= \langle\mathrm{Hg}(P_\chi),T_\mathfrak{a}^{\mathrm{Pic}}\mathrm{Hg}(P_\chi)\rangle_U^{\mathrm{NT}}\\ &= \langle\mathrm{Hg}(P_\chi),\mathrm{Hg}(P_{\chi,\mathfrak{a}})\rangle_U^{\mathrm{NT}}\\ &= -\sum_w d_w\cdot g(P_\chi,P^0_{\chi,\mathfrak{a}})_{U,w^{\mathrm{alg}}}\end{aligned}$$

where the sum is over all places w of F, and where for each w we fix an extension w^{alg} to F^{alg}. Exactly as in [34, Lemma 6.3.4] the nonarchimedean places w which split in E contribute derivations of principal series and $\Pi_{\overline{\chi}}\otimes|\cdot|^{1/2}$, and so we may omit such places in the above summation. Combining Corollaries 5.3.3, 5.4.3, and 5.5.1 with Proposition 2.6.3 we find

$$2^{[F:\mathbb{Q}]+1}|d|^{1/2}\langle T_\mathfrak{a}^{\mathrm{Alb}}\mathrm{Hg}(P_\chi),\mathrm{Hg}(P_\chi)\rangle_U^{\mathrm{NT}} = [\widehat{\mathcal{O}}_E^\times:U_T]H_F\lambda_U^{-1}\widehat{B}(\mathcal{O}_F;T_\mathfrak{a}\Phi_\mathfrak{r})$$

up to a sum of derivations of principal series, derivations of $\Pi_{\overline{\chi}} \otimes |\cdot|^{1/2}$, and functions of the form

$$\int_{\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_f)} (T_{\mathfrak{a}}\mathcal{P}_{\overline{\chi}})(x) \cdot g(x)\ dx \tag{5.6}$$

for $w \mid \mathfrak{d}\mathfrak{r}$ as in Corollary 5.4.3.

Let us consider (5.6) in more detail. Fix $w \mid \mathfrak{d}\mathfrak{r}$ and let $\tilde{U}$, $\tilde{G}$, and so on be as in §5.4. Let $S_{\tilde{U}} = \tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_f)/\tilde{U}$ as in §4.3. It follows from the Jacquet-Langlands correspondence that the $\mathbb{C}$-algebra generated by the operators $T_{\mathfrak{a}}$ acting on $L^2(S_{\tilde{U}})$ is a quotient of $\mathbb{T}_{\mathbb{C}}$. Thus it makes sense to form $e_\Pi \cdot \mathcal{P}_{\overline{\chi}} \in L^2(S_{\tilde{U}})$, which is nothing more than the projection of $\mathcal{P}_{\overline{\chi}}$ to the automorphic representation $\tilde{\Pi}$ of $\tilde{G}(\mathbb{A})$ whose Jacquet-Langlands lift is Π. By construction the function $e_\Pi \cdot \mathcal{P}_{\overline{\chi}}$ has character χ_w^{-1} under right multiplication by $T(F_w)$. On the other hand, if Π' is the automorphic representation of $G(\mathbb{A})$ whose Jacquet-Langlands lift is Π then Π' contains a nonzero vector on which $T(F_w)$ acts through χ_w^{-1} (as Π'_w admits a toric newvector in the sense of §4.2). Thus if $e_\Pi \mathcal{P}_{\overline{\chi}} \neq 0$ we would have nonzero vectors in both $\tilde{\Pi}_w$ and Π'_w on which $T(F_w)$ acts through χ_w^{-1}. This contradicts results of Saito, Tunnell, and Waldspurger (as described in [12, §10] or [13, Proposition 1.1], and using [32, Lemme 9(iii)] to relate $T(E_w)$-invariants to $T(E_w)$-coinvariants), and so $e_\Pi \mathcal{P}_{\overline{\chi}} = 0$.

We now deduce, using [35, Proposition 4.5.1] for the vanishing of derivations of principal series and theta series, that

$$2^{[F:\mathbb{Q}]+1}|d|^{1/2}\langle e_\Pi \mathrm{Hg}(P_\chi), \mathrm{Hg}(P_\chi)\rangle_U^{\mathrm{NT}} = [\widehat{\mathcal{O}}_E^\times : U_T] H_F \lambda_U^{-1} \widehat{B}(\mathcal{O}_F; e_\Pi \Phi_{\mathfrak{r}}).$$

As $e_\Pi \Phi_{\mathfrak{r}}$ is the projection of $\Phi_{\mathfrak{r}}$ to Π, the proof now follows from

$$\widehat{B}(\mathcal{O}_F; e_\Pi \Phi_{\mathfrak{r}}) \cdot ||\phi_\Pi^\#||^2_{K_0(\mathfrak{d}\mathfrak{r})} = 2^{|S|}\widehat{B}(\mathcal{O}_F; \phi_\Pi^\#) L'(1/2, \Pi \times \Pi_\chi)$$

as in the proof of Proposition 4.3.2. □

As above there is a ring homomorphism $\mathbb{T} \to \mathrm{End}(J_V)$ taking $T_{\mathfrak{a}} \mapsto T_{\mathfrak{a}}^{\mathrm{Alb}}$ and $\langle \mathfrak{a} \rangle \mapsto \langle \mathfrak{a} \rangle^{\mathrm{Alb}}$, and so $\mathbb{T}_{\mathbb{C}}$ acts on $J_V(E_\chi) \otimes_{\mathbb{Z}} \mathbb{C}$.

Theorem 5.6.2. *Abbreviate* $Q_{\chi,\Pi} = e_\Pi \mathrm{Hg}(Q_\chi) \in J_V(E_\chi) \otimes_{\mathbb{Z}} \mathbb{C}$.

$$\frac{L'(1/2, \Pi \times \Pi_\chi)}{||\phi_\Pi||^2_{K_0(\mathfrak{n})}} = \frac{2^{[F:\mathbb{Q}]+1}}{H_{F,\mathfrak{s}}\sqrt{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d}\mathfrak{c}^2)}} \langle Q_{\chi,\Pi}, Q_{\chi,\Pi}\rangle_V^{\mathrm{NT}}.$$

Proof. Recall the constants $\mathbf{a}_\Pi$, $\mathbf{b}_\Pi$, and $\mathbf{c}_\Pi$ of §4.6. The argument of [36, §17] gives the first equality of

$$\langle P_{\chi,\Pi}, P_{\chi,\Pi}\rangle_U^{\mathrm{NT}} \cdot \mathbf{c}_\Pi = \langle \pi^* Q_{\chi,\Pi}, \pi^* Q_{\chi,\Pi}\rangle_U^{\mathrm{NT}} = \deg(\pi) \cdot \langle Q_{\chi,\Pi}, Q_{\chi,\Pi}\rangle_V^{\mathrm{NT}}$$

where $\pi^* : J_V \to J_U$ is the morphism induced by the natural projection $\pi : X_U \to X_V$ of degree $[F^\times V : F^\times U] = [V : U]\lambda_V \lambda_U^{-1}$. It therefore follows from Proposition 5.6.1 that

$$\mathbf{a}_\Pi \mathbf{c}_\Pi \frac{2^{|S|} H_F [\widehat{\mathcal{O}}_E^\times : U_T]}{[V : U]\lambda_V} \frac{L'(1/2, \Pi \times \Pi_\chi)}{||\phi_\Pi^\#||^2_{K_0(\mathfrak{n})}} = \frac{\mathbf{b}_\Pi 2^{[F:\mathbb{Q}]+1}}{\sqrt{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d})}} \langle Q_{\chi,\Pi}, Q_{\chi,\Pi}\rangle_V^{\mathrm{NT}}$$

and so the theorem follows from the equality of rational functions $\kappa \prod \mathbf{a}_v \mathbf{c}_v = \prod \mathbf{b}_v$ proved in §4.6. □

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