

Singular Moduli Refined

Benjamin Howard^{*} Tonghai Yang[†]

Abstract

In this paper, we give a refinement of the work of Gross and Zagier on singular moduli. Let K_1 and K_2 be two imaginary quadratic fields with relatively prime discriminants d_1 and d_2 , and let $F = \mathbb{Q}(\sqrt{d_1 d_2})$. Hecke constructed a Hilbert modular Eisenstein series over F of weight 1 whose functional equation forces it to vanish at $s = 0$. For CM elliptic curves E_1 and E_2 with complex multiplication by \mathcal{O}_{K_1} and \mathcal{O}_{K_2} , we define an \mathcal{O}_F -module structure and \mathcal{O}_F -quadratic form \deg_{CM} on $\text{Hom}(E_1, E_2)$, which is totally positive definite and satisfies $\text{Tr}_{F/\mathbb{Q}} \deg_{\text{CM}} = \deg$. For each totally positive $\alpha \in F$ we consider the moduli stack \mathcal{X}_α of triples (E_1, E_2, j) with E_i as above and $j \in \text{Hom}(E_1, E_2)$ with $\deg_{\text{CM}}(j) = \alpha$. We prove that \mathcal{X}_α has dimension 0, and that its arithmetic degree is equal to the α -th Fourier coefficient of the central derivative of Hecke's Eisenstein series.

2000 Mathematics Subject Classification: 11G15, 11F41, 14K22

Keywords and Phrases: Singular moduli, arithmetic intersection, Eisenstein series

1 Introduction

Let K_1 and K_2 be non-isomorphic quadratic imaginary fields with discriminants d_1 and d_2 , respectively, and set $K = K_1 \otimes_{\mathbb{Q}} K_2$. Let F be the real quadratic subfield of K , set $D = \text{disc}(F)$, and let $\mathfrak{D} \subset \mathcal{O}_F$ be the different of F/\mathbb{Q} . Let $x \mapsto \bar{x}$ denote complex conjugation on K and set $\mathfrak{w}_i = |\mathcal{O}_{K_i}^\times|$. Let χ be the quadratic Hecke character of F associated to K , and let σ_1 and σ_2 be the two real embeddings of F . We assume throughout this paper that $\gcd(d_1, d_2) = 1$ so that K/F is unramified at all finite places, and $\mathcal{O}_{K_1} \otimes_{\mathbb{Z}} \mathcal{O}_{K_2}$ is the maximal order in K .

Almost one hundred years ago, Hecke constructed his famous Eisenstein

^{*}Supported in part by NSF grant DMS-0901753.

[†]Supported in part by grants DMS-0555503, NSFC-10628103.

series (see [4, (7.2)]) of parallel weight 1 for $\mathrm{SL}_2(\mathcal{O}_F)$

$$E^*(\tau_1, \tau_2, s) = D^{\frac{s+1}{2}} \left(\pi^{-\frac{s+2}{2}} \Gamma\left(\frac{s+2}{2}\right) \right)^2 \sum_{[\mathfrak{a}] \in \mathrm{CL}(F)} \chi(\mathfrak{a}) \mathbf{N}(\mathfrak{a})^{1+s} \\ \times \sum_{(0,0) \neq (m,n) \in \mathfrak{a}^2 / \mathcal{O}_F^\times} \frac{(v_1 v_2)^{\frac{s}{2}}}{(m(\tau_1, \tau_2) + n) |m(\tau_1, \tau_2) + n|^s}.$$

Here $\mathrm{CL}(F)$ is the ideal class group of F , $[\mathfrak{a}]$ denotes the class of the fractional ideal \mathfrak{a} , and

$$m(\tau_1, \tau_2) + n = (\sigma_1(m)\tau_1 + \sigma_1(n))(\sigma_2(m)\tau_2 + \sigma_2(n)).$$

Hecke showed that this sum, convergent for $\mathrm{Re}(s) \gg 0$, has meromorphic continuation to all s and defines a non-holomorphic Hilbert modular form of weight 1 for $\mathrm{SL}_2(\mathcal{O}_F)$ which is holomorphic (in the variable s) in a neighborhood of $s = 0$. The value $E^*(\tau_1, \tau_2, 0)$ at $s = 0$ is a holomorphic Hilbert modular form of weight 1 (Hecke's trick). He further computed the Fourier expansion of this holomorphic modular form. Unfortunately, he missed a sign in the calculation, and it turns out that $E^*(\tau_1, \tau_2, 0) = 0$ identically. In the early 1980's, Gross and Zagier took advantage of this fact to compute the central derivative at $s = 0$, and found that the Fourier coefficients of the diagonal restriction to the upper half plane are very closely related to the factorization of singular moduli (see [4]). Their result can be rephrased (see [22, Section 3] or Corollary D below for more details) in terms of arithmetic intersections as follows: if \mathcal{E} is the moduli stack of elliptic curves over \mathbb{Z} -schemes then the m -th Fourier coefficient of $E^{*'}(\tau, \tau, 0)$ is the arithmetic intersection on $\mathcal{E} \times \mathcal{E}$ of the m -th Hecke correspondence with the codimension two cycle of points representing pairs $(\mathbf{E}_1, \mathbf{E}_2)$ of elliptic curves with complex multiplication by \mathcal{O}_{K_1} and \mathcal{O}_{K_2} , respectively. One naturally asks, for $\alpha \in F^\times$ what is the arithmetic meaning of the α -th Fourier coefficient of the central derivative $E^{*'}(\tau_1, \tau_2, 0)$ itself, before one restricts to the diagonal $\tau_1 = \tau_2$? In another word, is there an arithmetic Siegel-Weil formula (in the sense of [10] or [12]) for this Hecke Eisenstein series? The purpose of this paper is to answer this question positively.

Let \mathcal{X} be the algebraic stack over \mathbb{Z} representing the functor that assigns to every scheme S the category $\mathcal{X}(S)$ of pairs $(\mathbf{E}_1, \mathbf{E}_2)$ in which each $\mathbf{E}_i = (E_i, \kappa_i)$ consists of an elliptic curve E_i over S and an action $\kappa_i : \mathcal{O}_{K_i} \rightarrow \mathrm{End}(E_i)$. For an object $(\mathbf{E}_1, \mathbf{E}_2)$ of $\mathcal{X}(S)$ let

$$L(\mathbf{E}_1, \mathbf{E}_2) = \mathrm{Hom}(E_1, E_2)$$

be the \mathbb{Z} -module of homomorphisms from E_1 to E_2 , equipped with the quadratic form deg . Let $[\cdot, \cdot]$ be the bilinear form associated to deg . The maximal order

$$\mathcal{O}_K = \mathcal{O}_{K_1} \otimes_{\mathbb{Z}} \mathcal{O}_{K_2}$$

acts on $L(\mathbf{E}_1, \mathbf{E}_2)$ by

$$(t_1 \otimes t_2) \bullet j = \kappa_2(t_2) \circ j \circ \kappa_1(\bar{t}_1)$$

($t_i \in \mathcal{O}_{K_i}$) making $L(\mathbf{E}_1, \mathbf{E}_2)$ into an \mathcal{O}_K -module. The action satisfies

$$[t \bullet j_1, j_2] = [j_1, \bar{t} \bullet j_2]$$

for all $t \in \mathcal{O}_K$, and it follows that if we view $L(\mathbf{E}_1, \mathbf{E}_2)$ as an \mathcal{O}_F -module then there is a unique \mathcal{O}_F -bilinear form

$$[\ , \]_{\text{CM}} : L(\mathbf{E}_1, \mathbf{E}_2) \times L(\mathbf{E}_1, \mathbf{E}_2) \rightarrow \mathfrak{D}^{-1}$$

satisfying $[j_1, j_2] = \text{Tr}_{F/\mathbb{Q}}[j_1, j_2]_{\text{CM}}$. If deg_{CM} is the totally positive definite F -quadratic form on $L(\mathbf{E}_1, \mathbf{E}_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to $[\ , \]_{\text{CM}}$ then

$$\text{deg}(j) = \text{Tr}_{F/\mathbb{Q}} \text{deg}_{\text{CM}}(j).$$

For any $\alpha \in F^\times$ let \mathcal{X}_α be the algebraic stack representing the functor that assigns to a scheme S the category $\mathcal{X}_\alpha(S)$ of triples $(\mathbf{E}_1, \mathbf{E}_2, j)$ in which $(\mathbf{E}_1, \mathbf{E}_2)$ is an object of $\mathcal{X}(S)$ and $j \in L(\mathbf{E}_1, \mathbf{E}_2)$ with $\text{deg}_{\text{CM}}(j) = \alpha$. It is clear that \mathcal{X}_α is empty unless α is totally positive.

For $\alpha \in F^\times$ totally positive define the *Arakelov degree*

$$\text{deg}(\mathcal{X}_\alpha) = \sum_p \log(p) \sum_{x \in [\mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})]} e_x^{-1} \cdot \text{length}(\mathcal{O}_{\mathcal{X}_\alpha, x}^{\text{sh}}) \quad (1.1)$$

where $[\mathcal{X}_\alpha(S)]$ is the set of isomorphism classes of objects in the category $\mathcal{X}_\alpha(S)$, $\mathcal{O}_{\mathcal{X}_\alpha, x}^{\text{sh}}$ is the strictly Henselian local ring of \mathcal{X}_α at x , and e_x is the order of the automorphism group of the triple $(\mathbf{E}_1, \mathbf{E}_2, j)$ corresponding to x . Define $\text{Diff}(\alpha)$ to be the set of finite primes \mathfrak{p} of F satisfying

$$\chi_{\mathfrak{p}}(\alpha \mathfrak{D}) = -1.$$

Using the product formula $\prod_w \chi_w(\alpha \sqrt{D}) = 1$ and the fact that $\mathfrak{D} = \sqrt{D} \mathcal{O}_F$, we see that $\text{Diff}(\alpha)$ has odd cardinality, and in particular is nonempty. If \mathfrak{b} is a fractional \mathcal{O}_F -ideal we define $\rho(\mathfrak{b})$ to be the number of ideals $\mathfrak{B} \subset \mathcal{O}_K$ satisfying $N_{K/F}(\mathfrak{B}) = \mathfrak{b}$. If ℓ is a rational prime we define $\rho_\ell(\mathfrak{b})$ to be the number of ideals $\mathfrak{B} \subset \mathcal{O}_{K, \ell}$ satisfying $N_{K_\ell/F_\ell}(\mathfrak{B}) = \mathfrak{b}_\ell$. Thus

$$\rho(\mathfrak{b}) = \prod_{\ell} \rho_\ell(\mathfrak{b}). \quad (1.2)$$

For the proof of the following theorem see Section 2.7.

Theorem A. *Suppose $\alpha \in F$ is totally positive. If $\alpha \in \mathfrak{D}^{-1}$ and $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ then \mathcal{X}_α has dimension zero, is supported in characteristic p (the rational prime below \mathfrak{p}), and satisfies*

$$\text{deg}(\mathcal{X}_\alpha) = \frac{1}{2} \text{ord}_{\mathfrak{p}}(\alpha \mathfrak{p} \mathfrak{D}) \rho(\alpha \mathfrak{D} \mathfrak{p}^{-1}) \cdot \log(p).$$

If $\alpha \notin \mathfrak{D}^{-1}$ or if $|\text{Diff}(\alpha)| > 1$, then $\mathcal{X}_\alpha = \emptyset$.

The functional equation forces $E^*(\tau_1, \tau_2, 0) = 0$, and the central derivative has a Fourier expansion

$$E^{*,\prime}(\tau_1, \tau_2, 0) = \sum_{\alpha \in \mathfrak{D}^{-1}} a_\alpha(v_1, v_2) \cdot q^\alpha$$

where $v_i = \text{Im}(\tau_i)$, $e(x) = e^{2\pi i x}$, and $q^\alpha = e(\sigma_1(\alpha)\tau_1 + \sigma_2(\alpha)\tau_2)$.

Theorem B. *Suppose $\alpha \in F$ is totally positive. If $\alpha \in \mathfrak{D}^{-1}$ and $\text{Diff}(\alpha) = \{\mathfrak{p}\}$, then $a_\alpha = a_\alpha(v_1, v_2)$ is independent of v_1, v_2 , and*

$$a_\alpha = 2\text{ord}_{\mathfrak{p}}(\alpha\mathfrak{p}\mathfrak{D})\rho(\alpha\mathfrak{D}\mathfrak{p}^{-1}) \cdot \log(p).$$

Here p is the rational prime below \mathfrak{p} . If $\alpha \notin \mathfrak{D}^{-1}$ or if $|\text{Diff}(\alpha)| > 1$, then $a_\alpha = 0$.

Theorem B is stated in a different form in [4], but without proof. We will give a sketch of the proof in Section 3.2. Combining the above theorems we obtain the following.

Theorem C. *Assume $\alpha \in F$ is totally positive. Then \mathcal{X}_α is a stack of dimension zero and*

$$4 \cdot \deg(\mathcal{X}_\alpha) = a_\alpha$$

where a_α is the α -th Fourier coefficient of $E^{*,\prime}(\tau_1, \tau_2, 0)$.

In Section 3.3 we give a slightly different and more conceptual proof of Theorem C, based on the Siegel-Weil formula, which we now outline. Fix a totally positive $\alpha \in F$. Assume that $\alpha \in \mathfrak{D}^{-1}$ and $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ (otherwise both sides of the equality of Theorem C are equal to zero). Let p be the rational prime lying below \mathfrak{p} , so that \mathcal{X}_α is supported in characteristic p . It is proved in Theorem 2.26 that

$$\text{length}(\mathcal{O}_{\mathcal{X}_\alpha, x}^{\text{sh}}) = \nu_{\mathfrak{p}}(\alpha)$$

for some explicit number $\nu_{\mathfrak{p}}(\alpha)$ independent of x , and so

$$\deg(\mathcal{X}_\alpha) = \nu_{\mathfrak{p}}(\alpha) \log(p) \sum_{x \in [\mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})]} e_x^{-1}.$$

Next, applying the Siegel-Weil formula, one can prove that the summation on the right is, up to a factor of $1/4$, the α -th Fourier coefficient of the value at $s = 0$ of a coherent (in the sense of Kudla [8]) Eisenstein series $E_\alpha^*(\tau, s, \phi^{(\mathfrak{p})})$; see Proposition 3.9 and the argument at the end of the paper. Therefore

$$\deg(\mathcal{X}_\alpha) \cdot q^\alpha = \frac{1}{4} \nu_{\mathfrak{p}}(\alpha) \log(p) \cdot E_\alpha^*(\tau, 0, \phi^{(\mathfrak{p})}).$$

On the other hand, $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ implies

$$E_\alpha^{*,\prime}(\tau, 0) = \frac{W_{\alpha, \mathfrak{p}}^{*,\prime}(1, 0)}{W_{\alpha, \mathfrak{p}}^*(1, 0, \phi_{\mathfrak{p}}^{(\mathfrak{p})})} E_\alpha^*(\tau, 0, \phi^{(\mathfrak{p})}),$$

where $W_{\alpha, \mathfrak{p}}^*(1, s)$ is the local Whittaker function of $E^*(\tau, s)$ at \mathfrak{p} , and similarly for $W_{\alpha, \mathfrak{p}}^*(1, s, \phi^{(\mathfrak{p})})$. Finally, explicit calculation shows (see Proposition 3.10 and the argument at the end of the paper)

$$\frac{W_{\alpha, \mathfrak{p}}^{*,'}(1, 0)}{W_{\alpha, \mathfrak{p}}^*(1, 0, \phi_{\mathfrak{p}}^{(\mathfrak{p})})} = \nu_{\mathfrak{p}}(\alpha) \log(p).$$

Therefore

$$E_{\alpha}^{*,'}(\tau, 0) = \nu_{\mathfrak{p}}(\alpha) \log p \cdot E_{\alpha}^*(\tau, 0, \phi^{(\mathfrak{p})}) = 4 \deg(\mathcal{X}_{\alpha}) \cdot q^{\alpha}.$$

This gives a proof of Theorem C without explicitly counting the number of points in $[\mathcal{X}_{\alpha}(\mathbb{F}_p^{\text{alg}})]$, and without explicitly computing the Fourier coefficient a_{α} .

By Theorem C, one sees that the generating function

$$\phi(\tau) = \sum_{\substack{\alpha \in \mathfrak{D}^{-1} \\ \alpha \gg 0}} \deg(\mathcal{X}_{\alpha}) \cdot q^{\alpha}$$

is the holomorphic part of a non-holomorphic Hilbert modular form of weight 1 for $\text{SL}_2(\mathcal{O}_F)$, namely $E^{*,'}(\tau_1, \tau_2, 0)$. One can also view the theorem as an arithmetic Siegel-Weil formula in the sense of [10] and [12]—giving an arithmetic interpretation of the central derivative of the incoherent Eisenstein series $E^*(\tau_1, \tau_2, s)$.

We now explain in what sense Theorem A is a refinement of the earlier work of Gross and Zagier on singular moduli. For a positive integer m let \mathcal{T}_m be the algebraic stack representing the functor that assigns to a scheme S the category of all triples $(\mathbf{E}_1, \mathbf{E}_2, j)$ where $(\mathbf{E}_1, \mathbf{E}_2)$ is an object of $\mathcal{X}(S)$ and $j \in \text{Hom}(E_1, E_2)$ satisfies $\deg(j) = m$. Directly from the moduli problems we have

$$\mathcal{T}_m = \bigsqcup_{\substack{\alpha \in F \\ \text{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \mathcal{X}_{\alpha}.$$

Combining this decomposition with the formula for $\deg(\mathcal{X}_{\alpha})$ of Theorem A one finds (see Corollary 2.29) a formula for $\deg(\mathcal{T}_m)$. This formula is precisely the main result of [4].

Corollary D (Gross-Zagier). *For any positive integer m we have*

$$\deg(\mathcal{T}_m) = \frac{1}{2} \sum_{\substack{\alpha \in \mathfrak{D}^{-1} \\ \text{Tr}_{F/\mathbb{Q}}(\alpha) = m \\ a \gg 0}} \sum_p \log p \sum_{\mathfrak{p}|p} \text{ord}_{\mathfrak{p}}(\alpha \mathfrak{D} \mathfrak{p}) \rho(\alpha \mathfrak{D} \mathfrak{p}^{-1})$$

where the middle summation is over those rational primes p that are nonsplit in both K_1 and K_2 .

This work grew out of the authors' attempts to understand Gross and Zagier's work on singular moduli from the perspective of Kudla's program [8, 10, 12] to relate arithmetic intersection multiplicities on Shimura varieties of orthogonal and unitary type to the Fourier coefficients of derivatives of Eisenstein series. On the occasion of his sixtieth birthday the authors wish to express to Steve Kudla both their deepest appreciation for his beautiful mathematics, and their gratitude for his influence on their own lives and work.

2 Moduli spaces of CM elliptic curves

In this section we study the moduli stack \mathcal{X}_α and prove Theorem A.

2.1 CM pairs

Let S be a scheme and R an order in a quadratic imaginary field. An *elliptic curve over S with complex multiplication by R* is a pair $\mathbf{E} = (E, \kappa)$ in which $E \rightarrow S$ is an elliptic curve and $\kappa : R \rightarrow \text{End}(E)$ is an action of R on E .

Definition 2.1. A *CM pair* over a scheme S is a pair $(\mathbf{E}_1, \mathbf{E}_2)$ in which \mathbf{E}_1 and \mathbf{E}_2 are elliptic curves over S with complex multiplication by \mathcal{O}_{K_1} and \mathcal{O}_{K_2} , respectively. An *isomorphism* between CM pairs $(\mathbf{E}'_1, \mathbf{E}'_2) \rightarrow (\mathbf{E}_1, \mathbf{E}_2)$ is a pair (f_1, f_2) in which each $f_i : E'_i \rightarrow E_i$ is an \mathcal{O}_{K_i} -linear isomorphism of elliptic curves.

For every CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ over a scheme S we abbreviate

$$L(\mathbf{E}_1, \mathbf{E}_2) = \text{Hom}(E_1, E_2),$$

where Hom means homomorphisms between elliptic curves over S in the usual sense, and set

$$V(\mathbf{E}_1, \mathbf{E}_2) = L(\mathbf{E}_1, \mathbf{E}_2) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Assuming that S is connected, the finite rank \mathbb{Z} -module $L(\mathbf{E}_1, \mathbf{E}_2)$ is equipped with the positive definite quadratic form $\deg(j)$. Denote by

$$\begin{aligned} [j_1, j_2] &= \deg(j_1 + j_2) - \deg(j_1) - \deg(j_2) \\ &= j_1^\vee \circ j_2 + j_2^\vee \circ j_1 \end{aligned}$$

the associated bilinear form. The maximal order $\mathcal{O}_K = \mathcal{O}_{K_1} \otimes_{\mathbb{Z}} \mathcal{O}_{K_2}$ acts on the \mathbb{Z} -module $L(\mathbf{E}_1, \mathbf{E}_2)$ by

$$(x_1 \otimes x_2) \bullet j = \kappa_2(x_2) \circ j \circ \kappa_1(\overline{x_1}). \quad (2.1)$$

By a *K -Hermitian form* on $V(\mathbf{E}_1, \mathbf{E}_2)$ we mean a function

$$\langle \cdot, \cdot \rangle : V(\mathbf{E}_1, \mathbf{E}_2) \times V(\mathbf{E}_1, \mathbf{E}_2) \rightarrow K$$

which is K -linear in the first variable and satisfies $\langle j_1, j_2 \rangle = \overline{\langle j_2, j_1 \rangle}$.

Proposition 2.2.

1. There is a unique F -bilinear form $[j_1, j_2]_{\text{CM}}$ on $V(\mathbf{E}_1, \mathbf{E}_2)$ satisfying

$$[j_1, j_2] = \text{Tr}_{F/\mathbb{Q}} [j_1, j_2]_{\text{CM}}.$$

2. The F -quadratic form

$$\text{deg}_{\text{CM}}(j) = \frac{1}{2} \cdot [j, j]_{\text{CM}}$$

is the unique F -quadratic form on $V(\mathbf{E}_1, \mathbf{E}_2)$ satisfying

$$\text{deg}(j) = \text{Tr}_{F/\mathbb{Q}} \text{deg}_{\text{CM}}(j).$$

3. There is a unique K -Hermitian form $\langle j_1, j_2 \rangle_{\text{CM}}$ on $V(\mathbf{E}_1, \mathbf{E}_2)$ satisfying

$$[j_1, j_2]_{\text{CM}} = \text{Tr}_{K/F} \langle j_1, j_2 \rangle_{\text{CM}}.$$

Proof. Suppose $j_1, j_2 \in V(\mathbf{E}_1, \mathbf{E}_2)$. If $x = x_1 \otimes x_2 \in K$ is nonzero then as elements of $\text{End}(E_1)$ we have

$$\begin{aligned} [x \bullet j_1, j_2] &= \kappa_1(x_1)^{-1} \circ [x \bullet j_1, j_2] \circ \kappa_1(x_1) \\ &= j_1^\vee \circ \kappa_2(\bar{x}_2) \circ j_2 \circ \kappa_1(x_1) + \kappa_1(\bar{x}_1) \circ j_2^\vee \circ \kappa_2(x_2) \circ j_1 \\ &= [j_1, \bar{x} \bullet j_2]. \end{aligned}$$

Thus for all $x \in K$ we have $[x \bullet j_1, j_2] = [j_1, \bar{x} \bullet j_2]$. All of the claims now follow from this property and some elementary linear algebra; in particular from the fact that if M/L is a finite separable extension of fields, then for any finite dimensional M -vector space V the trace $\text{Tr}_{M/L}$ induces an isomorphism $\text{Hom}_M(V, M) \rightarrow \text{Hom}_L(V, L)$. \square

Thus the complex multiplication of \mathbf{E}_1 and \mathbf{E}_2 endows the set $V(\mathbf{E}_1, \mathbf{E}_2)$ not only with a K -action, but with an F -quadratic form deg_{CM} which refines the usual notion of degree.

To understand the moduli space of CM pairs over schemes we use the language of stacks and groupoids as in [18]. Given a CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ over a scheme S and a morphism of schemes $T \rightarrow S$ there is an evident notion of the pullback CM pair $(\mathbf{E}_1, \mathbf{E}_2)_{/T}$.

Definition 2.3. Define \mathcal{X} to be the category whose objects are CM pairs over schemes. In the category \mathcal{X} an arrow $(\mathbf{E}'_1, \mathbf{E}'_2) \rightarrow (\mathbf{E}_1, \mathbf{E}_2)$ between CM pairs defined over schemes T and S , respectively, is a morphism of schemes $T \rightarrow S$ together with an isomorphism (in the sense of Definition 2.1) of CM pairs over T

$$(\mathbf{E}'_1, \mathbf{E}'_2) \cong (\mathbf{E}_1, \mathbf{E}_2)_{/T}.$$

Thus \mathcal{X} is a category fibered in groupoids over the category of schemes. For a scheme S the fiber $\mathcal{X}(S)$ is the category of CM pairs over S , and arrows in this category are isomorphisms in the sense of Definition 2.1.

Definition 2.4. For every $m \in \mathbb{Q}$ define \mathcal{T}_m to be the category, fibered in groupoids over schemes, of triples $(\mathbf{E}_1, \mathbf{E}_2, j)$ in which $(\mathbf{E}_1, \mathbf{E}_2)$ is a CM pair over a scheme S and $j \in L(\mathbf{E}_1, \mathbf{E}_2)$ satisfies $\deg(j) = m$ on every connected component of S .

Definition 2.5. For every $\alpha \in F$ define \mathcal{X}_α to be the category of triples $(\mathbf{E}_1, \mathbf{E}_2, j)$ in which $(\mathbf{E}_1, \mathbf{E}_2)$ is a CM pair over a scheme and $j \in L(\mathbf{E}_1, \mathbf{E}_2)$ satisfies $\deg_{\text{CM}}(j) = \alpha$ on every connected component of S .

The categories \mathcal{X} , \mathcal{T}_m , and \mathcal{X}_α are algebraic stacks in the sense of [18] (also known as Deligne-Mumford stacks) of finite type over $\text{Spec}(\mathbb{Z})$. Briefly, one knows that the category \mathcal{E} of elliptic curves over schemes is an algebraic stack of finite type over $\text{Spec}(\mathbb{Z})$, and the relative representability of each of \mathcal{X} , \mathcal{T}_m , and \mathcal{X}_α over $\mathcal{E} \times_{\mathbb{Z}} \mathcal{E}$ is proved using the methods and results of [5, Chapter 6]. For every $m \in \mathbb{Q}$ there is an evident decomposition

$$\mathcal{T}_m = \bigsqcup_{\substack{\alpha \in F \\ \text{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \mathcal{X}_\alpha. \quad (2.2)$$

If S is a scheme and \mathcal{C} is any one of \mathcal{X} , \mathcal{T}_m , or \mathcal{X}_α then $[\mathcal{C}(S)]$ denotes the set of isomorphism classes of objects in the category $\mathcal{C}(S)$.

2.2 The support of \mathcal{X}_α

Given $\alpha \in F^\times$ define a nondegenerate \mathbb{Q} -quadratic form Q_α on K by

$$Q_\alpha(x) = \text{Tr}_{F/\mathbb{Q}}(\alpha x \bar{x}).$$

For each place $\ell \leq \infty$ of \mathbb{Q} let $\text{hasse}_\ell(\cdot)$ be the Hasse invariant on \mathbb{Q}_ℓ -quadratic spaces (the invariant ϵ of [15, Chapter IV.2]) and let $(\cdot, \cdot)_\ell$ be the usual Hilbert symbol $\mathbb{Q}_\ell^\times \times \mathbb{Q}_\ell^\times \rightarrow \{\pm 1\}$. Define the *local invariant* $\text{inv}_\ell(\alpha) = \pm 1$ by

$$\text{inv}_\ell(\alpha) = \text{hasse}_\ell(K_\ell, Q_\alpha) \cdot (-1, -1)_\ell.$$

Define the *modified local invariant* of α by

$$\text{inv}_\ell^*(\alpha) = \begin{cases} \text{inv}_\ell(\alpha) & \text{if } \ell < \infty \\ -\text{inv}_\ell(\alpha) & \text{if } \ell = \infty. \end{cases}$$

Define the *support* of α to be the finite set of rational primes

$$\text{Sppt}(\alpha) = \{\ell \leq \infty : \text{inv}_\ell^*(\alpha) = -1\},$$

and note that the product formula $\prod_{\ell \leq \infty} \text{inv}_\ell(\alpha) = 1$ implies that $\text{Sppt}(\alpha)$ has odd cardinality.

A CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ over an algebraically closed field k of nonzero characteristic is *supersingular* if the underlying elliptic curves E_1 and E_2 are supersingular in the usual sense. If this is the case then E_1 and E_2 are isogenous, and there is an isomorphism of \mathbb{Q} -vector spaces $V(\mathbf{E}_1, \mathbf{E}_2) \cong \text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ for any supersingular elliptic curve E over k . In particular $V(\mathbf{E}_1, \mathbf{E}_2)$ has dimension four as a \mathbb{Q} -vector space, and so has dimension one over K . Hermitian K -modules of dimension one are easy to classify, and we find that for some $\beta \in F^\times$ there is an isomorphism of F -quadratic spaces

$$(V(\mathbf{E}_1, \mathbf{E}_2), \text{deg}_{\text{CM}}) \cong (K, \beta \cdot \text{Nm}_{K/F}). \quad (2.3)$$

The element β is determined up to multiplication by a norm from K^\times , and as the \mathbb{Q} -quadratic form deg is positive definite, β is totally positive. In the next subsection we will compute the image of β in $\widehat{F}^\times / \text{Nm}_{K/F}(\widehat{K}^\times)$, which is enough to determine the isomorphism class of the quadratic space (2.3); see Theorem 2.12 below.

Proposition 2.6. *Suppose k is an algebraically closed field of characteristic $p \geq 0$ and $\alpha \in F^\times$. If $(\mathbf{E}_1, \mathbf{E}_2, j) \in \mathcal{X}_\alpha(k)$ then*

1. $p > 0$ and the CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ is supersingular;
2. p is nonsplit in both K_1 and K_2 ;
3. there are isomorphisms of quadratic spaces over \mathbb{Q}

$$(K, Q_\alpha) \cong (V(\mathbf{E}_1, \mathbf{E}_2), \text{deg}) \cong (H, \text{Nm})$$

where H is the rational quaternion algebra over \mathbb{Q} of discriminant p and Nm is the reduced norm on H ;

4. $\text{Sppt}(\alpha) = \{p\}$ and α is totally positive.

Proof. Suppose that $p = 0$. As $j : E_1 \rightarrow E_2$ is a nonzero isogeny, κ_1 , κ_2 , and j determine isomorphisms

$$K_1 \cong \text{End}(E_1) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \text{End}(E_2) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_2, \quad (2.4)$$

contrary to our hypotheses on K_1 and K_2 . Thus k has characteristic $p > 0$. The existence of the isogeny j implies that the elliptic curves E_1 and E_2 are either both supersingular or both ordinary. If they are both ordinary then again (2.4) gives a contradiction. Thus $(\mathbf{E}_1, \mathbf{E}_2)$ is supersingular. In particular

$$\text{End}(E_1) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H \cong \text{End}(E_2) \otimes_{\mathbb{Z}} \mathbb{Q}$$

as \mathbb{Q} -algebras, and as K_1 and K_2 both embed into H we see that p is nonsplit in both K_1 and K_2 . By hypothesis the quadratic space (2.3) represents α , and so there is a $u \in K^\times$ such that $\alpha = \beta \cdot \text{Nm}_{K/F}(u)$. Thus we have an isomorphism of F -quadratic spaces

$$(V(\mathbf{E}_1, \mathbf{E}_2), \text{deg}_{\text{CM}}) \cong (K, \alpha \cdot \text{Nm}_{K/F})$$

and so also an isomorphism of \mathbb{Q} -quadratic spaces

$$(V(\mathbf{E}_1, \mathbf{E}_2), \deg) \cong (K, Q_\alpha).$$

Fix an isomorphism of \mathbb{Q} -algebras $H \cong \text{End}(E_1) \otimes_{\mathbb{Z}} \mathbb{Q}$. The function $f \mapsto f^\vee \circ j$ defines an isomorphism of \mathbb{Q} -quadratic spaces

$$(V(\mathbf{E}_1, \mathbf{E}_2), \deg) \cong (H, b^{-1} \cdot \text{Nm})$$

where $b = \deg(j)$, and as b lies in the image of the reduced norm $H^\times \rightarrow \mathbb{Q}^\times$ there is an isomorphism of \mathbb{Q} -quadratic spaces $(H, b^{-1} \cdot \text{Nm}) \cong (H, \text{Nm})$. It only remains to prove that $\text{Sppt}(\alpha) = \{p\}$. Using the isomorphism $(K, Q_\alpha) \cong (H, \text{Nm})$ already proved we find

$$\text{inv}_\ell(\alpha) = \text{hasse}_\ell(H_\ell, \text{Nm}) \cdot (-1, -1)_\ell.$$

By direct calculation of the Hasse invariant of (H_ℓ, Nm) it follows that

$$\text{inv}_\ell(\alpha) = \begin{cases} -1 & \text{if } \ell \in \{p, \infty\} \\ 1 & \text{otherwise.} \end{cases}$$

In particular $\text{inv}_\ell^*(\alpha) = -1$ if and only if $\ell = p$. \square

Corollary 2.7. *Suppose $\alpha \in F^\times$. If $\mathcal{X}_\alpha \neq \emptyset$ then α is totally positive and $\text{Sppt}(\alpha) = \{p\}$ for a finite prime p . Furthermore all geometric points of \mathcal{X}_α lie in characteristic p and are supersingular.*

Proof. This is immediate from Proposition 2.6. \square

2.3 Local quadratic spaces

Fix a prime p that is nonsplit in both K_1 and K_2 , and a supersingular CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ over $\mathbb{F}_p^{\text{alg}}$. For $i \in \{1, 2\}$ the action

$$\kappa_i^{\text{Lie}} : \mathcal{O}_{K_i} \rightarrow \text{End}_{\mathbb{F}_p^{\text{alg}}}(\text{Lie}(E_i)) \cong \mathbb{F}_p^{\text{alg}}$$

induces a homomorphism $\mathcal{O}_K \cong \mathcal{O}_{K_1} \otimes_{\mathbb{Z}} \mathcal{O}_{K_2} \rightarrow \mathbb{F}_p^{\text{alg}}$ defined by

$$t_1 \otimes t_2 \mapsto \kappa_1^{\text{Lie}}(t_1) \cdot \kappa_2^{\text{Lie}}(t_2). \quad (2.5)$$

The kernel of this map is denoted \mathfrak{q} .

Definition 2.8. The prime \mathfrak{p} of F below \mathfrak{q} is the *reflex prime* of $(\mathbf{E}_1, \mathbf{E}_2)$.

Remark 2.9. If p is inert in both K_1 and K_2 then p is split in F , and each prime of F above p is inert in K ; in particular there are two possibilities for the reflex prime of a supersingular CM pair $(\mathbf{E}_1, \mathbf{E}_2)$. If instead p is ramified in one of K_1 or K_2 then p is ramified in F , and the unique prime of F above p is inert in K ; in particular there is only one possibility for the reflex prime of $(\mathbf{E}_1, \mathbf{E}_2)$.

For every prime ℓ abbreviate

$$\begin{aligned} L_\ell(\mathbf{E}_1, \mathbf{E}_2) &= L(\mathbf{E}_1, \mathbf{E}_2) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \\ V_\ell(\mathbf{E}_1, \mathbf{E}_2) &= V(\mathbf{E}_1, \mathbf{E}_2) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \end{aligned}$$

Lemma 2.10. *Suppose ℓ is a prime different from p . For some $\beta \in F_\ell^\times$ satisfying $\beta \mathcal{O}_{F,\ell} = \mathfrak{D}_\ell^{-1}$ there is a K_ℓ -linear isomorphism of F_ℓ -quadratic spaces*

$$(V_\ell(\mathbf{E}_1, \mathbf{E}_2), \deg_{\text{CM}}) \cong (K_\ell, \beta \cdot \text{Nm}_{K_\ell/F_\ell})$$

taking $L_\ell(\mathbf{E}_1, \mathbf{E}_2)$ isomorphically to $\mathcal{O}_{K,\ell}$.

Proof. The existence of the desired isomorphism for some choice of $\beta \in F_\ell^\times$ is clear from (2.3). We must determine the fractional $\mathcal{O}_{F,\ell}$ -ideal $\beta \mathcal{O}_{F,\ell}$. Any choice of \mathbb{Z}_ℓ -bases for the Tate modules $\text{Ta}_\ell(E_1)$ and $\text{Ta}_\ell(E_2)$ determines isomorphisms of \mathbb{Z}_ℓ -modules

$$L_\ell(\mathbf{E}_1, \mathbf{E}_2) \cong \text{Hom}_{\mathbb{Z}_\ell}(\text{Ta}_\ell(E_1), \text{Ta}_\ell(E_2)) \cong M_2(\mathbb{Z}_\ell),$$

which identify the quadratic form \deg with the quadratic form $u \cdot \det$ for some $u \in \mathbb{Z}_\ell^\times$. It follows that the \mathbb{Z}_ℓ -lattice $L_\ell(\mathbf{E}_1, \mathbf{E}_2)$ is self dual relative to \deg , and hence that the $\mathcal{O}_{F,\ell}$ -bilinear form of Proposition 2.2

$$[\cdot, \cdot]_{\text{CM}} : L_\ell(\mathbf{E}_1, \mathbf{E}_2) \times L_\ell(\mathbf{E}_1, \mathbf{E}_2) \rightarrow \mathfrak{D}_\ell^{-1}$$

is a perfect pairing. This implies that the $\mathcal{O}_{F,\ell}$ -bilinear form

$$\mathcal{O}_{K,\ell} \times \mathcal{O}_{K,\ell} \rightarrow \mathfrak{D}_\ell^{-1}$$

defined by $\beta \cdot \text{Tr}_{K_\ell/F_\ell}(x\bar{y})$ is a perfect pairing. As K/F is unramified, it follows that β generates \mathfrak{D}_ℓ^{-1} . \square

Lemma 2.11. *For some $\beta \in F_p^\times$ satisfying $\beta \mathcal{O}_{F,p} = \mathfrak{p} \mathfrak{D}_p^{-1}$ there is a K_p -linear isomorphism of F_p -quadratic spaces*

$$(V_p(\mathbf{E}_1, \mathbf{E}_2), \deg_{\text{CM}}) \cong (K_p, \beta \cdot \text{Nm}_{K_p/F_p})$$

taking $L_p(\mathbf{E}_1, \mathbf{E}_2)$ isomorphically to $\mathcal{O}_{K,p}$. Here \mathfrak{p} is the reflex prime of $(\mathbf{E}_1, \mathbf{E}_2)$.

Proof. The existence of the desired isomorphism for some choice of $\beta \in F_p^\times$ is clear from (2.3). We must determine the fractional $\mathcal{O}_{F,p}$ -ideal $\beta \mathcal{O}_{F,p}$. The proof is similar to the proof of Lemma 2.10, but with ℓ -adic Tate modules replaced by (covariant) Dieudonné modules. Fix any supersingular elliptic curve E over $\mathbb{F}_p^{\text{alg}}$. By choosing prime to p -isogenies $E_i \rightarrow E$ we may reduce to the case where the CM elliptic curves \mathbf{E}_1 and \mathbf{E}_2 have the same underlying elliptic curve E . Let D be the Dieudonné module of E and set $\Delta = \text{End}(D)$, the maximal order in a quaternion division algebra over \mathbb{Q}_p . In this way we obtain an isomorphism of \mathbb{Z}_p -quadratic spaces

$$(L_p(\mathbf{E}_1, \mathbf{E}_2), \deg) \cong (\Delta, \text{Nm})$$

where Nm is the reduced norm on Δ . Denoting by \mathfrak{m}_Δ the unique maximal ideal of Δ , the dual lattice of Δ relative to Nm is \mathfrak{m}_Δ^{-1} . The dual lattice of $L_p(\mathbf{E}_1, \mathbf{E}_2)$ with respect to deg is

$$L_p(\mathbf{E}_1, \mathbf{E}_2)^\vee = \{j \in V_p(\mathbf{E}_1, \mathbf{E}_2) : [j, j']_{\text{CM}} \in \mathfrak{D}_p^{-1} \quad \forall j' \in L_p(\mathbf{E}_1, \mathbf{E}_2)\},$$

and we have $\mathcal{O}_{K,p}$ -linear isomorphisms

$$\beta^{-1}\mathfrak{D}^{-1}\mathcal{O}_{K,p}/\mathcal{O}_{K,p} \cong L_p(\mathbf{E}_1, \mathbf{E}_2)^\vee/L_p(\mathbf{E}_1, \mathbf{E}_2) \cong \mathfrak{m}_\Delta^{-1}/\Delta. \quad (2.6)$$

As a group $\Delta/\mathfrak{m}_\Delta \cong \mathbb{F}_{p^2}$, and so $[\mathcal{O}_{K,p} : \beta\mathfrak{D}\mathcal{O}_{K,p}] = p^2$.

Suppose first that p is ramified in one of K_1 or K_2 , so that $p\mathcal{O}_F = \mathfrak{p}^2$. Using Remark 2.9 and the equality $[\mathcal{O}_{K,p} : \beta\mathfrak{D}\mathcal{O}_{K,p}] = p^2$ we immediately deduce that $\beta\mathfrak{D}_p = \mathfrak{p}\mathcal{O}_{F,p}$, and we are done.

Now suppose that p is unramified in K_1 and K_2 , and recall that \mathcal{O}_K acts on Δ by

$$(t_1 \otimes t_2) \bullet j = \kappa_2(t_2) \circ j \circ \kappa_1(\bar{t}_1).$$

If we fix a uniformizing parameter $\Pi \in \Delta$ in such a way that $\kappa_1(\bar{t}_1)\Pi = \Pi\kappa_1(t_1)$ for every $t_1 \in \mathcal{O}_{K_1}$, then for any $u \in \kappa_1(\mathcal{O}_{K_1}) \subset \Delta$ we have

$$(t_1 \otimes t_2) \bullet u\Pi^{-1} = \kappa_2(t_2)u\Pi^{-1}\kappa_1(\bar{t}_1) = \kappa_2(t_2)\kappa_1(t_1) \cdot u\Pi^{-1}.$$

As $\mathfrak{m}_\Delta^{-1}/\Delta$ is generated by such elements $u\Pi^{-1}$, \mathcal{O}_K acts on $\mathfrak{m}_\Delta^{-1}/\Delta$ through left multiplication by the composition

$$\mathcal{O}_K \rightarrow \Delta \rightarrow \Delta/\mathfrak{m}_\Delta$$

where the first arrow is $t_1 \otimes t_2 \mapsto \kappa_2(t_2)\kappa_1(t_1)$. On the other hand, the action

$$\Delta \rightarrow \text{End}_{\mathbb{F}_p^{\text{alg}}}(\text{Lie}(E)) \cong \mathbb{F}_p^{\text{alg}}$$

determines an isomorphism $\Delta/\mathfrak{m}_\Delta \cong \mathbb{F}_{p^2}$ which allows us to identify κ_i^{Lie} with the composition

$$\mathcal{O}_{K_i} \xrightarrow{\kappa_i} \Delta \rightarrow \Delta/\mathfrak{m}_\Delta \rightarrow \mathbb{F}_{p^2}.$$

The conclusion of all of this is that the action of \mathcal{O}_K on $\mathfrak{m}_\Delta^{-1}/\Delta$ factors through the kernel \mathfrak{q} of the map (2.5). Returning to (2.6) we find that

$$\beta^{-1}\mathfrak{D}^{-1}\mathcal{O}_{K,p}/\mathcal{O}_{K,p} \cong \mathcal{O}_{K,p}/\mathfrak{q}$$

as \mathcal{O}_K -modules, and the relation $\beta\mathcal{O}_{F,p} = \mathfrak{p}\mathfrak{D}_p^{-1}$ follows easily. \square

Theorem 2.12. *For any finite idele $\beta \in \widehat{F}^\times$ satisfying $\beta\widehat{\mathcal{O}}_F = \mathfrak{p}\mathfrak{D}^{-1}\widehat{\mathcal{O}}_F$ there is a \widehat{K} -linear isomorphism of \widehat{F} -quadratic spaces*

$$(\widehat{V}(\mathbf{E}_1, \mathbf{E}_2), \text{deg}_{\text{CM}}) \cong (\widehat{K}, \beta \cdot \text{Nm}_{K/F})$$

taking $\widehat{L}(\mathbf{E}_1, \mathbf{E}_2)$ isomorphically to $\widehat{\mathcal{O}}_K$.

Proof. Combining Lemma 2.10 with Lemma 2.11 shows that the claim is true for some $\beta \in \widehat{F}^\times$ satisfying $\beta\widehat{\mathcal{O}}_F = \mathfrak{p}\mathfrak{D}^{-1}\widehat{\mathcal{O}}_F$. The surjectivity of the norm map $\widehat{\mathcal{O}}_K^\times \rightarrow \widehat{\mathcal{O}}_F^\times$ implies that the claim is true for all such β . \square

2.4 Group actions

For any sets $Y \subset X$ the characteristic function of Y is denoted $\mathbf{1}_Y$. For $i \in \{1, 2\}$ define an algebraic group over \mathbb{Q} by

$$T_i(A) = (K_i \otimes_{\mathbb{Q}} A)^\times$$

for any \mathbb{Q} -algebra A . Let $\nu_i : T_i \rightarrow \mathbb{G}_m$ be the norm $\nu_i(t_i) = t_i \bar{t}_i$ and define

$$T(A) = \{(t_1, t_2) \in T_1(A) \times T_2(A) : \nu_1(t_1) = \nu_2(t_2)\}.$$

Define an algebraic group S over \mathbb{Q} by

$$S(A) = \{z \in (K \otimes_{\mathbb{Q}} A)^\times : \text{Nm}_{K/F}(z) = 1\}.$$

There is an evident character $\nu : T \rightarrow \mathbb{G}_m$ defined by the relations

$$\nu_1(t_1) = \nu(t) = \nu_2(t_2)$$

for $t = (t_1, t_2) \in T(R)$, and a homomorphism $\eta : T \rightarrow S$ defined by

$$\eta(t) = \nu(t)^{-1} \cdot (t_1 \otimes t_2). \quad (2.7)$$

Let $U \subset T(\mathbb{A}_f)$ be the compact open subgroup

$$U = T(\mathbb{A}_f) \cap (\hat{\mathcal{O}}_{K_1}^\times \times \hat{\mathcal{O}}_{K_2}^\times)$$

and let $V \subset S(\mathbb{A}_f)$ be the image of U under $\eta : T(\mathbb{A}_f) \rightarrow S(\mathbb{A}_f)$. The groups U and V factor as $U = \prod_{\ell} U_{\ell}$ and $V = \prod_{\ell} V_{\ell}$ for compact open subgroups $U_{\ell} \subset T(\mathbb{Q}_{\ell})$ and $V_{\ell} \subset S(\mathbb{Q}_{\ell})$.

Proposition 2.13. *If k is a field of characteristic 0, the ring of adeles \mathbb{A} , or the ring of finite adeles \mathbb{A}_f then the sequence*

$$1 \rightarrow k^\times \rightarrow T(k) \xrightarrow{\eta} S(k) \rightarrow 1$$

is exact, where $k^\times \rightarrow T(k)$ is the diagonal inclusion.

Proof. If k is an algebraically closed of characteristic 0 then the claim is proved by explicit calculation after choosing diagonalizations $T(k) \cong (k^\times)^3$ and $S(k) \cong (k^\times)^2$, and we leave this as an exercise for the reader. The exactness for an arbitrary field of characteristic 0 is immediate from the algebraically closed case and Hilbert's Theorem 90. The proof for $k = \mathbb{A}$ is proved the same way, using the adelic form of Hilbert's Theorem 90 [13, Corollary 8.1.3], and the exactness for $k = \mathbb{A}$ implies the exactness for $k = \mathbb{A}_f$. \square

Using Proposition 2.13 and the inclusion $\mathbb{A}_f^\times \subset T(\mathbb{Q})U$, it follows that the homomorphism (2.7) induces an isomorphism

$$T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / U \cong S(\mathbb{Q}) \backslash S(\mathbb{A}_f) / V. \quad (2.8)$$

For $i \in \{1, 2\}$ let $\text{Pic}(\mathcal{O}_{K_i})$ be the ideal class group of K_i and set

$$\Gamma = \text{Pic}(\mathcal{O}_{K_1}) \times \text{Pic}(\mathcal{O}_{K_2}).$$

Define a homomorphism

$$T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / U \rightarrow \Gamma \tag{2.9}$$

by sending $(t_1, t_2) \in T(\mathbb{A}_f)$ to the pair of ideal classes $(\mathfrak{a}_1, \mathfrak{a}_2) \in \Gamma$ determined by $\mathfrak{a}_i \widehat{\mathcal{O}}_{K_i} = t_i \widehat{\mathcal{O}}_{K_i}$.

Proposition 2.14. *The homomorphism (2.9) is an isomorphism.*

Proof. If we identify $\text{Pic}(\mathcal{O}_{K_i}) \cong K_i^\times \backslash \widehat{K}_i^\times / \widehat{\mathcal{O}}_{K_i}$ in the usual way then the map (2.9) is identified with the map

$$T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / U \rightarrow (K_1^\times \backslash \widehat{K}_1^\times / \widehat{\mathcal{O}}_{K_1}) \times (K_2^\times \backslash \widehat{K}_2^\times / \widehat{\mathcal{O}}_{K_2})$$

defined by $(t_1, t_2) \mapsto (t_1, t_2)$, and the injectivity follows easily.

The surjectivity of (2.9) is less obvious. For $i \in \{1, 2\}$ fix a fractional \mathcal{O}_{K_i} -ideal \mathfrak{a}_i , set $a_i = \text{Nm}_{K_i/\mathbb{Q}}(\mathfrak{a}_i)$, and define a quadratic form on the \mathbb{Q} -vector space K_i

$$Q_i(x) = a_i \cdot \text{Nm}_{K_i/\mathbb{Q}}(x).$$

Let W be the \mathbb{Q} -vector space $K_1 \oplus K_2$ endowed with the quadratic form

$$Q(x_1, x_2) = Q_1(x_1) - Q_2(x_2).$$

The claim is that (W, Q) represents 0, and by the Hasse-Minkowski theorem it suffices to prove this everywhere locally. As $W \otimes_{\mathbb{Q}} \mathbb{R}$ has signature $(2, 2)$ it clearly represents 0. Fix a prime $\ell < \infty$. The quadratic space W_ℓ has discriminant $d_1 d_2 \in \mathbb{Q}_\ell^\times / (\mathbb{Q}_\ell^\times)^2$ and Hasse invariant

$$(a_1, d_1)_\ell \cdot (a_2, d_2)_\ell \cdot (d_1, -d_2)_\ell \cdot (-1, -1)_\ell.$$

If $d_1 d_2$ is not a square in \mathbb{Q}_ℓ^\times then W_ℓ represents 0 by [15, Chapter IV.2.2]. Thus we may assume that $d_1 = d_2$ up to a square in \mathbb{Q}_ℓ^\times . As a_i is the norm of a fractional ideal in $K_{i,\ell}$ we may factor $a_i = u_i \cdot b_i$ with b_i equal to the norm of some element in K_ℓ^\times and $u_i \in \mathbb{Z}_\ell^\times$. As we assume that $\gcd(d_1, d_2) = 1$, at least one of K_1 and K_2 is unramified at ℓ . Thus u_1 is either a norm from $K_{1,\ell}^\times$ or a norm from $K_{2,\ell}^\times$, and so either $(u_1, d_1)_\ell = 1$ or $(u_1, d_2)_\ell = 1$. But $(u_1, d_1)_\ell = (u_1, d_2)_\ell$ as $d_1 = d_2$ up to a square. Thus we have

$$(a_1, d_1)_\ell = (u_1, d_1)_\ell = 1.$$

The same argument shows that $(a_2, d_2)_\ell = 1$, and as $(d_1, -d_2)_\ell = 1$ is obvious we find that the Hasse invariant of W_ℓ is $(-1, -1)_\ell$. Again by [15, Chapter IV.2.2] the quadratic space W_ℓ represents 0. Having proved that the quadratic space (W, Q) represents 0, we deduce that there is an $m \in \mathbb{Q}^\times$ that is represented

both by Q_1 and by Q_2 . Choosing $r_i \in K_i^\times$ such that $Q_1(r_1) = m = Q_2(r_2)$ we see that the fractional ideal $\mathfrak{b}_i = \mathfrak{a}_i r_i$ lies in the same ideal class as \mathfrak{a}_i , and that

$$\mathrm{Nm}_{K_1/\mathbb{Q}}(\mathfrak{b}_1) = \mathrm{Nm}_{K_2/\mathbb{Q}}(\mathfrak{b}_2). \quad (2.10)$$

Thus we have proved that every element of Γ has the form $(\mathfrak{b}_1, \mathfrak{b}_2)$ with \mathfrak{b}_1 and \mathfrak{b}_2 satisfying (2.10). Now choose $t_i \in \widehat{K}_i^\times$ satisfying

$$t_i \widehat{\mathcal{O}}_{K,i}^\times = \mathfrak{b}_i \widehat{\mathcal{O}}_{K,i}^\times.$$

The relation (2.10) implies that there is a $u \in \widehat{\mathbb{Z}}^\times$ such that

$$\mathrm{Nm}_{K_1/\mathbb{Q}}(t_1) = u \cdot \mathrm{Nm}_{K_2/\mathbb{Q}}(t_2).$$

The hypothesis $\mathrm{gcd}(d_1, d_2) = 1$ implies that

$$\widehat{\mathbb{Z}}^\times = \mathrm{Nm}_{K_1/\mathbb{Q}}(\widehat{\mathcal{O}}_{K_1}^\times) \cdot \mathrm{Nm}_{K_2/\mathbb{Q}}(\widehat{\mathcal{O}}_{K_2}^\times).$$

Factoring u as the product of the norm of some $v_1^{-1} \in \widehat{\mathcal{O}}_{K_1}^\times$ and the norm of some $v_2 \in \widehat{\mathcal{O}}_{K_2}^\times$ we may then replace t_i by $t_i v_i$ so that $(t_1, t_2) \in T(\mathbb{A}_f)$. This proves the surjectivity of (2.9), and completes the proof. \square

For any scheme S the group Γ acts on the set $[\mathcal{X}(S)]$ on the right by Serre's tensor construction [1, Section 7]

$$(\mathbf{E}_1, \mathbf{E}_2) \otimes (\mathfrak{a}_1, \mathfrak{a}_2) = (\mathbf{E}_1 \otimes \mathfrak{a}_1, \mathbf{E}_2 \otimes \mathfrak{a}_2)$$

(the tensor products on the right are over \mathcal{O}_{K_1} and \mathcal{O}_{K_2} , respectively).

Remark 2.15. The classical theory of complex multiplication implies that the action of Γ on $[\mathcal{X}(\mathbb{C})]$ breaks $[\mathcal{X}(\mathbb{C})]$ into a disjoint union of four simply transitive orbits. The orbits are indexed by the set of pairs

$$\{(\pi_1, \pi_2) : \pi_i \in \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(K_i, \mathbb{C})\},$$

and the isomorphism class of a CM pair $(\mathbf{E}_1, \mathbf{E}_2) \in \mathcal{X}(\mathbb{C})$ lies in the orbit indexed by (π_1, π_2) if and only if the action of K_i on the 1-dimensional \mathbb{C} -vector space $\mathrm{Lie}(E_i)$ is through π_i for each $i \in \{1, 2\}$.

Suppose $(\mathbf{E}_1, \mathbf{E}_2)$ is a supersingular CM pair over an algebraically closed field of nonzero characteristic. It follows from (2.3) and [6, Corollary V.6.1.3] that the restriction of the action (2.1) to the subgroup $S(\mathbb{Q}) \subset K^\times$ identifies

$$S \cong \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(V(\mathbf{E}_1, \mathbf{E}_2), \mathrm{deg}_{\mathrm{CM}}).$$

The group $T(\mathbb{Q})$ then acts on $V(\mathbf{E}_1, \mathbf{E}_2)$ through orthogonal transformations by composing with the homomorphism $\eta : T \rightarrow S$, and this action is given by the simple formula

$$t \bullet j = \kappa_2(t_2) \circ j \circ \kappa_1(t_1)^{-1}$$

for $t = (t_1, t_2) \in T(\mathbb{Q})$. To understand the relation between the action of T on $V(\mathbf{E}_1, \mathbf{E}_2)$ and the action of Γ on the set of all supersingular CM pairs, fix a $t = (t_1, t_2) \in T(\mathbb{A}_f)$ and let $(\mathbf{a}_1, \mathbf{a}_2)$ be the image of t under (2.9). For $i \in \{1, 2\}$ there is an \mathcal{O}_K -linear quasi-isogeny

$$f_i \in \text{Hom}(\mathbf{E}_i, \mathbf{E}_i \otimes \mathbf{a}_i) \otimes_{\mathbb{Z}} \mathbb{Q}$$

defined by $f_i(x) = x \otimes 1$, and the \widehat{K} -linear isomorphism of \widehat{F} -quadratic spaces

$$V(\mathbf{E}_1, \mathbf{E}_2) \cong V(\mathbf{E}_1 \otimes \mathbf{a}_1, \mathbf{E}_2 \otimes \mathbf{a}_2)$$

defined by $j \mapsto f_2 \circ j \circ f_1^{-1}$ identifies $\widehat{L}(\mathbf{E}_1 \otimes \mathbf{a}_1, \mathbf{E}_2 \otimes \mathbf{a}_2)$ with the $\widehat{\mathcal{O}}_K$ -submodule

$$t \bullet \widehat{L}(\mathbf{E}_1, \mathbf{E}_2) = \{\kappa_2(t_2) \circ j \circ \kappa_1(t_1)^{-1} : j \in \widehat{L}(\mathbf{E}_1, \mathbf{E}_2)\}$$

of $\widehat{V}(\mathbf{E}_1, \mathbf{E}_2)$.

As above, let $(\mathbf{E}_1, \mathbf{E}_2)$ be a supersingular CM pair over an algebraically closed field of nonzero characteristic. Given a prime ℓ and an $\alpha \in F_\ell^\times$, the *orbital integral* at ℓ is defined by

$$O_\ell(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \sum_{t \in \mathbb{Q}_\ell^\times \setminus T(\mathbb{Q}_\ell)/U_\ell} \mathbf{1}_{L_\ell(\mathbf{E}_1, \mathbf{E}_2)}(t^{-1} \bullet j) \quad (2.11)$$

if there exists a $j \in V_\ell(\mathbf{E}_1, \mathbf{E}_2)$ satisfying $\deg_{\text{CM}}(j) = \alpha$. If no such j exists then set $O_\ell(\alpha, \mathbf{E}_1, \mathbf{E}_2) = 0$. As $T(\mathbb{Q}_\ell)$ acts transitively on the set of all $j \in V_\ell(\mathbf{E}_1, \mathbf{E}_2)$ for which $\deg_{\text{CM}}(j) = \alpha$, the orbital integral does not depend on the choice of j used in its definition. If $t \in T(\mathbb{A}_f)$ has image $(\mathbf{a}_1, \mathbf{a}_2) \in \Gamma$ under the isomorphism (2.9) then

$$\begin{aligned} O_\ell(\alpha, \mathbf{E}_1 \otimes \mathbf{a}_1, \mathbf{E}_2 \otimes \mathbf{a}_2) &= \sum_{s \in \mathbb{Q}_\ell^\times \setminus T(\mathbb{Q}_\ell)/U_\ell} \mathbf{1}_{L_\ell(\mathbf{E}_1 \otimes \mathbf{a}_1, \mathbf{E}_2 \otimes \mathbf{a}_2)}(s^{-1} \bullet j) \\ &= \sum_{s \in \mathbb{Q}_\ell^\times \setminus T(\mathbb{Q}_\ell)/U_\ell} \mathbf{1}_{t \bullet L_\ell(\mathbf{E}_1, \mathbf{E}_2)}(s^{-1} \bullet j) \\ &= O_\ell(\alpha, \mathbf{E}_1, \mathbf{E}_2), \end{aligned}$$

and so the orbital integral is constant on Γ -orbits.

Lemma 2.16. *Let k be an algebraically closed field and recall $\mathbf{w}_i = |\mathcal{O}_{K_i}^\times|$. Every $x \in [\mathcal{X}(k)]$ has trivial stabilizer in Γ and satisfies*

$$|\text{Aut}_{\mathcal{X}(k)}(x)| = \mathbf{w}_1 \mathbf{w}_2.$$

Proof. Suppose we have a $(\mathbf{a}_1, \mathbf{a}_2) \in \Gamma$ and a CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ defined over k with the property that

$$(\mathbf{E}_1, \mathbf{E}_2) \cong (\mathbf{E}_1 \otimes \mathbf{a}_1, \mathbf{E}_2 \otimes \mathbf{a}_2).$$

In particular there is an isomorphism of \mathcal{O}_{K_i} -modules

$$\mathrm{Hom}_{\mathcal{O}_{K_i}}(E_i, E_i) \cong \mathrm{Hom}_{\mathcal{O}_{K_i}}(E_i, E_i \otimes \mathfrak{a}_i)$$

and hence, by [1, Lemma 7.14],

$$\mathrm{End}_{\mathcal{O}_{K_i}}(E_i) \cong \mathrm{End}_{\mathcal{O}_{K_i}}(E_i) \otimes \mathfrak{a}_i.$$

Both as a ring and as an \mathcal{O}_{K_i} -module $\mathrm{End}_{\mathcal{O}_{K_i}}(E_i) \cong \mathcal{O}_{K_i}$, and so $\mathfrak{a}_i \cong \mathcal{O}_{K_i}$ as an \mathcal{O}_{K_i} -module. Thus \mathfrak{a}_i is a principal ideal. The equality $|\mathrm{Aut}_{\mathcal{X}(k)}(x)| = \mathbf{w}_1 \mathbf{w}_2$ is clear from $\mathrm{Aut}_{\mathcal{O}_{K_i}}(E_i) \cong \mathcal{O}_{K_i}^\times$. \square

Lemma 2.17. *We have the equalities*

$$|S(\mathbb{Q}) \cap V| = |(T(\mathbb{Q}) \cap U) / \{\pm 1\}| = \frac{\mathbf{w}_1 \mathbf{w}_2}{2}.$$

Proof. The relation $T(\mathbb{Q}) \cap U = \mathcal{O}_{K_1}^\times \times \mathcal{O}_{K_2}^\times$ implies the second equality in the statement of the lemma. By Proposition 2.13 the kernel of

$$T(\mathbb{Q}) \cap U \xrightarrow{\eta} S(\mathbb{Q}) \cap V$$

is $\mathbb{Q}^\times \cap U = \{\pm 1\}$, and so $|S(\mathbb{Q}) \cap V| \geq \mathbf{w}_1 \mathbf{w}_2 / 2$. On the other hand,

$$S(\mathbb{Q}) \cap V \subset S(\mathbb{Q}) \cap \widehat{\mathcal{O}}_K^\times = \mu_K$$

(the group of roots of unity in K), and so

$$|S(\mathbb{Q}) \cap V| \leq |\mu_K| = \frac{\mathbf{w}_1 \mathbf{w}_2}{2}.$$

\square

Proposition 2.18. *Suppose we are given a totally positive $\alpha \in F^\times$, a prime p nonsplit in both K_1 and K_2 , and a supersingular CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ over $\mathbb{F}_p^{\mathrm{alg}}$. Then*

$$\sum_{(\mathfrak{a}_1, \mathfrak{a}_2) \in \Gamma} |\{j \in L(\mathbf{E}_1 \otimes \mathfrak{a}_1, \mathbf{E}_2 \otimes \mathfrak{a}_2) : \deg_{\mathrm{CM}}(j) = \alpha\}| = \frac{\mathbf{w}_1 \mathbf{w}_2}{2} \prod_{\ell < \infty} O_\ell(\alpha, \mathbf{E}_1, \mathbf{E}_2).$$

Proof. Using the isomorphisms (2.8) and (2.9), the left hand side of the desired equality is equal to

$$\begin{aligned} & \sum_{(\mathfrak{a}_1, \mathfrak{a}_2) \in \Gamma} \sum_{\substack{j \in V(\mathbf{E}_1 \otimes \mathfrak{a}_1, \mathbf{E}_2 \otimes \mathfrak{a}_2) \\ \deg_{\mathrm{CM}}(j) = \alpha}} \mathbf{1}_{L(\mathbf{E}_1 \otimes \mathfrak{a}_1, \mathbf{E}_2 \otimes \mathfrak{a}_2)}(j) & (2.12) \\ &= \sum_{t \in T(\mathbb{Q}) \setminus T(\mathbb{A}_f) / U} \sum_{\substack{j \in V(\mathbf{E}_1, \mathbf{E}_2) \\ \deg_{\mathrm{CM}}(j) = \alpha}} \mathbf{1}_{t \bullet \widehat{L}(\mathbf{E}_1, \mathbf{E}_2)}(j) \\ &= \sum_{s \in S(\mathbb{Q}) \setminus S(\mathbb{A}_f) / V} \sum_{\substack{j \in V(\mathbf{E}_1, \mathbf{E}_2) \\ \deg_{\mathrm{CM}}(j) = \alpha}} \mathbf{1}_{s \bullet \widehat{L}(\mathbf{E}_1, \mathbf{E}_2)}(j). \end{aligned}$$

Let us assume there is some $j_0 \in V(\mathbf{E}_1, \mathbf{E}_2)$ for which $\deg_{\text{CM}}(j_0) = \alpha$. As the group $S(\mathbb{Q})$ acts simply transitively on the set of all such j_0 , the final expression in (2.12) may be rewritten as

$$\begin{aligned}
& \sum_{s \in S(\mathbb{Q}) \backslash S(\mathbb{A}_f) / V} \sum_{\substack{j \in V(\mathbf{E}_1, \mathbf{E}_2) \\ \deg_{\text{CM}}(j) = \alpha}} \mathbf{1}_{s \bullet \widehat{L}(\mathbf{E}_1, \mathbf{E}_2)}(j) \\
&= \sum_{s \in S(\mathbb{Q}) \backslash S(\mathbb{A}_f) / V} \sum_{\gamma \in S(\mathbb{Q})} \mathbf{1}_{s \bullet \widehat{L}(\mathbf{E}_1, \mathbf{E}_2)}(\gamma^{-1} j_0) \\
&= |S(\mathbb{Q}) \cap V| \sum_{s \in S(\mathbb{A}_f) / V} \mathbf{1}_{s \bullet \widehat{L}(\mathbf{E}_1, \mathbf{E}_2)}(j_0) \\
&= \frac{\mathbf{w}_1 \mathbf{w}_2}{2} \prod_{\ell} O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2). \tag{2.13}
\end{aligned}$$

In the final equality we have used Proposition 2.13 and Lemma 2.17. If no such j_0 exists then (by the Hasse-Minkowski Theorem) both the first and last expression in (2.13) vanish. \square

2.5 Calculation of orbital integrals

Fix a prime p that is nonsplit in both K_1 and K_2 , and a supersingular CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ over $\mathbb{F}_p^{\text{alg}}$. In this subsection we will evaluate the local orbital integral of (2.11) for every prime ℓ , and so obtain an explicit formula for the left hand side of the equality of Proposition 2.18.

Let $\mathfrak{p} \subset \mathcal{O}_F$ be the reflex prime of $(\mathbf{E}_1, \mathbf{E}_2)$ in the sense of Definition 2.8. Fix a finite idele $\beta \in \widehat{F}^{\times}$ such that $\beta \widehat{\mathcal{O}}_F = \mathfrak{p} \mathfrak{D}^{-1} \widehat{\mathcal{O}}_F$. As in the introduction, for any prime ℓ and any fractional $\mathcal{O}_{F, \ell}$ -ideal \mathfrak{b} we define $\rho_{\ell}(\mathfrak{b})$ to be the number of ideals $\mathfrak{B} \subset \mathcal{O}_{K, \ell}$ for which $\text{Nm}_{K/F}(\mathfrak{B}) = \mathfrak{b}$.

Lemma 2.19. *If $\alpha \in F_{\ell}^{\times}$ and $\ell \neq p$ then*

$$O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \rho_{\ell}(\alpha \mathfrak{D}_{\ell}).$$

Proof. Fix an isomorphism

$$(V_{\ell}(\mathbf{E}_1, \mathbf{E}_2), \deg_{\text{CM}}) \cong (K_{\ell}, \beta_{\ell} \cdot \text{Nm}_{K_{\ell}/F_{\ell}})$$

as in Lemma 2.10. Proposition 2.13 implies that $(t_1, t_2) \mapsto \nu(t)^{-1}(t_1 \otimes t_2)$ defines an isomorphism

$$\mathbb{Q}_{\ell}^{\times} \backslash T(\mathbb{Q}_{\ell}) / U_{\ell} \rightarrow S(\mathbb{Q}_{\ell}) / V_{\ell} \tag{2.14}$$

which allows us to rewrite the orbital integral (2.11) as

$$O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \sum_{s \in S(\mathbb{Q}_{\ell}) / V_{\ell}} \mathbf{1}_{\mathcal{O}_{K, \ell}}(s^{-1} \phi) \tag{2.15}$$

where $\phi \in K_{\ell}$ satisfies $\text{Nm}_{K_{\ell}/F_{\ell}}(\phi) = \alpha \beta_{\ell}^{-1}$. If no such ϕ exists then of course the orbital integral vanishes.

Suppose first that ℓ is inert in both K_1 and K_2 , so that

$$\mathcal{O}_{K,\ell} \cong \mathbb{Z}_{\ell^2} \times \mathbb{Z}_{\ell^2} \quad \mathcal{O}_{F,\ell} \cong \mathbb{Z}_{\ell} \times \mathbb{Z}_{\ell}.$$

In this case $\mathbb{Q}_{\ell}^{\times} \backslash T(\mathbb{Q}_{\ell})/U_{\ell} = \{1\}$ and (2.15) shows that $O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = 1$ if there is a $\phi \in K_{\ell}$ satisfying $\text{Nm}_{K_{\ell}/F_{\ell}}(\phi) = \alpha\beta_{\ell}^{-1}$. Otherwise $O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = 0$. It follows that $O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \rho_{\ell}(\alpha\mathfrak{D}_{\ell})$ as both sides are equal to 1 if $\text{ord}_w(\alpha\beta_{\ell}^{-1})$ is even and nonnegative for both places w of F above ℓ , and otherwise both sides are zero.

Suppose next that ℓ is inert in K_1 and is ramified in K_2 . Then $F_{\ell}/\mathbb{Q}_{\ell}$ is a ramified field extension and K_{ℓ}/F_{ℓ} is an unramified field extension. Again one has $\mathbb{Q}_{\ell}^{\times} \backslash T(\mathbb{Q}_{\ell})/U_{\ell} = \{1\}$ and (2.15) shows that $O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = 1$ if there is a $\phi \in K_{\ell}$ satisfying $\text{Nm}_{K_{\ell}/F_{\ell}}(\phi) = \alpha\beta_{\ell}^{-1}$. Otherwise $O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = 0$. It follows that $O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \rho_{\ell}(\alpha\mathfrak{D}_{\ell})$, as both sides are equal to 1 if $\text{ord}_w(\alpha\beta_{\ell}^{-1})$ is even and nonnegative for the unique place w of F above ℓ , and otherwise both sides are zero. The case of ℓ ramified in K_1 and inert in K_2 is identical.

Suppose next that ℓ is split in K_1 and nonsplit in K_2 . Fix an isomorphism $\mathcal{O}_{K_1,\ell} \cong \mathbb{Z}_{\ell} \times \mathbb{Z}_{\ell}$ and a uniformizer $\varpi \in \mathcal{O}_{K_2,\ell}$. Let σ be the nontrivial Galois automorphism of $K_{2,\ell}$ and define

$$t_1 = (1, \text{Nm}_{K_{2,\ell}/\mathbb{Q}_{\ell}}(\varpi)) \in K_{1,\ell}^{\times} \quad t_2 = \varpi^{\sigma} \in K_{2,\ell}^{\times}.$$

Then $\mathbb{Q}_{\ell}^{\times} \backslash T(\mathbb{Q}_{\ell})/U_{\ell}$ is the infinite cyclic group generated by $t = (t_1, t_2)$. If we identify

$$K_{\ell} \cong K_{1,\ell} \otimes_{\mathbb{Q}_{\ell}} K_{2,\ell} \cong K_{2,\ell} \times K_{2,\ell}$$

using $(x_1, x_2) \otimes y \mapsto (x_1y, x_2y^{\sigma})$ then

$$F_{\ell} \cong \{(a, b) \in K_{2,\ell} \times K_{2,\ell} : a = b\} \quad (2.16)$$

and

$$S(\mathbb{Q}_{\ell}) \cong \{(a, b) \in K_{2,\ell}^{\times} \times K_{2,\ell}^{\times} : ab = 1\}.$$

Using the isomorphism (2.14) and the above generator $t \in \mathbb{Q}_{\ell}^{\times} \backslash T(\mathbb{Q}_{\ell})/U_{\ell}$ we find that $S(\mathbb{Q}_{\ell})/V_{\ell}$ is the infinite cyclic group generated by (ϖ, ϖ^{-1}) . It now follows from (2.15) that

$$O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \sum_{i=-\infty}^{\infty} \mathbf{1}_{\mathcal{O}_{K_2,\ell}}(\varpi^i \phi_1) \cdot \mathbf{1}_{\mathcal{O}_{K_2,\ell}}(\varpi^{-i} \phi_2)$$

where $(\phi_1, \phi_2) \in K_{2,\ell}^{\times} \times K_{2,\ell}^{\times}$ is any element that satisfies

$$(\phi_1 \phi_2, \phi_1 \phi_2) = \alpha\beta_{\ell}^{-1}$$

under the identification (2.16). If we let w be the unique place of F above ℓ then $O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \rho_{\ell}(\alpha\mathfrak{D}_{\ell})$ as both sides are $1 + \text{ord}_w(\alpha\mathfrak{D}_{\ell})$ if $\text{ord}_w(\alpha\mathfrak{D}_{\ell}) \geq 0$, and otherwise both sides are zero. The case of ℓ nonsplit in K_1 and split in K_2 is identical.

Finally suppose that ℓ is split in both K_1 and K_2 and fix isomorphisms

$$K_1 \cong \mathbb{Q}_\ell \times \mathbb{Q}_\ell \quad K_2 \cong \mathbb{Q}_\ell \times \mathbb{Q}_\ell.$$

Define

$$\rho_{i,j} = (\ell^i, \ell^j) \in \mathbb{Q}_\ell^\times \times \mathbb{Q}_\ell^\times.$$

The group $\mathbb{Q}_\ell^\times \backslash T(\mathbb{Q}_\ell) / U_\ell$ is then isomorphic to the quotient of

$$\{(\rho_{a,b}, \rho_{c,d}) \in K_{1,\ell}^\times \times K_{2,\ell}^\times : a + b = c + d\}$$

by the subgroup $\{(\rho_{a,b}, \rho_{c,d}) \in K_{1,\ell}^\times \times K_{2,\ell}^\times : a = b = c = d\}$. If we identify

$$\mathcal{O}_{K,\ell} \cong \mathcal{O}_{K_1,\ell} \otimes \mathcal{O}_{K_2,\ell} \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell \times \mathbb{Z}_\ell \times \mathbb{Z}_\ell \quad (2.17)$$

via $(x_1, x_2) \otimes (y_1, y_2) \mapsto (x_1 y_1, x_2 y_2, x_1 y_2, x_2 y_1)$ then

$$\mathcal{O}_{F,\ell} = \{(z_1, z_2, z_3, z_4) \in \mathbb{Z}_\ell \times \mathbb{Z}_\ell \times \mathbb{Z}_\ell \times \mathbb{Z}_\ell : z_1 = z_2, z_3 = z_4\}$$

and

$$S(\mathbb{Q}_\ell) \cong \{(z_1, z_2, z_3, z_4) \in \mathbb{Z}_\ell \times \mathbb{Z}_\ell \times \mathbb{Z}_\ell \times \mathbb{Z}_\ell : z_1 z_2 = 1, z_3 z_4 = 1\}.$$

The isomorphism (2.14) takes $(\rho_{a,b}, \rho_{c,d})$ to the quadruple $(\ell^i, \ell^{-i}, \ell^j, \ell^{-j}) \in S(\mathbb{Q}_\ell)$ where $i = c - b = a - d$ and $j = d - b = a - c$, and a complete set of coset representatives for $S(\mathbb{Q}_\ell) / V_\ell$ is given by the set $\{(\ell^i, \ell^{-i}, \ell^j, \ell^{-j}) : i, j \in \mathbb{Z}\}$. It now follows from (2.15) that

$$O_\ell(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \sum_{-\infty < i, j < \infty} \mathbf{1}_{\mathbb{Z}_\ell}(\ell^i \phi_1) \cdot \mathbf{1}_{\mathbb{Z}_\ell}(\ell^{-i} \phi_2) \cdot \mathbf{1}_{\mathbb{Z}_\ell}(\ell^j \phi_3) \cdot \mathbf{1}_{\mathbb{Z}_\ell}(\ell^{-j} \phi_4)$$

where $(\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbb{Z}_\ell \times \mathbb{Z}_\ell \times \mathbb{Z}_\ell \times \mathbb{Z}_\ell \cong \mathcal{O}_{F,\ell}$ satisfies

$$(\phi_1 \phi_2, \phi_1 \phi_2, \phi_3 \phi_4, \phi_3 \phi_4) = \alpha \beta_\ell^{-1}$$

under (2.17). If we let w_1, w_2 be the two places of F above ℓ then $O_\ell(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \rho_\ell(\alpha \mathfrak{D}_\ell)$ as both sides are

$$(1 + \text{ord}_{w_1}(\alpha \mathfrak{D}_\ell))(1 + \text{ord}_{w_2}(\alpha \mathfrak{D}_\ell))$$

if $\text{ord}_{w_1}(\alpha \mathfrak{D}_\ell) \geq 0$ and $\text{ord}_{w_2}(\alpha \mathfrak{D}_\ell) \geq 0$, and otherwise both sides are zero. \square

Lemma 2.20. *For any $\alpha \in F_p^\times$*

$$O_p(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \rho_p(\alpha \mathfrak{p}^{-1} \mathfrak{D}_p).$$

Proof. As p is unramified in at least one of K_1 or K_2 , it is easy to see that

$$\mathbb{Q}_p^\times \backslash T(\mathbb{Q}_p) / U_p = \{1\}.$$

Thus the orbital integral $O_p(\alpha, \mathbf{E}_1, \mathbf{E}_2)$ is 1 if there is a $j \in L_p(\mathbf{E}_1, \mathbf{E}_2)$ satisfying $\deg_{\text{CM}}(j) = \alpha$ and is 0 otherwise. Using the model of Lemma 2.11 we see that $O_p(\alpha, \mathbf{E}_1, \mathbf{E}_2) = 1$ if and only if there is a $j \in \mathcal{O}_{K,p}$ satisfying

$$\text{Nm}_{K_p/F_p}(j) = \alpha\beta_p^{-1}.$$

As each prime of F above p is inert in K , such a j exists if and only if $\text{ord}_w(\alpha\beta_p^{-1})$ is even and nonnegative for each place w of F above p . Using

$$\text{ord}_w(\alpha\beta_p^{-1}) = \text{ord}_w(\alpha\mathfrak{p}^{-1}\mathfrak{D}_p)$$

we find that both sides of the desired equality are 1 if $\text{ord}_w(\alpha\mathfrak{p}^{-1}\mathfrak{D}_p)$ is even and nonnegative for both places w of F above p , and otherwise both sides of the equality are 0. \square

Recall from the introduction that for any fractional \mathcal{O}_F -ideal \mathfrak{b} , $\rho(\mathfrak{b})$ is defined to be the number of ideals $\mathfrak{B} \subset \mathcal{O}_K$ for which $\text{Nm}_{K/F}(\mathfrak{B}) = \mathfrak{b}$.

Theorem 2.21. *For any $\alpha \in F^\times$ we have*

$$\prod_{\ell} O_{\ell}(\alpha, \mathbf{E}_1, \mathbf{E}_2) = \rho(\alpha\mathfrak{D}\mathfrak{p}^{-1}).$$

Proof. This is clear from Lemmas 2.19 and 2.20, and the product formula (1.2). \square

2.6 Deformation theory

Let p be a prime that is nonsplit in both K_1 and K_2 . This implies that all CM pairs over $\mathbb{F}_p^{\text{alg}}$ are supersingular. Let $W = W(\mathbb{F}_p^{\text{alg}})$ be the ring of Witt vectors of $\mathbb{F}_p^{\text{alg}}$, and let $\mathbb{Z}_{p^2} \subset W$ be the ring of Witt vectors of \mathbb{F}_{p^2} . Denote by \mathcal{CLN} be the category of complete local Noetherian W -algebras with residue field $\mathbb{F}_p^{\text{alg}}$. Let \mathfrak{g} be the unique (up to isomorphism) connected p -divisible group of height 2 and dimension 1 over $\mathbb{F}_p^{\text{alg}}$, and set $\Delta = \text{End}(\mathfrak{g})$. Thus \mathfrak{g} is isomorphic to the p -divisible group of any supersingular elliptic curve over $\mathbb{F}_p^{\text{alg}}$, and Δ is the maximal order in a quaternion division algebra over \mathbb{Q}_p . Let $x \mapsto x^t$ denote the main involution on Δ , let $\mathfrak{m}_{\Delta} \subset \Delta$ be the maximal ideal, and let ord_{Δ} be the valuation on Δ defined by

$$\text{ord}_{\Delta}(j) = k \iff j \in \mathfrak{m}_{\Delta}^k \setminus \mathfrak{m}_{\Delta}^{k+1}.$$

For any \mathbb{Z}_p -subalgebra $\mathcal{O} \subset \Delta$ denote by $\text{Def}(\mathfrak{g}, \mathcal{O})$ the functor from \mathcal{CLN} to the category of sets that assigns to an object R of \mathcal{CLN} the set of isomorphism classes of deformations of \mathfrak{g} , with its \mathcal{O} -action, to R . Suppose that $\mathcal{O} = \mathcal{O}_L$ is the maximal order in a quadratic field extension of \mathbb{Q}_p and let $\pi_L \in \mathcal{O}_L$ be a uniformizer. Let \mathcal{W}_L be the completion of the ring of integers of the maximal unramified extension of L , and choose a continuous ring homomorphism $W \rightarrow$

\mathcal{W}_L . By Lubin-Tate theory (see for example [17, Theorem 3.8]) the deformation functor $\text{Def}(\mathfrak{g}, \mathcal{O}_L)$ is represented by \mathcal{W}_L , so that

$$\text{Def}(\mathfrak{g}, \mathcal{O}_L)(R) \cong \text{Hom}_{\mathcal{C}\mathcal{L}\mathcal{N}}(\mathcal{W}_L, R).$$

Let \mathfrak{G} be the universal deformation of \mathfrak{g} , with its \mathcal{O}_L -action, to \mathcal{W}_L , and let \mathfrak{G}_k be the reduction of \mathfrak{G} to $\mathcal{W}_L/\pi_L^k \mathcal{W}_L$. It is a result of Gross (see [3, Proposition 3.3] or [20, Theorem 1.4]) that the reduction map $\text{End}(\mathfrak{G}_k) \rightarrow \text{End}(\mathfrak{g})$ identifies

$$\text{End}(\mathfrak{G}_k) \cong \mathcal{O}_L + \pi_L^{k-1} \Delta. \quad (2.18)$$

Given any $j \in \Delta \setminus \mathcal{O}_L$, it follows from [14, Proposition 2.9] that the functor $\text{Def}(\mathfrak{g}, \mathcal{O}_L[j])$ is represented by $\mathcal{W}_L/\pi_L^k \mathcal{W}_L$ where k is the largest integer such that j lifts (necessarily uniquely) to an endomorphism of \mathfrak{G}_k .

Given a CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ over $\mathbb{F}_p^{\text{alg}}$ let $\text{Def}(\mathbf{E}_1, \mathbf{E}_2)$ be the functor that assigns to every object R of $\mathcal{C}\mathcal{L}\mathcal{N}$ the set of isomorphism classes of deformations of the pair $(\mathbf{E}_1, \mathbf{E}_2)$ to R . Fix isomorphisms of p -divisible groups

$$E_1[p^\infty] \cong \mathfrak{g} \cong E_2[p^\infty].$$

Let $\mathcal{O}_{L_i} \subset \Delta$ be the image of $\kappa_i : \mathcal{O}_{K_{i,p}} \rightarrow \Delta$ and let $L_i \cong K_{i,p}$ be the fraction field of \mathcal{O}_{L_i} . It follows from the Serre-Tate theorem and the comments of the preceding paragraph that the functor

$$\text{Def}(\mathbf{E}_1, \mathbf{E}_2) \cong \text{Def}(\mathfrak{g}, \mathcal{O}_{L_1}) \times \text{Def}(\mathfrak{g}, \mathcal{O}_{L_2}),$$

is represented by $\mathcal{W}_{L_1} \widehat{\otimes}_W \mathcal{W}_{L_2}$.

Proposition 2.22. *Let \mathfrak{p} be a prime of F above p . There are $2 \cdot |\Gamma|$ isomorphism classes of CM pairs over $\mathbb{F}_p^{\text{alg}}$ having reflex prime \mathfrak{p} .*

Proof. Let $(\mathbf{E}_1, \mathbf{E}_2)$ be a CM pair over $\mathbb{F}_p^{\text{alg}}$. Suppose first that p is unramified in both K_1 and K_2 , so that $\text{Def}(\mathbf{E}_1, \mathbf{E}_2)$ is represented by $W \widehat{\otimes}_W W \cong W$. If R is any object of $\mathcal{C}\mathcal{L}\mathcal{N}$ then

$$\text{Def}(\mathbf{E}_1, \mathbf{E}_2)(R) \cong \text{Hom}_{\mathcal{C}\mathcal{L}\mathcal{N}}(W, R)$$

consists of a single point. From this it follows easily that the pair $(\mathbf{E}_1, \mathbf{E}_2)$ admits a unique lift to characteristic 0, and that the reduction map

$$[\mathcal{X}(\mathbb{C}_p)] \rightarrow [\mathcal{X}(\mathbb{F}_p^{\text{alg}})]$$

is a bijection. As explained in Remark 2.15 there are $4 \cdot |\Gamma|$ isomorphism classes of CM pairs over \mathbb{C}_p , and hence also $4 \cdot |\Gamma|$ isomorphism classes of CM pairs over $\mathbb{F}_p^{\text{alg}}$. If $(\mathbf{E}_1, \mathbf{E}_2)$ has reflex prime \mathfrak{p} then we may consider the pair $(\mathbf{E}_1, \mathbf{E}'_2)$ obtained by replacing $\mathbf{E}_2 = (E_2, \kappa_2)$ by the pair $\mathbf{E}'_2 = (E_2, \kappa'_2)$, where $\kappa'_2(x) = \kappa_2(\bar{x})$. It is easy to check that the CM pair $(\mathbf{E}_1, \mathbf{E}'_2)$ has reflex prime $\mathfrak{p}' \neq \mathfrak{p}$, and hence half of the $4 \cdot |\Gamma|$ isomorphism classes of supersingular CM pairs have reflex prime \mathfrak{p} , and half have reflex prime \mathfrak{p}' .

Now suppose that p is ramified in one of K_1 or K_2 , and for simplicity suppose it is K_1 . Then $\text{Def}(\mathbf{E}_1, \mathbf{E}_2)$ represented by $\mathcal{W}_{L_1} \widehat{\otimes}_W \mathcal{W}_{L_2} \cong \mathcal{W}_{L_1}$, and so for any object R of $\mathcal{C}\mathcal{L}\mathcal{N}$

$$\text{Def}(\mathbf{E}_1, \mathbf{E}_2)(R) \cong \text{Hom}_{\mathcal{C}\mathcal{L}\mathcal{N}}(\mathcal{W}_{L_1}, R).$$

Assuming that R is large enough to contain a subring isomorphic to \mathcal{W}_{L_1} , the set on the right consists of two points (which are interchanged by pre-composing with the nontrivial element of $\text{Aut}(\mathcal{W}_{L_1}/W)$). It follows from this that $(\mathbf{E}_1, \mathbf{E}_2)$ admits precisely two nonisomorphic deformations to characteristic 0, and that the reduction map

$$[\mathcal{X}(\mathbb{C}_p)] \rightarrow [\mathcal{X}(\mathbb{F}_p^{\text{alg}})]$$

is two-to-one. As in the preceding paragraph, there are $4 \cdot |\Gamma|$ isomorphism classes of CM pairs over \mathbb{C}_p , and therefore there are half as many over $\mathbb{F}_p^{\text{alg}}$. As p is ramified in F ($p\mathcal{O}_F = \mathfrak{p}^2$), we find that there are $2 \cdot |\Gamma|$ isomorphism classes of CM pairs over $\mathbb{F}_p^{\text{alg}}$, all having reflex prime \mathfrak{p} . \square

Fix a CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ over $\mathbb{F}_p^{\text{alg}}$ and let \mathfrak{p} be the reflex prime of $(\mathbf{E}_1, \mathbf{E}_2)$. Given any nonzero $j \in L(\mathbf{E}_1, \mathbf{E}_2)$ let $\text{Def}(\mathbf{E}_1, \mathbf{E}_2, j)$ be the functor that assigns to every object R of $\mathcal{C}\mathcal{L}\mathcal{N}$ the set of isomorphism classes of deformations of $(\mathbf{E}_1, \mathbf{E}_2, j)$ to R . As above choose an isomorphism of p -divisible groups $E_i[p^\infty] \cong \mathfrak{g}$. Such choices determine an isomorphism

$$L_p(\mathbf{E}_1, \mathbf{E}_2) \cong \Delta$$

of \mathbb{Z}_p -modules, and allow us to view deg as a \mathbb{Z}_p -quadratic form on Δ . The quadratic form deg on Δ is a \mathbb{Z}_p^\times -multiple of the reduced norm.

Lemma 2.23. *If p is inert in both K_1 and K_2 then the deformation functor $\text{Def}(\mathbf{E}_1, \mathbf{E}_2, j)$ is represented by a local Artinian W -algebra of length*

$$\frac{\text{ord}_{\mathfrak{p}}(\text{deg}_{\text{CM}}(j)) + 1}{2}. \quad (2.19)$$

Proof. The isomorphisms $E_i[p^\infty] \cong \mathfrak{g}$ may be chosen so that $\kappa_1 : \mathcal{O}_{K_1, p} \rightarrow \Delta$ and $\kappa_2 : \mathcal{O}_{K_2, p} \rightarrow \Delta$ have the same image $\mathcal{O}_L \cong \mathbb{Z}_{p^2}$, and we may fix a uniformizing parameter $\Pi \in \Delta$ with the property $u\Pi = \Pi u^t$ for every $u \in \mathcal{O}_L$. There is then a decomposition of left \mathcal{O}_L -modules

$$\Delta = \Delta_+ \oplus \Delta_-$$

where $\Delta_+ = \mathcal{O}_L$ and $\Delta_- = \mathcal{O}_L\Pi$, and this decomposition is orthogonal with respect to the quadratic form deg on Δ . Now define $f_{\pm} : \mathcal{O}_{K, p} \rightarrow \mathcal{O}_L$ by

$$\begin{aligned} f_+(x_1 \otimes x_2) &= \kappa_2(x_2)\kappa_1(\bar{x}_1) \\ f_-(x_1 \otimes x_2) &= \kappa_2(x_2)\kappa_1(x_1) \end{aligned}$$

and denote by Ψ the isomorphism

$$\Psi = f_+ \times f_- : \mathcal{O}_{K, p} \rightarrow \mathcal{O}_L \times \mathcal{O}_L.$$

Let $\mathfrak{p}_- = \mathfrak{p}$ be the reflex prime of the pair $(\mathbf{E}_1, \mathbf{E}_2)$, and let \mathfrak{p}_+ be the other prime of F above p . The map f_{\pm} factors through the completion of $\mathcal{O}_{K,p}$ at the prime \mathfrak{p}_{\pm} . The action (2.1) of \mathcal{O}_K on Δ is by (for $j = j_+ + j_-$)

$$x \bullet j = f_+(x)j_+ + f_-(x)j_-$$

and the $\mathcal{O}_{F,p}$ -quadratic form \deg_{CM} on Δ takes the explicit form

$$\Psi(\deg_{\text{CM}}(j)) = (\deg(j_+), \deg(j_-)).$$

It follows that

$$\begin{aligned} \text{ord}_{\mathfrak{p}_+}(\deg_{\text{CM}}(j)) &= \text{ord}_p(\deg(j_+)) \\ \text{ord}_{\mathfrak{p}_-}(\deg_{\text{CM}}(j)) &= \text{ord}_p(\deg(j_-)). \end{aligned}$$

In particular for any positive $k \in \mathbb{Z}$ we have

$$\begin{aligned} j \in \mathcal{O}_L + p^{k-1}\Delta &\iff j_- \in p^{k-1}\mathcal{O}_L\Pi \\ &\iff \text{ord}_p(\deg(j_-)) \geq 2k - 1 \\ &\iff \frac{\text{ord}_{\mathfrak{p}}(\deg_{\text{CM}}(j)) + 1}{2} \geq k. \end{aligned}$$

The deformation functor $\text{Def}(\mathfrak{g}, \mathcal{O}_L)$ is represented by W , and hence

$$\text{Def}(\mathbf{E}_1, \mathbf{E}_2) \cong \text{Def}(\mathfrak{g}, \mathcal{O}_L) \times \text{Def}(\mathfrak{g}, \mathcal{O}_L)$$

is represented by $W \widehat{\otimes}_W W \cong W$. If we let \mathfrak{G} be the universal deformation of \mathfrak{g} with its \mathcal{O}_L -action then the p -divisible group of the universal deformation of $(\mathbf{E}_1, \mathbf{E}_2)$ is identified with $(\mathfrak{G}, \mathfrak{G})$, and the deformation functor of the triple $(\mathbf{E}_1, \mathbf{E}_2, j)$ is represented by $W/p^k W$ where k is the largest integer for which j lifts to an endomorphism of the reduction $\mathfrak{G} \otimes_W W/p^k W$. Combining (2.18) with the calculation of preceding paragraph, this k is given by the formula (2.19). \square

Lemma 2.24. *If p is ramified in one of K_1 or K_2 then the deformation functor $\text{Def}(\mathbf{E}_1, \mathbf{E}_2, j)$ is represented by a local Artinian W -algebra of length*

$$\frac{\text{ord}_{\mathfrak{p}}(\deg_{\text{CM}}(j)) + \text{ord}_{\mathfrak{p}}(\mathfrak{D}) + 1}{2}.$$

Proof. The prime p is ramified in one of K_1 or K_2 and is inert in the other. For simplicity let us assume that p is ramified in K_2 and inert in K_1 . Let \mathcal{O}_{L_i} be the image of $\kappa_i : \mathcal{O}_{K_i,p} \rightarrow \Delta$. If we choose a uniformizing parameter $\varpi \in \mathcal{O}_{K_2,p}$ then $\pi = 1 \otimes \varpi$ is a uniformizing parameter of $\mathcal{O}_{K,p}$ and $\Pi = \kappa_2(\varpi)$ is a uniformizing parameter of both \mathcal{O}_{L_2} and Δ . The action (2.1) of $\pi \in \mathcal{O}_{K,p}$ on $L_p(\mathbf{E}_1, \mathbf{E}_2) \cong \Delta$ is simply left multiplication by Π . Let $\beta \in \mathcal{O}_{F,p}$ be as in Lemma 2.11, and choose an $\mathcal{O}_{K,p}$ -linear isomorphism

$$(\mathcal{O}_{K,p}, \beta \cdot \text{Nm}_{K/F}) \cong (\Delta, \deg_{\text{CM}}).$$

As this isomorphism carries $\pi^k \mathcal{O}_{K,p}$ isomorphically to $\Pi^k \Delta$, if we view j as both an element of Δ and an element of $\mathcal{O}_{K,p}$ we have

$$\begin{aligned} \text{ord}_\Delta(j) &= \text{ord}_\mathfrak{p}(j) \\ &= \frac{\text{ord}_\mathfrak{p}(\text{Nm}_{K/F}(j))}{2} \\ &= \frac{\text{ord}_\mathfrak{p}(\beta \cdot \text{Nm}_{K/F}(j)) + \text{ord}_\mathfrak{p}(\mathfrak{D}) - 1}{2} \\ &= \frac{\text{ord}_\mathfrak{p}(\text{deg}_{\text{CM}}(j)) + \text{ord}_\mathfrak{p}(\mathfrak{D}) - 1}{2}. \end{aligned}$$

The functor $\text{Def}(\mathfrak{g}, \mathcal{O}_{L_i})$ is represented by \mathcal{W}_{L_i} . In particular $\text{Def}(\mathfrak{g}, \mathcal{O}_{L_1})$ is represented by W , and so \mathfrak{g} , with its \mathcal{O}_{L_1} -action, admits a unique deformation to \mathcal{W}_{L_2} corresponding to the unique element of $\text{Hom}_{\mathcal{C}\mathcal{L}\mathcal{N}}(W, \mathcal{W}_{L_2})$. Call this deformation $\mathfrak{G}^{(1)}$, let $\mathfrak{G}^{(2)}$ be the universal deformation of \mathfrak{g} to \mathcal{W}_{L_2} , and let $\mathfrak{G}_m^{(i)}$ be the reduction of $\mathfrak{G}^{(i)}$ to $\mathcal{W}_{L_2}/(\pi^m)$. The deformation functor $\text{Def}(\mathbf{E}_1, \mathbf{E}_2)$ is then represented by $W \widehat{\otimes}_W \mathcal{W}_{L_2} \cong \mathcal{W}_{L_2}$, the p -divisible group of the universal deformation of the pair $(\mathbf{E}_1, \mathbf{E}_2)$ is $(\mathfrak{G}^{(1)}, \mathfrak{G}^{(2)})$, and the functor $\text{Def}(\mathbf{E}_1, \mathbf{E}_2, j)$ is represented by $\mathcal{W}_{L_2}/(\pi^k)$ where k is the largest integer such that the homomorphism $j : \mathfrak{g} \rightarrow \mathfrak{g}$ lifts to a homomorphism $\mathfrak{G}_k^{(1)} \rightarrow \mathfrak{G}_k^{(2)}$.

Factor $j = \Pi^m \cdot u$ with $u \in \Delta^\times$ and $m = \text{ord}_\Delta(j)$. Suppose u lifts to a homomorphism $\mathfrak{G}_k^{(1)} \rightarrow \mathfrak{G}_k^{(2)}$. This lift is necessarily an isomorphism, and as $\Pi \in \mathcal{O}_{L_2}$ lifts to an endomorphism of $\mathfrak{G}^{(2)}$ we find that $u^{-1} \circ \Pi \circ u$ also lifts to an endomorphism of $\mathfrak{G}_k^{(1)}$. But \mathcal{O}_{L_1} and $u^{-1} \circ \Pi \circ u$ generate all of Δ as a \mathbb{Z}_p -algebra (as do any subalgebra isomorphic to \mathbb{Z}_{p^2} and uniformizer of Δ). Thus every element of Δ lifts to an endomorphism of $\mathfrak{G}_k^{(1)}$. It follows from (2.18) that the functor $\text{Def}(\mathfrak{g}, \Delta)$ is represented by $\mathbb{F}_p^{\text{alg}}$, and we deduce that $k = 1$. Thus u lifts to an endomorphism $\mathfrak{G}_1^{(1)} \rightarrow \mathfrak{G}_1^{(2)}$ but not to $\mathfrak{G}_2^{(1)} \rightarrow \mathfrak{G}_2^{(2)}$. It now follows immediately from [20, Proposition 5.2] that $j = \Pi^m \cdot u$ lifts to an endomorphism $\mathfrak{G}_{m+1}^{(1)} \rightarrow \mathfrak{G}_{m+1}^{(2)}$ but not to $\mathfrak{G}_{m+2}^{(1)} \rightarrow \mathfrak{G}_{m+2}^{(2)}$, and so the functor $\text{Def}(\mathbf{E}_1, \mathbf{E}_2, j)$ is represented by $\mathcal{W}_{L_2}/(\pi^{m+1})$ where

$$m + 1 = \text{ord}_\Delta(j) + 1 = \frac{\text{ord}_\mathfrak{p}(\text{deg}_{\text{CM}}(j)) + \text{ord}_\mathfrak{p}(\mathfrak{D}) + 1}{2}.$$

□

Let \mathcal{C} be any algebraic stack over $\text{Spec}(\mathbb{Z})$ and suppose $x \in \mathcal{C}(\mathbb{F}_p^{\text{alg}})$ is a geometric point. An *étale neighborhood* of x is a commutative diagram in the category of algebraic stacks

$$\begin{array}{ccc} & & U \\ & \nearrow \tilde{x} & \downarrow \\ \text{Spec}(\mathbb{F}_p^{\text{alg}}) & \xrightarrow{x} & \mathcal{C} \end{array} \quad (2.20)$$

in which U is a scheme and the vertical arrow is an étale morphism. The *strictly Henselian local ring* of \mathcal{C} at x is the direct limit

$$\mathcal{O}_{\mathcal{C},x}^{\text{sh}} = \varinjlim_{(U,\tilde{x})} \mathcal{O}_{U,\tilde{x}}$$

over all étale neighborhoods of x , where $\mathcal{O}_{U,\tilde{x}}$ is the usual local ring of the scheme U at (the image of) \tilde{x} . As the name suggests, $\mathcal{O}_{\mathcal{C},x}^{\text{sh}}$ is a strictly Henselian local ring, and has residue field $\mathbb{F}_p^{\text{alg}}$. If $T \subset W$ is a subring that is étale as a \mathbb{Z} -algebra then $U = \mathcal{C} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(T)$ is naturally an étale neighborhood of x , and so $\mathcal{O}_{\mathcal{C},x}^{\text{sh}}$ is a T -algebra. As the union of all such T is dense in W , the completed strictly Henselian local ring $\widehat{\mathcal{O}}_{\mathcal{C},x}^{\text{sh}}$ is naturally a W -algebra.

Proposition 2.25. *Suppose $\alpha \in F^\times$ satisfies $\text{Sppt}(\alpha) = \{p\}$. For any $x \in \mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})$ the strictly Henselian local ring of \mathcal{X}_α at x is Artinian of length*

$$\nu_{\mathfrak{p}}(\alpha) = \frac{\text{ord}_{\mathfrak{p}}(\alpha\mathfrak{D}) + 1}{2}$$

where \mathfrak{p} is the reflex prime of the pair $(\mathbf{E}_1, \mathbf{E}_2)$ underlying the triple $x = (\mathbf{E}_1, \mathbf{E}_2, j)$.

Proof. Suppose R is any local Artinian W -algebra with residue field $\mathbb{F}_p^{\text{alg}}$, and $z \in \text{Def}(\mathbf{E}_1, \mathbf{E}_2, j)(R)$. Then z determines a point $z \in \mathcal{X}_\alpha(R)$ whose image under the reduction map $\mathcal{X}_\alpha(R) \rightarrow \mathcal{X}_\alpha(\mathbb{F}_p)$ is x , and so we have a commutative diagram

$$\begin{array}{ccc} \text{Spec}(R) & & \\ \uparrow & \searrow z & \\ \text{Spec}(\mathbb{F}_p^{\text{alg}}) & \xrightarrow{x} & \mathcal{X}_\alpha \end{array}$$

Given an étale neighborhood (2.20) of x there is a unique $\tilde{z} : \text{Spec}(R) \rightarrow U$ making the diagram

$$\begin{array}{ccc} \text{Spec}(R) & \xrightarrow{\tilde{z}} & U \\ \uparrow & \searrow & \downarrow \\ \text{Spec}(\mathbb{F}_p^{\text{alg}}) & \xrightarrow{x} & \mathcal{X}_\alpha \end{array}$$

commute (the morphism $U \rightarrow \mathcal{X}_\alpha$ is formally étale). The morphism of schemes \tilde{z} induces a ring homomorphism $\tilde{z} : \mathcal{O}_{U,\tilde{x}} \rightarrow R$, and by varying the étale neighborhood we obtain a map $\tilde{z} : \mathcal{O}_{\mathcal{X}_\alpha,x}^{\text{sh}} \rightarrow R$ that induces the identity on residue fields. In particular \tilde{z} extends uniquely to $\tilde{z} \in \text{Hom}_{\mathcal{CLN}}(\widehat{\mathcal{O}}_{\mathcal{X}_\alpha,x}^{\text{sh}}, R)$. It is now easy to check that the construction $z \mapsto \tilde{z}$ establishes a bijection

$$\text{Def}(\mathbf{E}_1, \mathbf{E}_2, j)(R) \rightarrow \text{Hom}_{\mathcal{CLN}}(\widehat{\mathcal{O}}_{\mathcal{X}_\alpha,x}^{\text{sh}}, R)$$

for every Artinian R in \mathcal{CLN} , and that the functor $\text{Def}(\mathbf{E}_1, \mathbf{E}_2, j)$ is represented by the completed strictly Henselian local ring $\widehat{\mathcal{O}}_{\mathcal{X}_\alpha,x}^{\text{sh}}$. The claim is now immediate from Lemmas 2.23 and 2.24. \square

2.7 Final formula

Recall our notation: χ is the quadratic Hecke character associated to the extension K/F ; if $\alpha \in F^\times$ is totally positive then $\text{Diff}(\alpha)$ is the set of all finite primes \mathfrak{p} of F such that $\chi_{\mathfrak{p}}(\alpha\mathfrak{D}) = -1$; for a fractional \mathcal{O}_F -ideal \mathfrak{b} we denote by $\rho(\mathfrak{b})$ the number of ideals $\mathfrak{B} \subset \mathcal{O}_E$ satisfying $\text{Nm}_{K/F}(\mathfrak{B}) = \mathfrak{b}$.

Theorem 2.26. *Suppose $\alpha \in F^\times$. If $|\text{Diff}(\alpha)| > 1$ then $\mathcal{X}_\alpha = \emptyset$. If $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ and $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$ then \mathcal{X}_α is supported in characteristic p , the strictly Henselian local ring of every geometric point $x \in \mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})$ is Artinian of length*

$$\nu_{\mathfrak{p}}(\alpha) = \frac{1}{2} \cdot \text{ord}_{\mathfrak{p}}(\alpha\mathfrak{p}\mathfrak{D}),$$

and the CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ underlying the triple $x = (\mathbf{E}_1, \mathbf{E}_2, j)$ has reflex prime \mathfrak{p} .

Proof. Suppose $\mathcal{X}_\alpha \neq \emptyset$, and recall from Corollary 2.7 that \mathcal{X}_α is supported in a single nonzero characteristic p . Fix a triple $(\mathbf{E}_1, \mathbf{E}_2, j) \in \mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})$ and let \mathfrak{p} be the reflex prime of $(\mathbf{E}_1, \mathbf{E}_2)$. As p is nonsplit in both K_1 and K_2 it follows that \mathfrak{p} is inert in K , and so for every finite place \mathfrak{l} of F

$$\chi_{\mathfrak{l}}(\mathfrak{p}) = \begin{cases} -1 & \text{if } \mathfrak{l} = \mathfrak{p} \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand Theorem 2.12 implies that $(\widehat{K}, \beta x\bar{x})$ represents α for any $\beta \in \widehat{F}^\times$ satisfying $\beta\widehat{\mathcal{O}}_F = \mathfrak{p}\mathfrak{D}^{-1}\widehat{\mathcal{O}}_F$. This implies that $\chi_{\mathfrak{l}}(\alpha) = \chi_{\mathfrak{l}}(\mathfrak{p}\mathfrak{D}^{-1})$ for every finite place \mathfrak{l} of F , and we deduce that $\text{Diff}(\alpha) = \{\mathfrak{p}\}$. We have now shown that if $\mathcal{X}_\alpha \neq \emptyset$ and is supported in characteristic p then $\text{Diff}(\alpha)$ contains a single prime \mathfrak{p} . This prime \mathfrak{p} lies above p and is equal to the reflex prime of every triple $(\mathbf{E}_1, \mathbf{E}_2, j) \in \mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})$. The stated formula for the length of the strictly Henselian local ring of a geometric point is now just a restatement of Proposition 2.25. \square

Theorem 2.27. *Suppose $\alpha \in F^\times$ is totally positive. If $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ then*

$$\deg(\mathcal{X}_\alpha) = \frac{1}{2} \cdot \log(p) \cdot \text{ord}_{\mathfrak{p}}(\alpha\mathfrak{p}\mathfrak{D}) \cdot \rho(\alpha\mathfrak{D}\mathfrak{p}^{-1})$$

where $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$, and the left hand side is the Arakelov degree of (1.1).

Proof. Let $[\mathcal{X}(\mathbb{F}_p^{\text{alg}})]_{\mathfrak{p}}$ denote the subset of $[\mathcal{X}(\mathbb{F}_p^{\text{alg}})]$ consisting of isomorphism classes of supersingular CM pairs with reflex prime \mathfrak{p} . Combining Theorem 2.26

with Lemma 2.16 results in

$$\begin{aligned}
\deg(\mathcal{X}_\alpha) &= \log(p) \sum_{x \in [\mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})]} e_x^{-1} \cdot \text{length}(\mathcal{O}_{\mathcal{X}_\alpha, x}^{\text{sh}}) \\
&= \log(p) \cdot \nu_{\mathfrak{p}}(\alpha) \sum_{(\mathbf{E}_1, \mathbf{E}_2, j) \in [\mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})]} \frac{1}{|\text{Aut}(\mathbf{E}_1, \mathbf{E}_2, j)|} \\
&= \log(p) \cdot \nu_{\mathfrak{p}}(\alpha) \sum_{(\mathbf{E}_1, \mathbf{E}_2) \in [\mathcal{X}(\mathbb{F}_p^{\text{alg}})]_{\mathfrak{p}}} \sum_{\substack{j \in L(\mathbf{E}_1, \mathbf{E}_2) \\ \deg_{\text{CM}}(j) = \alpha}} \frac{1}{\mathbf{w}_1 \mathbf{w}_2}.
\end{aligned}$$

Now applying Proposition 2.18 and Theorem 2.21 to the final expression we find

$$\begin{aligned}
\deg(\mathcal{X}_\alpha) &= \log(p) \cdot \frac{\nu_{\mathfrak{p}}(\alpha)}{2 \cdot |\Gamma|} \sum_{(\mathbf{E}_1, \mathbf{E}_2) \in [\mathcal{X}(\mathbb{F}_p^{\text{alg}})]_{\mathfrak{p}}} \prod_{\ell < \infty} O_\ell(\alpha, \mathbf{E}_1, \mathbf{E}_2) \\
&= \log(p) \cdot \frac{\nu_{\mathfrak{p}}(\alpha)}{2 \cdot |\Gamma|} \sum_{(\mathbf{E}_1, \mathbf{E}_2) \in [\mathcal{X}(\mathbb{F}_p^{\text{alg}})]_{\mathfrak{p}}} \rho(\alpha \mathfrak{D} \mathfrak{p}^{-1}),
\end{aligned}$$

and applying Proposition 2.22 to this equality results in

$$\deg(\mathcal{X}_\alpha) = \log(p) \cdot \nu_{\mathfrak{p}}(\alpha) \cdot \rho(\alpha \mathfrak{D} \mathfrak{p}^{-1})$$

as desired. \square

If \mathfrak{b} is any fractional \mathcal{O}_F -ideal and p is a prime that is nonsplit in both K_1 and K_2 we set

$$f_p(\mathfrak{b}) = \sum_{\mathfrak{p}} \text{ord}_{\mathfrak{p}}(\mathfrak{b} \mathfrak{p}) \cdot \rho(\mathfrak{b} \mathfrak{p}^{-1})$$

where the sum is over the primes \mathfrak{p} of F above p . If p is a prime that splits in either K_1 or K_2 we set $f_p(\mathfrak{b}) = 0$. It is clear from the definition that $f_p(\mathfrak{b}) = 0$ unless $\mathfrak{b} \subset \mathcal{O}_F$.

Theorem 2.28. *For any totally positive $\alpha \in F^\times$ the stack \mathcal{X}_α has Arakelov degree*

$$\deg(\mathcal{X}_\alpha) = \frac{1}{2} \cdot \sum_p f_p(\alpha \mathfrak{D}) \log(p). \quad (2.21)$$

Proof. Suppose $\mathfrak{q} \in \text{Diff}(\alpha)$. Then \mathfrak{q} is inert in K and $\text{ord}_{\mathfrak{q}}(\alpha \mathfrak{D})$ is odd. If \mathfrak{p} is any prime of F that satisfies $\rho(\alpha \mathfrak{D} \mathfrak{p}^{-1}) \neq 0$, then $\text{ord}_{\mathfrak{q}}(\alpha \mathfrak{D} \mathfrak{p}^{-1})$ is even and so $\mathfrak{p} = \mathfrak{q}$. Thus

$$\frac{1}{2} \cdot \sum_p f_p(\alpha \mathfrak{D}) \log(p) = \frac{1}{2} \cdot \text{ord}_{\mathfrak{q}}(\alpha \mathfrak{D}) \rho(\alpha \mathfrak{D} \mathfrak{q}^{-1}) \log(q)$$

where $q\mathbb{Z} = \mathfrak{q} \cap \mathbb{Z}$. In particular, if $\text{Diff}(\alpha) = \{\mathfrak{q}\}$ then the desired equality follows from Theorem 2.27.

Now suppose $|\text{Diff}(\alpha)| > 1$. The claim is that both sides of (2.21) are equal to 0. The vanishing of the left hand side is precisely the final claim of Theorem 2.26. To see that the right hand side vanishes, suppose $\mathfrak{r}, \mathfrak{q} \in \text{Diff}(\alpha)$ with $\mathfrak{r} \neq \mathfrak{q}$. The hypothesis $\mathfrak{r} \in \text{Diff}(\alpha)$ implies that $\text{ord}_{\mathfrak{r}}(\alpha \mathfrak{D})$ is odd. Therefore $\text{ord}_{\mathfrak{r}}(\alpha \mathfrak{D} \mathfrak{q}^{-1})$ is also odd and so $\rho(\alpha \mathfrak{D} \mathfrak{q}^{-1}) = 0$. The calculation of the previous paragraph now shows that the right hand side of (2.21) is equal to 0. \square

Corollary 2.29 (Gross-Zagier). *For every $m \in \mathbb{Q}^\times$*

$$\deg(\mathcal{T}_m) = \frac{1}{2} \sum_{\substack{\alpha \in \mathfrak{D}^{-1} \\ \text{Tr}_{F/\mathbb{Q}}(\alpha) = m \\ \alpha \gg 0}} \sum_p f_p(\alpha \mathfrak{D}) \log(p).$$

Proof. This is immediate from Theorem 2.28 and the decomposition (2.2). \square

3 Eisenstein Series

Let $\psi_{\mathbb{Q}} = \prod_p \psi_{\mathbb{Q}_p}$ be the unique unramified additive character $\mathbb{Q} \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$ satisfying $\psi_{\mathbb{R}}(x) = e(x) = e^{2\pi i x}$, and let $\psi_F = \psi_{\mathbb{Q}} \circ \text{Tr}_{F/\mathbb{Q}}$. Denote by σ_1 and σ_2 the two real embeddings of F , let χ be the quadratic Hecke character of F associated to K , and let $\sqrt{D} \in F$ be a fixed square root of D , so that $\sqrt{D} \mathcal{O}_F = \mathfrak{D}$.

3.1 Incoherent quadratic spaces and Hecke's Eisenstein series

Let $W = K$ viewed as an F -quadratic space with the quadratic form $Q(x) = \frac{1}{\sqrt{D}} x \bar{x}$. Define a collection $\{\mathcal{C}_v\}$ of F_v -quadratic spaces, one for each place v of F , by taking $\mathcal{C}_{\mathfrak{p}} = W_{\mathfrak{p}}$ for every finite prime \mathfrak{p} of F , and taking \mathcal{C}_{σ_i} to be of signature $(2, 0)$ for the two infinite primes σ_1 and σ_2 of F . The product $\mathcal{C} = \prod_v \mathcal{C}_v$ is then an *incoherent* quadratic space over \mathbb{A}_F in the sense of [8, Definition 2.1]. For each place v of F let $\mathfrak{S}(\mathcal{C}_v)$ be the space of Schwartz functions on \mathcal{C}_v , and let $\mathfrak{S}(\mathcal{C}) = \otimes_v \mathfrak{S}(\mathcal{C}_v)$ be the space of Schwartz functions on \mathcal{C} . By means of the Weil representation $\omega = \omega_{\mathcal{C}, \psi_F}$ (see for example [7]), one has an $\text{SL}_2(\mathbb{A}_F)$ -equivariant map

$$\lambda : \mathfrak{S}(\mathcal{C}) \rightarrow I(0, \chi), \quad \lambda(\phi)(g) = \omega(g)\phi(0).$$

Here $I(s, \chi) = \text{Ind}_{B(\mathbb{A}_F)}^{\text{SL}_2(\mathbb{A}_F)}(\chi)$ is the induced representation of $\text{SL}_2(\mathbb{A}_F)$ consisting of smooth functions $\Phi(g, s)$ on $\text{SL}_2(\mathbb{A}_F)$ satisfying

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|_{\mathbb{A}}^{s+1} \Phi(g, s), \quad b \in \mathbb{A}_F, a \in \mathbb{A}_F^\times,$$

and

$$B = NM = \{n(b)m(a) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : b \in F, a \in F^\times\}$$

viewed as an algebraic group over F . We say $\Phi \in I(s, \chi)$ is *standard* if $\Phi(g, s)$ is independent of s for g in the maximal compact subgroup $\text{SL}_2(\widehat{\mathcal{O}}_F) \times \text{SO}_2(F_\infty)$.

We say Φ is factorizable if $\Phi = \otimes \Phi_v$ is the product of local sections. For a factorizable standard section $\Phi \in I(s, \chi)$ the Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in B(F) \backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s)$$

is absolutely convergent when $\mathrm{Re}(s)$ is sufficiently large, and has meromorphic continuation with a functional equation in $s \mapsto -s$. Moreover, it is holomorphic along the unitary axis $\mathrm{Re}(s) = 0$.

Remark 3.1. Following [8, Definition 5.1, (5.4)], we denote by $\mathrm{Diff}(\mathcal{C}, \alpha)$ the set of places v of F at which \mathcal{C}_v does not represent α . Then $\mathrm{Diff}(\mathcal{C}, \alpha) = \mathrm{Diff}(\alpha)$ for every totally positive $\alpha \in F$, where $\mathrm{Diff}(\alpha)$ is the set defined in the introduction.

Remark 3.2. The incoherent quadratic space \mathcal{C} is closely related to the quadratic spaces studied in Section 2. Let p be a prime that is nonsplit in K_1 and K_2 , let $(\mathbf{E}_1, \mathbf{E}_2)$ be a supersingular CM pair over $\mathbb{F}_p^{\mathrm{alg}}$, and let \mathfrak{p} be the reflex prime of $(\mathbf{E}_1, \mathbf{E}_2)$ in the sense of Definition 2.8. By Theorem 2.12, for every place v of F

$$(V(\mathbf{E}_1, \mathbf{E}_2) \otimes_F F_v, \mathrm{deg}_{\mathrm{CM}}) \cong \mathcal{C}_v \iff v \neq \mathfrak{p}$$

(see also Lemma 3.8 below).

For $\phi \in \mathfrak{S}(\mathcal{C})$ let $\Phi_\phi \in I(s, \chi)$ be the standard section associated to $\lambda(\phi)$, characterized by $\Phi_\phi(g, 0) = \lambda(\phi)$, and abbreviate

$$E(g, s, \phi) = E(g, s, \Phi_\phi).$$

We now choose a particular $\phi_v^+ \in \mathfrak{S}(\mathcal{C}_v)$ for every place v of F : for a finite prime \mathfrak{p} set $\phi_{\mathfrak{p}}^+ = \mathbf{1}_{\mathcal{O}_{K, \mathfrak{p}}}$, and for $l \in \{1, 2\}$ set $\phi_{\sigma_l}^+ = e^{-2\pi Q_{\sigma_l}(x)}$, where Q_{σ_l} is the quadratic form on \mathcal{C}_{σ_l} . We define $\phi^{\mathcal{C}} = \otimes_v \phi_v^+ \in \mathfrak{S}(\mathcal{C})$ and let

$$\Phi^{\mathcal{C}} = \otimes_v \Phi_v^+ \in I(s, \chi)$$

be the associated standard section, where Φ_v^+ is the standard section associated to ϕ_v^+ . The following is well-known.

Lemma 3.3.

1. For all s and $l \in \{1, 2\}$, $\Phi_{\sigma_l}^+(g, s)$ is the normalized eigenfunction of $\mathrm{SO}_2(F_{\sigma_l})$ of weight 1, i.e. $\Phi_{\sigma_l}^+(1, s) = 1$ and

$$\Phi_{\sigma_i}^+(gk_\theta, s) = \Phi_{\sigma_i}^+(g, s) \cdot e^{i\theta}, \quad k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

2. For all s and all finite primes \mathfrak{p} , $\Phi_{\mathfrak{p}}^+(g, s)$ is the spherical section in $I(s, \chi_{\mathfrak{p}})$, i.e. $\Phi_{\mathfrak{p}}^+(1, s) = 1$ and

$$\Phi_{\mathfrak{p}}^+(gk, s) = \Phi_{\mathfrak{p}}^+(g, s), \quad k \in \mathrm{SL}_2(\mathcal{O}_{F_{\mathfrak{p}}}).$$

Proof. Claim (1) follows from [16, Lemma 1.2]. For (2), let $\psi'_p(x) = \psi_p(\frac{1}{\sqrt{D}}x)$ so that ψ'_p is unramified. Let $V'_p = K_p$ with $Q'(x) = x\bar{x}$ for $x \in K_p$. Then

$$\omega_{V_p, \psi_p} = \omega_{V'_p, \psi'_p}.$$

Now (2) follows from [21, Proposition 2.1]. \square

Let \mathbb{H} be the complex upper half-plane. For $\tau = (\tau_1, \tau_2) \in \mathbb{H}^2$ write $\tau_l = u_l + iv_l$, set

$$g_{\tau_l} = n(u_l) \cdot m(\sqrt{v_l}) \in \mathrm{SL}_2(\mathbb{R}),$$

and view $g_\tau = (g_{\tau_1}, g_{\tau_2})$ as an element of

$$\mathrm{SL}_2(F_{\sigma_1}) \times \mathrm{SL}_2(F_{\sigma_2}) \subset \mathrm{SL}_2(\mathbb{A}_F).$$

Let

$$E^*(\tau, s, \phi^C) = \Lambda(s+1, \chi)E(\tau, s, \phi^C) = \Lambda(s+1, \chi)(v_1v_2)^{-\frac{1}{2}}E(g_\tau, s, \Phi^C) \quad (3.1)$$

be the normalized Eisenstein series of weight 1, where

$$\Lambda(s, \chi) = D^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s+1)^2L(s, \chi) = \Lambda(s, \chi_1)\Lambda(s, \chi_2).$$

Here χ_i is the quadratic Dirichlet character associated to K_i ,

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right),$$

and

$$\Lambda(s, \chi_i) = |d_i|^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s+1)L(s, \chi_i)$$

is the completed Dirichlet L -function of χ_i . This Eisenstein series is, up to scalar normalization, Hecke's famous Eisenstein series.

Proposition 3.4. For $\tau = (\tau_1, \tau_2) \in \mathbb{H}^2$,

$$E^*(\tau, s, \phi^C) = E^*(\tau, s)$$

where $E^*(\tau, s) = E^*(\tau_1, \tau_2, s)$ is the normalized Hecke Eisenstein series defined in the introduction.

Proof. Let $I(F)$ be the group of fractional ideals of F , and for any nonzero vector $(c, d) \in F^2$ define $I(c, d) = \mathcal{O}_F c + \mathcal{O}_F d \in I(F)$. Given $\gamma \in \mathrm{SL}_2(F)$ write

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \quad (3.2)$$

and define $f(\gamma) = I(c, d)$. Then [2, Section 1.3] implies that $f : \mathrm{SL}_2(F) \rightarrow I(F)$ induces a bijection

$$B(F) \backslash \mathrm{SL}_2(F) / \mathrm{SL}_2(\mathcal{O}_F) \rightarrow \mathrm{CL}(F). \quad (3.3)$$

For each ideal $\mathfrak{a} \in I(F)$ fix a $g_{\mathfrak{a}} \in \mathrm{SL}_2(F)$ satisfying $f(g_{\mathfrak{a}}) = \mathfrak{a}$. The bijectivity of (3.3) implies that the function

$$g_{\mathfrak{a}}\mathrm{SL}_2(\mathcal{O}_F) \rightarrow \{(c, d) \in \mathfrak{a}^2 : I(c, d) = \mathfrak{a}\}$$

defined by $\gamma \mapsto (c, d)$ induces a bijection

$$B(F) \backslash B(F)g_{\mathfrak{a}}\mathrm{SL}_2(\mathcal{O}_F) \rightarrow \{(c, d) \in \mathfrak{a}^2 : I(c, d) = \mathfrak{a}\} / \mathcal{O}_F^\times.$$

Now we are ready to prove the proposition using the usual unfolding technique. Recall from (3.1)

$$E(\tau, s, \phi^{\mathcal{C}}) = (v_1 v_2)^{-\frac{1}{2}} \sum_{\gamma \in B(F) \backslash \mathrm{SL}_2(F)} \Phi^{\mathcal{C}}(\gamma g_{\tau}, s).$$

For $\gamma \in \mathrm{SL}_2(F)$ written as (3.2), let $\widehat{\gamma}$ denote the image of γ in $\mathrm{SL}_2(\widehat{F})$, and let $\gamma_l = \sigma_l(\gamma)$ be the image of γ in $\mathrm{SL}_2(F_{\sigma_l}) = \mathrm{SL}_2(\mathbb{R})$. If we factor

$$\widehat{\gamma} = n(b)m(a)k$$

with $b \in \widehat{F}$, $a \in \widehat{F}^\times$, and $k \in \mathrm{SL}_2(\widehat{\mathcal{O}}_F)$ then $f(\gamma)$ is the ideal of F associated to a^{-1} , and

$$\widehat{\Phi}^{\mathcal{C}}(\widehat{\gamma}, s) = \chi(a)|a|^{s+1} = \chi(f(\gamma))N(f(\gamma))^{s+1}$$

by Lemma 3.3. Next, given $\tau_l = u_l + iv_l \in \mathbb{H}$ there is a $\tau'_l = u'_l + iv'_l \in \mathbb{H}$ such that

$$\gamma_l g_{\tau_l} = g_{\tau'_l} k_{\theta_l}, \quad k_{\theta_l} = \begin{pmatrix} \cos \theta_l & \sin \theta_l \\ -\sin \theta_l & \cos \theta_l \end{pmatrix}.$$

Writing $c_l = \sigma_l(c)$ and $d_l = \sigma_l(d)$ we have

$$v'_l = \frac{v_l}{|c_l \tau_l + d_l|^2} \quad e^{i\theta_l} = \frac{|c_l \tau_l + d_l|}{c_l \tau_l + d_l},$$

so Lemma 3.3 implies

$$\Phi_{\sigma_l}^+(\gamma_l g_{\tau_l}, s) = (v'_l)^{\frac{s+1}{2}} e^{i\theta_l} = \frac{v_l^{\frac{s+1}{2}}}{(c_l \tau_l + d_l)|c_l \tau_l + d_l|^s}.$$

It now follows that

$$\begin{aligned} E(\tau, s, \phi^{\mathcal{C}}) &= \sum_{[\mathfrak{a}] \in \mathrm{CL}(F)} \sum_{\gamma \in B(F) \backslash B(F)g_{\mathfrak{a}}\mathrm{SL}_2(\mathcal{O}_F)} (v_1 v_2)^{-\frac{1}{2}} \Phi^{\mathcal{C}}(\gamma g_{\tau}, s) \\ &= \sum_{[\mathfrak{a}] \in \mathrm{CL}(F)} \chi(\mathfrak{a}) N(\mathfrak{a})^{s+1} \sum_{\gamma \in B(F) \backslash B(F)g_{\mathfrak{a}}\mathrm{SL}_2(\mathcal{O}_F)} \frac{(v_1 v_2)^{\frac{s}{2}}}{(c\tau + d)|c\tau + d|^s} \\ &= \sum_{[\mathfrak{a}] \in \mathrm{CL}(F)} \chi(\mathfrak{a}) N(\mathfrak{a})^{s+1} \sum_{\substack{(c,d) \in \mathfrak{a}^2 / \mathcal{O}_F^\times \\ I(c,d) = \mathfrak{a}}} \frac{(v_1 v_2)^{\frac{s}{2}}}{(c\tau + d)|c\tau + d|^s}. \end{aligned}$$

On the other hand, for any $0 \neq (c, d) \in \mathfrak{a}^2$, one has $0 \neq I(c, d) \subset \mathfrak{a}$ and so

$$\begin{aligned}
& \sum_{[\mathfrak{a}] \in \text{CL}(F)} \chi(\mathfrak{a}) N(\mathfrak{a})^{s+1} \sum_{0 \neq (c, d) \in \mathfrak{a}^2 / \mathcal{O}_F^\times} \frac{(v_1 v_2)^{\frac{s}{2}}}{(c\tau + d)|c\tau + d|^s} \\
&= \sum_{[\mathfrak{a}] \in \text{CL}(F)} \sum_{\substack{\mathfrak{b} \subset \mathfrak{a} \\ \mathfrak{b} \neq 0}} \chi(\mathfrak{a}) N(\mathfrak{a})^{s+1} \sum_{\substack{(c, d) \in \mathfrak{b}^2 / \mathcal{O}_F^\times \\ I(c, d) = \mathfrak{b}}} \frac{(v_1 v_2)^{\frac{s}{2}}}{(c\tau + d)|c\tau + d|^s} \\
&= \sum_{[\mathfrak{b}] \in \text{CL}(F)} \chi(\mathfrak{b}) N(\mathfrak{b})^{s+1} \sum_{\substack{(c, d) \in \mathfrak{b}^2 / \mathcal{O}_F^\times \\ I(c, d) = \mathfrak{b}}} \frac{(v_1 v_2)^{\frac{s}{2}}}{(c\tau + d)|c\tau + d|^s} \sum_{\mathfrak{c} \subset \mathcal{O}_F} \chi(\mathfrak{c}) N(\mathfrak{c})^{-s-1}.
\end{aligned}$$

Therefore

$$\sum_{[\mathfrak{a}] \in \text{CL}(F)} \chi(\mathfrak{a}) N(\mathfrak{a})^{s+1} \sum_{0 \neq (c, d) \in \mathfrak{a}^2 / \mathcal{O}_F^\times} \frac{(v_1 v_2)^{\frac{s}{2}}}{(c\tau + d)|c\tau + d|^s} = L(s, \chi) E(\tau, s, \phi^{\mathcal{C}})$$

and the proposition is clear. \square

We need some more notation. One has the Fourier expansion:

$$E^*(\tau, s, \phi^{\mathcal{C}}) = \sum_{\alpha \in F} E_\alpha^*(\tau, s, \phi^{\mathcal{C}}),$$

where for $\alpha \neq 0$

$$E_\alpha^*(\tau, s, \phi^{\mathcal{C}}) = \prod_{\mathfrak{p} < \infty} W_{\alpha, \mathfrak{p}}^*(1, s, \phi_{\mathfrak{p}}^+) \prod_{l=1}^2 W_{\alpha, \sigma_l}^*(\tau_l, s, \phi_{\sigma_l}^+),$$

and

$$E_0^*(\tau, s, \phi^{\mathcal{C}}) = \Lambda(s+1, \chi) (v_1 v_2)^{\frac{s}{2}} + \prod_{\mathfrak{p} < \infty} W_{0, \mathfrak{p}}^*(1, s, \phi_{\mathfrak{p}}^+) \prod_{l=1}^2 W_{0, \sigma_l}^*(\tau_l, s, \phi_{\sigma_l}^+).$$

Here (for all $\alpha \in F$)

$$\begin{aligned}
W_{\alpha, \mathfrak{p}}^*(1, s, \phi_{\mathfrak{p}}^+) &= L(s+1, \chi_{\mathfrak{p}}) |D|_{\mathfrak{p}}^{-\frac{s+1}{2}} W_{\alpha, \mathfrak{p}}(1, s, \phi_{\mathfrak{p}}^+) \\
&= L(s+1, \chi_{\mathfrak{p}}) |D|_{\mathfrak{p}}^{-\frac{s+1}{2}} \int_{F_{\mathfrak{p}}} \Phi_{\mathfrak{p}}^+(wn(b), s) \psi_{\mathfrak{p}}(-\alpha_{\mathfrak{p}} b) db,
\end{aligned}$$

where db is the Haar measure on $F_{\mathfrak{p}}$ self-dual with respect to $\psi_{\mathfrak{p}}$, $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and

$$W_{\alpha, \sigma_l}^*(\tau_l, s, \phi_{\sigma_l}^+) = v_l^{-\frac{1}{2}} \Gamma_{\mathbb{R}}(s+2) W_{\alpha, \sigma_l}(g_{\tau_l}, s, \phi_{\sigma_l}^+)$$

with

$$W_{\alpha, \sigma_l}(g_{\tau_l}, s, \phi_{\sigma_l}^+) = \int_{\mathbb{R}} \Phi_{\sigma_l}^+(wn(b)g_{\tau_l}, s) \psi_{\sigma_l}(-\alpha_l b) db.$$

3.2 Explicit Calculations

We now record the results of [21, Proposition 2.1] and [21, Proposition 2.2] for the convenience of the reader.

Proposition 3.5. *Let \mathfrak{p} be a finite prime of F .*

1. *For all $\alpha \in F$*

$$W_{\alpha, \mathfrak{p}}^*(1, s, \phi_{\mathfrak{p}}^+) = |D|_{\mathfrak{p}}^{-\frac{s}{2}} \mathbf{1}_{\mathfrak{D}_{\mathfrak{p}}^{-1}}(\alpha) \rho_{\mathfrak{p}}(\alpha \mathfrak{D}, s).$$

Here $\mathfrak{D}_{\mathfrak{p}} = \mathfrak{D} \otimes_{\mathcal{O}_F} \mathcal{O}_{F, \mathfrak{p}}$, and

$$\rho_{\mathfrak{p}}(\mathfrak{a}, s) = \sum_{r=0}^{\text{ord}_{\mathfrak{p}}(\mathfrak{a})} (\chi_{\mathfrak{p}}(\mathfrak{p}) N(\mathfrak{p})^{-s})^r.$$

In particular, $\rho_{\mathfrak{p}}(\mathfrak{a}) = \rho_{\mathfrak{p}}(\mathfrak{a}, 0)$ and

$$W_{0, \mathfrak{p}}^*(1, s, \phi_{\mathfrak{p}}^+) = |D|_{\mathfrak{p}}^{-\frac{s}{2}} L(s, \chi_{\mathfrak{p}}).$$

2. *For all $\alpha \in \mathfrak{D}_{\mathfrak{p}}^{-1}$*

$$W_{\alpha, \mathfrak{p}}^*(1, 0, \phi_{\mathfrak{p}}^+) = \rho_{\mathfrak{p}}(\alpha \mathfrak{D}).$$

It is zero if and only if $\chi_{\mathfrak{p}}(\alpha \mathfrak{D}) = -1$, i.e. if and only if K/F is inert at \mathfrak{p} and $\text{ord}_{\mathfrak{p}}(\alpha \mathfrak{D})$ is odd. When this is the case

$$W_{\alpha, \mathfrak{p}}^{*,'}(1, 0, \phi_{\mathfrak{p}}^+) = -\frac{1}{2} \text{ord}_{\mathfrak{p}}(\alpha \mathfrak{p} \mathfrak{D}) \log N(\mathfrak{p}).$$

Proof. (sketch) Let $\psi'_{\mathfrak{p}}(x) = \psi_{\mathfrak{p}}(\frac{1}{\sqrt{D}}x)$. Then $\psi'_{\mathfrak{p}}$ is an unramified additive character of $F_{\mathfrak{p}}$, so

$$\begin{aligned} W_{\alpha, \mathfrak{p}}(g, s, \phi_{\mathfrak{p}}^+, \psi_{F_{\mathfrak{p}}}) &= \int_{F_{\mathfrak{p}}} \Phi_{\mathfrak{p}}^+(wn(b)g, s) \psi_{F_{\mathfrak{p}}}(-\alpha b) d_{\psi_{F_{\mathfrak{p}}}} b \\ &= |D|_{\mathfrak{p}}^{\frac{1}{2}} \int_{F_{\mathfrak{p}}} \Phi_{\mathfrak{p}}^+(wn(b)g, s) \psi'_{\mathfrak{p}}(-\alpha \sqrt{D}b) d_{\psi'_{\mathfrak{p}}} b \\ &= |D|_{\mathfrak{p}}^{\frac{1}{2}} W_{\alpha \sqrt{D}, \mathfrak{p}}(g, s, \phi_{\mathfrak{p}}^+, \psi'_{\mathfrak{p}}). \end{aligned}$$

Here we include the additive character in the notation to indicate the dependence of the Whittaker function on the additive character, and $d_{\psi}b$ is the Haar measure with respect to ψ . Now the proposition follows from [21, Proposition 2.1]. \square

Proposition 3.6. *Suppose $\tau = (\tau_1, \tau_2) \in \mathbb{H}^2$ and write $\tau_l = u_l + iv_l$.*

1. *One has*

$$W_{\alpha, \sigma_l}^*(\tau, 0, \phi_{\sigma_l}^+) = \begin{cases} -2ie(\sigma_l(\alpha)\tau_l) & \text{if } \sigma_l(\alpha) > 0, \\ -i & \text{if } \alpha = 0, \\ 0 & \text{if } \sigma_l(\alpha) < 0. \end{cases}$$

2. When $\sigma_l(\alpha) < 0$, one has

$$W_{\alpha, \sigma_l}^{*, \prime}(\tau, 0, \phi_{\sigma_l}^+) = -ie(\sigma_l(\alpha)\tau_l)\beta_1(4\pi|\sigma_l(\alpha)|v_l),$$

where

$$\beta_1(x) = \int_1^\infty e^{-ux} \frac{du}{u}, \quad x > 0$$

is a partial Gamma function.

3. One has

$$W_{0, \sigma_l}^*(\tau_l, s, \phi_{\sigma_l}^+) = v_l^{-\frac{s}{2}} \Gamma_{\mathbb{R}}(s).$$

Proof. See [11, Proposition 15.1]. □

Simple calculation using the above propositions gives the following theorem (see also [4, Page 215]).

Theorem 3.7. *One has $E^*(\tau, 0, \phi^{\mathcal{C}}) = 0$, and*

$$E^{*, \prime}(\tau, 0, \phi^{\mathcal{C}}) = \sum_{\alpha \in \mathfrak{D}^{-1}} a_\alpha(v_1, v_2) q^\alpha$$

where v_l is the imaginary part of τ_l , $q^\alpha = e(\sigma_1(\alpha)\tau_1 + \sigma_2(\alpha)\tau_2)$, and $a_\alpha(v_1, v_2)$ is as follows.

1. Assume α is totally positive, and recall that $\text{Diff}(\alpha)$ has odd cardinality. If $|\text{Diff}(\alpha)| > 1$, then $a_\alpha(v_1, v_2) = 0$. If $\text{Diff}(\alpha) = \{\mathfrak{p}\}$, then $a_\alpha = a_\alpha(v_1, v_2)$ is independent of v_1, v_2 and is equal to

$$a_\alpha = 2\text{ord}_{\mathfrak{p}}(\alpha\mathfrak{p}\mathfrak{D})\rho(\alpha\mathfrak{D}\mathfrak{p}^{-1}) \log N(\mathfrak{p}).$$

2. When $\sigma_k(\alpha) > 0 > \sigma_l(\alpha)$ with $\{k, l\} = \{1, 2\}$, one has

$$a_\alpha(v_1, v_2) = 2\rho(\alpha\mathfrak{D})\beta_1(4\pi|\sigma_l(\alpha)|v_l).$$

3. The constant term is

$$a_0(v_1, v_2) = 2\Lambda(0, \chi) \left(-\frac{\Lambda'(0, \chi)}{\Lambda(0, \chi)} + \frac{1}{2} \log(v_1 v_2) \right).$$

4. When α is totally negative, $a_\alpha(v_1, v_2) = 0$.

Proof. (sketch) Recall that for $\alpha \in F^\times$

$$E_\alpha^*(\tau, s, \phi^{\mathcal{C}}) = \prod_{\mathfrak{p} < \infty} W_{\alpha, \mathfrak{p}}^*(1, s, \phi_{\mathfrak{p}}^+) \prod_{l=1}^2 W_{\alpha, \sigma_l}^*(\tau_l, s, \phi_{\sigma_l}^+),$$

which is zero unless $\alpha \in \mathfrak{D}^{-1}$ by Proposition 3.5. Assume $\alpha \in \mathfrak{D}^{-1}$ is totally positive. Proposition 3.5 implies $W_{\alpha, \mathfrak{p}}^*(1, 0, \phi_{\mathfrak{p}}^+) = 0$ if $\mathfrak{p} \in \text{Diff}(\alpha)$, and so

$E_{\alpha}^{*,\prime}(\tau, 0, \phi^{\mathcal{C}}) = 0$ when $|\text{Diff}(\alpha)| > 1$. Still assuming that α is totally positive, when $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ one has, by Propositions 3.5 and 3.6,

$$\begin{aligned} E_{\alpha}^{*,\prime}(\tau, 0, \phi^{\mathcal{C}}) &= W_{\alpha, \mathfrak{p}}^{*,\prime}(1, 0, \phi_{\mathfrak{p}}^+) \prod_{\mathfrak{q} \neq \mathfrak{p}} W_{\alpha, \mathfrak{q}}^*(1, 0, \phi_{\mathfrak{q}}^+) \prod_{l=1}^2 W_{\alpha, \sigma_l}^*(\tau_l, 0, \phi_{\sigma_l}^+) \\ &= 2\text{ord}_{\mathfrak{p}}(\alpha \mathfrak{D}) \log N(\mathfrak{p}) \cdot \rho(\alpha \mathfrak{D} \mathfrak{p}^{-1}) \cdot q^{\alpha} \end{aligned}$$

as claimed. Here we have used the fact $\rho_{\mathfrak{p}}(\alpha \mathfrak{D} \mathfrak{p}^{-1}) = 1$ when $\text{Diff}(\alpha) = \{\mathfrak{p}\}$, and

$$\rho(\alpha \mathfrak{D} \mathfrak{p}^{-1}) = \prod_{\mathfrak{q} < \infty} \rho_{\mathfrak{q}}(\alpha \mathfrak{D} \mathfrak{p}^{-1}).$$

The other cases are similar and left to the reader. \square

Proof of Theorems B and C. Fix a totally positive $\alpha \in F^{\times}$. Assuming that $\alpha \in \mathfrak{D}^{-1}$ and $\text{Diff}(\alpha) = \{\mathfrak{p}\}$, Theorem B is just a restatement of the first claim of Theorem 3.7. Note we have used the fact that $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ implies that \mathfrak{p} is inert in K . This can only happen if the prime p of \mathbb{Q} below \mathfrak{p} is nonsplit in both K_1 and K_2 , which implies $N(\mathfrak{p}) = p$ by Remark 2.9. If $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ but $\alpha \notin \mathfrak{D}^{-1}$ then $\rho(\alpha \mathfrak{D} \mathfrak{p}^{-1}) = 0$, and so $a_{\alpha}(v_1, v_2) = 0$ by the first claim of Theorem 3.7. If $|\text{Diff}(\alpha)| > 1$ then, again by the first claim of Theorem 3.7, $a_{\alpha}(v_1, v_2) = 0$. This completes the proof of Theorem B. Theorem C follows immediately from Theorems A and B. \square

3.3 A conceptual proof of Theorem C

In this subsection we give a more conceptual proof of Theorem C which is based on the Siegel-Weil formula (as opposed to the explicit calculation of both sides of the stated equality).

For a finite prime \mathfrak{p} of F inert in K , let $W_{\mathfrak{p}}^{\pm}$ be the binary quadratic space $K_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$ with quadratic form

$$Q_{\mathfrak{p}}^+(x) = \frac{1}{\sqrt{D}} x \bar{x}, \quad Q_{\mathfrak{p}}^-(x) = \frac{\pi_{\mathfrak{p}}}{\sqrt{D}} x \bar{x},$$

where $\pi_{\mathfrak{p}}$ is a uniformizer of $F_{\mathfrak{p}}$. Notice that $\mathcal{C}_{\mathfrak{p}} \cong W_{\mathfrak{p}}^+$. Let $(W^{(\mathfrak{p})}, Q^{(\mathfrak{p})})$ be the global (totally positive definite) F -quadratic space obtained from \mathcal{C} by changing $\mathcal{C}_{\mathfrak{p}} = W_{\mathfrak{p}}^+$ to $W_{\mathfrak{p}}^-$ and leaving the other local quadratic spaces $\mathcal{C}_v = W_v^+$ unchanged.

Lemma 3.8.

1. Let $(\mathbf{E}_1, \mathbf{E}_2) \in [\mathcal{X}(\mathbb{F}_p^{\text{alg}})]$ be a supersingular CM pair, and let \mathfrak{p} be the reflex prime of $(\mathbf{E}_1, \mathbf{E}_2)$. Then there is an isomorphism of \mathbb{A}_F -quadratic spaces

$$(V(\mathbf{E}_1, \mathbf{E}_2) \otimes_F \mathbb{A}_F, \text{deg}_{\text{CM}}) \cong (W^{(\mathfrak{p})} \otimes_F \mathbb{A}_F, Q^{(\mathfrak{p})})$$

that maps $\widehat{L}(\mathbf{E}_1, \mathbf{E}_2)$ onto $\widehat{\mathcal{O}}_K$. In particular, there is an isomorphism of F -quadratic spaces

$$(V(\mathbf{E}_1, \mathbf{E}_2), \deg_{\text{CM}}) \cong (W^{(\mathfrak{p})}, Q^{(\mathfrak{p})}).$$

2. If $(\mathbf{E}_1, \mathbf{E}_2, j) \in [\mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})]$ then $\text{Diff}(\alpha) = \{\mathfrak{p}\}$, where \mathfrak{p} is the reflex prime of $(\mathbf{E}_1, \mathbf{E}_2)$. In particular, if $|\text{Diff}(\alpha)| > 1$, then \mathcal{X}_α is empty.

Proof. Part (1) follows from Theorem 2.12, together with the Hasse-Minkowski Theorem. Next, $(\mathbf{E}_1, \mathbf{E}_2, j) \in [\mathcal{X}_\alpha(\mathbb{F}_p^{\text{alg}})]$ implies that there is a $j \in L(\mathbf{E}_1, \mathbf{E}_2)$ with $\deg_{\text{CM}}(j) = \alpha$. By (1) there is $z \in \widehat{K}$ such that $Q^{(\mathfrak{p})}(z) = \alpha$. This implies that $\text{Diff}(\alpha) = \{\mathfrak{p}\}$. \square

Proposition 3.9. *Let \mathfrak{p} be the reflex prime of a supersingular CM pair $(\mathbf{E}_1, \mathbf{E}_2) \in \mathcal{X}(\mathbb{F}_p^{\text{alg}})$, and let*

$$\phi^{(\mathfrak{p})} = \mathbf{1}_{\widehat{\mathcal{O}}_K} \otimes \phi_{\sigma_1}^+ \otimes \phi_{\sigma_2}^+ \in \mathfrak{S}(W^{(\mathfrak{p})} \otimes_F \mathbb{A}_F).$$

Then for every totally positive $\alpha \in F$ one has

$$E_\alpha^*(\tau, 0, \phi^{(\mathfrak{p})})q^{-\alpha} = \frac{8}{\mathbf{w}_1 \mathbf{w}_2} \sum_{(\mathfrak{a}_1, \mathfrak{a}_2) \in \Gamma} \sum_{\substack{j \in L(\mathbf{E}_1 \otimes \mathfrak{a}_1, \mathbf{E}_2 \otimes \mathfrak{a}_2) \\ \deg_{\text{CM}}(j) = \alpha}} 1.$$

Here

$$E^*(\tau, s, \phi^{(\mathfrak{p})}) = \Lambda(s+1, \chi) E(\tau, s, \phi^{(\mathfrak{p})})$$

is a (non-holomorphic) Hilbert modular form of weight 1 defined as in (3.1).

We remark that the left hand side in the formula depends only on \mathfrak{p} , and not on the pair $(\mathbf{E}_1, \mathbf{E}_2)$.

Proof. As $W^{(\mathfrak{p})}$ is a K -vector space the algebraic group S defined in Section 2.4 acts on $W^{(\mathfrak{p})}$, and this action identifies $S \cong \text{Res}_{F/\mathbb{Q}} \text{SO}(W^{(\mathfrak{p})})$. Let

$$\theta(g, h, \phi^{(\mathfrak{p})}) = \sum_{j \in W^{(\mathfrak{p})}} \omega_{W^{(\mathfrak{p})}, \psi_F}(g) \phi^{(\mathfrak{p})}(h^{-1}j)$$

be the theta kernel; here $g \in \text{SL}_2(\mathbb{A}_F)$ and $h \in S(\mathbb{A})$. Let

$$\theta(g, \phi^{(\mathfrak{p})}) = \int_{[S]} \theta(g, h, \phi^{(\mathfrak{p})}) dh$$

be the associated theta integral, where $[S] = S(\mathbb{Q}) \backslash S(\mathbb{A})$, and dh is an $S(\mathbb{A})$ -invariant measure on $[S]$. Then

$$\theta(\tau, \phi^{(\mathfrak{p})}) = (v_1 v_2)^{-\frac{1}{2}} \theta(g_\tau, \phi^{(\mathfrak{p})})$$

is a holomorphic Hilbert modular form of weight 1. Moreover, the Siegel-Weil formula ([19], [9]) asserts

$$E(\tau, 0, \phi^{(\mathfrak{p})}) = C_1 \theta(\tau, \phi^{(\mathfrak{p})})$$

for some constant C_1 depending on the measure dh (in fact $C_1 = 2 \cdot \text{Vol}([S])^{-1}$, but we won't need this). Next, by Lemma 3.8, there is an isomorphism $\widehat{V}(\mathbf{E}_1, \mathbf{E}_2) \cong \widehat{W}^{(\mathfrak{p})}$ that maps $\widehat{L}(\mathbf{E}_1, \mathbf{E}_2)$ onto $\widehat{\mathcal{O}}_K$. It follows that

$$\int_{[S]} \sum_{\substack{j \in W^{(\mathfrak{p})} \\ Q^{(\mathfrak{p})}(j) = \alpha}} \phi^{(\mathfrak{p})}(h^{-1}j) dh = \int_{[S]} \sum_{\substack{j \in V(\mathbf{E}_1, \mathbf{E}_2) \\ \deg_{\text{CM}}(j) = \alpha}} \mathbf{1}_{\widehat{L}(\mathbf{E}_1, \mathbf{E}_2)}(h^{-1}j) dh$$

is the α -th Fourier coefficient of $\theta(\tau, \phi^{(\mathfrak{p})})$.

Recall from Section 2.4 that Γ acts on $[\mathcal{X}(\mathbb{F}_p^{\text{alg}})]$. Using the map (2.9) we obtain an action of $T(\widehat{\mathbb{Q}})$ on $[\mathcal{X}(\mathbb{F}_p^{\text{alg}})]$, and this action factors through the map $\eta : T(\widehat{\mathbb{Q}}) \rightarrow S(\widehat{\mathbb{Q}})$ of (2.7). It is easy to see that $\mathbf{1}_{\widehat{L}(\mathbf{E}_1, \mathbf{E}_2)}$ is invariant under $V = \eta(U)$, so there is a constant C , independent of α , such that

$$\begin{aligned} E_\alpha(\tau, 0, \phi^{(\mathfrak{p})}) q^{-\alpha} &= C \sum_{h \in S(\mathbb{Q}) \backslash S(\widehat{\mathbb{Q}}) / \eta(U)} \sum_{\substack{j \in V(\mathbf{E}_1, \mathbf{E}_2) \\ \deg_{\text{CM}}(j) = \alpha}} \mathbf{1}_{\widehat{L}(\mathbf{E}_1, \mathbf{E}_2)}(h^{-1}j) \\ &= C \sum_{t \in T(\mathbb{Q}) \backslash T(\widehat{\mathbb{Q}}) / U} \sum_{\substack{j \in V(\mathbf{E}_1, \mathbf{E}_2) \\ \deg_{\text{CM}}(j) = \alpha}} \mathbf{1}_{t \bullet \widehat{L}(\mathbf{E}_1, \mathbf{E}_2)}(j) \\ &= C \sum_{(\mathfrak{a}_1, \mathfrak{a}_2) \in \Gamma} \sum_{\substack{j \in L(\mathbf{E}_1 \otimes \mathfrak{a}_1, \mathbf{E}_2 \otimes \mathfrak{a}_2) \\ \deg_{\text{CM}}(j) = \alpha}} 1. \end{aligned}$$

The last identity follows from Proposition 2.14 and the discussion following Remark 2.15. Notice that the Eisenstein series has constant term $E_0(\tau, 0, \phi^{(\mathfrak{p})}) = 2$ (see for example [21, Theorem 1.2]). Taking $\alpha = 0$ on both sides, one sees that

$$C = \frac{2}{|\Gamma|} = \frac{2}{h_1 h_2}$$

where h_i is the class number of K_i . Now the proposition follows from the class number formula

$$\Lambda(1, \chi) = \Lambda(1, \chi_1) \Lambda(1, \chi_2) = \frac{4h_1 h_2}{\mathfrak{w}_1 \mathfrak{w}_2},$$

in which χ_i is the quadratic Dirichlet character associated to K_i/\mathbb{Q} . Alternatively, one can find C by tracking the Haar measures involved. \square

Proposition 3.10. *Assume $\mathfrak{p} \in \text{Diff}(\alpha)$.*

1. *One has $W_{\alpha, \mathfrak{p}}^*(1, s, \phi_{\mathfrak{p}}^{(\mathfrak{p})}) = 0$ unless $\alpha \in \mathfrak{D}_{\mathfrak{p}}^{-1}$. In such a case, one has*

$$W_{\alpha, \mathfrak{p}}^*(1, 0, \phi_{\mathfrak{p}}^{(\mathfrak{p})}) = -1.$$

2. One has

$$W_{\alpha, \mathfrak{p}}^{*,'}(1, 0, \phi_{\mathfrak{p}}^{\mathcal{C}}) = \nu_{\mathfrak{p}}(\alpha) W_{\alpha, \mathfrak{p}}^*(1, 0, \phi_{\mathfrak{p}}^{(\mathfrak{p})}) \log N(\mathfrak{p}).$$

Here

$$\nu_{\mathfrak{p}}(\alpha) = \frac{1}{2} \text{ord}_{\mathfrak{p}}(\alpha \mathfrak{p} \mathfrak{D})$$

as in Proposition 2.25.

Proof. The first equality of (1) follows from [21, Proposition 2.2] (and the argument in the proof of Proposition 3.5). The rest follows from Proposition 3.5. Notice that both sides of the stated equality of (2) are zero if $\alpha \notin \mathfrak{D}_{\mathfrak{p}}^{-1}$. \square

Proof of Theorem C. Assume first that $\alpha \in F^{\times}$ is totally positive and $\text{Diff}(\alpha) = \{\mathfrak{p}\}$. Let p be the rational prime below \mathfrak{p} . As \mathfrak{p} is inert in K , p is nonsplit in both K_1 and K_2 . Given a (necessarily supersingular) CM pair $(\mathbf{E}_1, \mathbf{E}_2)$ over $\mathbb{F}_p^{\text{alg}}$, write $\mathfrak{p}(\mathbf{E}_1, \mathbf{E}_2)$ for its reflex prime. Recall from Theorem 2.26 that for any geometric point $x \in \mathcal{X}_{\alpha}(\mathbb{F}_p^{\text{alg}})$ representing a triple $(\mathbf{E}_1, \mathbf{E}_2, j)$ we have $\mathfrak{p}(\mathbf{E}_1, \mathbf{E}_2) = \mathfrak{p}$ and

$$\text{length}(\mathcal{O}_{\mathcal{X}_{\alpha}, x}^{\text{sh}}) = \nu_{\mathfrak{p}}(\alpha) = \frac{1}{2} \text{ord}_{\mathfrak{p}}(\alpha \mathfrak{p} \mathfrak{D}).$$

Therefore

$$\begin{aligned} \deg(\mathcal{X}_{\alpha}) &= \nu_{\mathfrak{p}}(\alpha) \log(p) \sum_{(\mathbf{E}_1, \mathbf{E}_2, j) \in [\mathcal{X}_{\alpha}(\mathbb{F}_p^{\text{alg}})]} \frac{1}{|\text{Aut}(\mathbf{E}_1, \mathbf{E}_2, j)|} \\ &= \nu_{\mathfrak{p}}(\alpha) \log(p) \sum_{(\mathbf{E}_1, \mathbf{E}_2) \in [\mathcal{X}(\mathbb{F}_p^{\text{alg}})]} \sum_{\substack{j \in L(\mathbf{E}_1, \mathbf{E}_2) \\ \deg_{\text{CM}}(j) = \alpha}} \frac{1}{|\text{Aut}(\mathbf{E}_1, \mathbf{E}_2)|}. \end{aligned}$$

Lemma 2.16 then implies

$$\deg(\mathcal{X}_{\alpha}) = \frac{\nu_{\mathfrak{p}}(\alpha)}{\mathbf{w}_1 \mathbf{w}_2} \log(p) \sum_{\substack{(\mathbf{E}_1, \mathbf{E}_2) \in [\mathcal{X}(\mathbb{F}_p^{\text{alg}})] \\ \mathfrak{p}(\mathbf{E}_1, \mathbf{E}_2) = \mathfrak{p}}} \sum_{\substack{j \in L(\mathbf{E}_1, \mathbf{E}_2) \\ \deg_{\text{CM}}(j) = \alpha}} 1.$$

By Propositions 2.22 and Lemma 2.16, Γ acts on the set of $(\mathbf{E}_1, \mathbf{E}_2) \in [\mathcal{X}(\mathbb{F}_p^{\text{alg}})]$ with reflex prime \mathfrak{p} freely with two orbits, so Proposition 3.9 implies

$$\deg(\mathcal{X}_{\alpha}) = \frac{1}{4} \nu_{\mathfrak{p}}(\alpha) \log(p) E_{\alpha}^*(\tau, 0, \phi^{(\mathfrak{p})}) q^{-\alpha}.$$

Next, Proposition 3.10 implies

$$\begin{aligned} E_{\alpha}^{*,'}(\tau, 0, \phi^{\mathcal{C}}) &= W_{\alpha, \mathfrak{p}}^{*,'}(1, 0, \phi_{\mathfrak{p}}^{\mathcal{C}}) \prod_{\mathfrak{q} \neq \mathfrak{p}} W_{\alpha, \mathfrak{q}}^*(1, 0, \phi_{\mathfrak{q}}^{\mathcal{C}}) \prod_{l=1}^2 W_{\alpha, \sigma_l}^*(\tau_l, 0, \phi_{\sigma_l}^{\mathcal{C}}) \\ &= \nu_{\mathfrak{p}}(\alpha) \log N(\mathfrak{p}) \cdot E_{\alpha}^*(\tau, 0, \phi^{(\mathfrak{p})}). \end{aligned}$$

Notice that $N(\mathfrak{p}) = p$ by Remark 2.9, so we have

$$4 \deg(\mathcal{X}_\alpha) \cdot q^\alpha = E_\alpha^{*,'}(\tau, 0, \phi^C) = E_\alpha^{*,'}(\tau, 0)$$

by Proposition 3.4, as claimed in Theorem C.

Next, if $|\text{Diff}(\alpha)| > 1$ then $\text{ord}_{s=0} E_\alpha^*(\tau, s, \phi^C) > 1$ by Proposition 3.5, while \mathcal{X}_α is empty by Theorem 2.26. Thus both sides of the desired equality are zero. \square

One advantage of the conceptual proof is that Theorem C can be proved for $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ by only doing local calculations at \mathfrak{p} . For example one does not need to compute (as in Section 2.3) the local structure of $(V(\mathbf{E}_1, \mathbf{E}_2), \deg_{\text{CM}})$ at primes other than \mathfrak{p} , nor any of the orbital integrals of Section 2.5, nor does one need to compute local Whittaker functions at primes other than \mathfrak{p} . The obvious disadvantage is that one does not obtain the explicit values for $\deg(\mathcal{X}_\alpha)$ and a_α of Theorems A and B.

References

- [1] B. Conrad. Gross-Zagier revisited. In *Heegner points and Rankin L-series*, volume 49 of *Math. Sci. Res. Inst. Publ.*, pages 67–163. Cambridge Univ. Press, Cambridge, 2004. With an appendix by W. R. Mann.
- [2] P. Garrett. *Holomorphic Hilbert Modular Forms*. Wadsworth & Brooks/Cole, Pacific Grove, California, 1990.
- [3] B. Gross. On canonical and quasicanonical liftings. *Invent. Math.*, 84(2):321–326, 1986.
- [4] B. Gross and D. Zagier. On singular moduli. *J. Reine Angew. Math.*, 355:191–220, 1985.
- [5] H. Hida. *p-adic automorphic forms on Shimura varieties*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2004.
- [6] M. Knus. *Quadratic and Hermitian forms over rings*, volume 294 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991. With a foreword by I. Bertuccioni.
- [7] S. Kudla. Splitting metaplectic covers of dual reductive pairs, *Israel J. Math.* 87 (1994), 361–401
- [8] S. Kudla. Central derivatives of Eisenstein series and height pairings. *Ann. of Math. (2)* 146 (1997), 545–646.
- [9] S. Kudla. *Integrals of Borchers forms*, *Compositio Math.* 137 (2003), 293–349.

- [10] S. Kudla. Special cycles and derivatives of Eisenstein series, in Heegner points and Rankin L-series, 243-270, *Math. Sci. Res. Inst. Publ.*, 49, Cambridge Univ. Press, Cambridge, 2004.
- [11] S. Kudla, M. Rapoport, and T.H. Yang. Derivatives of Eisenstein series and Faltings heights, *Compositio Math.*, 140(2004), 887-951.
- [12] S. Kudla, M. Rapoport, and T.H. Yang. Modular forms and special cycles on Shimura curves, *Annals of Mathematics Studies Series 161*, Princeton University Press, 2006.
- [13] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000.
- [14] M. Rapoport and Th. Zink. *Period spaces for p-divisible groups*, volume 141 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.
- [15] J.-P. Serre. *A Course in Arithmetic*. Springer-Verlag, New York, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.
- [16] T. Shintani. On construction of holomorphic cusp forms of half integral weight, *Nagoya Math. J.* 58(1975), 83-126.
- [17] E. Viehmann and K. Ziegler. Formal moduli of formal \mathcal{O}_K -modules. *Astérisque*, (312):57–66, 2007.
- [18] A. Vistoli. Intersection theory on algebraic stacks and on their moduli spaces. *Invent. Math.*, 97(3):613–670, 1989.
- [19] A. Weil. Sur la formule de Siegel dans la théorie des groupes classiques *Acta. Math.*, (113): 1–87, 1965.
- [20] S. Wewers. Canonical and quasi-canonical liftings. *Astérisque*, (312):67–86, 2007.
- [21] T. H. Yang. CM number fields and modular forms, *Quarterly Jour. Pure Appl. Math. Special issue in memory of A. Borel* 1(2005), 305-340
- [22] T. H. Yang. An arithmetic intersection formula on Hilbert modular surface. *To appear in Amer. J. Math.*, pp32.