

# Complete Numerical Isolation of Real Zeros in General Triangular Systems <sup>1)</sup>

Jin-San Cheng<sup>1</sup>, Xiao-Shan Gao<sup>1</sup> and Chee-Keng Yap<sup>2,3</sup>

<sup>1</sup> KLMM, Institute of Systems Science, AMSS,  
Academia Sinica, Beijing 100080, China

<sup>2</sup> Courant Institute of Mathematical Sciences  
New York University, 251 Mercer Street  
New York, NY 10012, USA

<sup>3</sup> Korea Institute for Advanced Study, Seoul, Korea

**Abstract.** We consider the computational problem of isolating all the real zeros of a zero-dimensional triangular polynomial system  $F_n \subseteq \mathbb{Z}[x_1, \dots, x_n]$ . We present a complete numerical algorithm for this problem. Our system  $F_n$  is general, with no further assumptions. In particular, our algorithm is the first to successfully treat multiple zeros in such systems. A key idea is to introduce evaluation and separation bounds, which are used in conjunction with sleeve bounds to detect zeros of even multiplicity. Our algorithm assumes a computational model of bigfloats with exact ring operations. We have implemented our algorithm and promising experimental results are shown.

**Keywords.** *system of polynomial equations, triangular polynomial system, zero-dimensional system, isolating interval, real zero isolation, complete numerical algorithms, sleeve bound, evaluation bound, separation bound.*

## 1. Introduction

Many problems in the computational sciences and engineering can be reduced to the problem of solving polynomial equations. There are two basic approaches to solving such polynomial systems – numerically or algebraically. Usually, the numerical methods have no global guarantees of correctness. Algebraic methods for solving polynomial systems include Gröbner bases [6], characteristic sets [19, 15], CAD (Cylinder Algebraic Decomposition) [2, 3], or resultants [1, 17]. One general idea in polynomial equation solving is to reduce the original system into a triangular system. Zero-dimensional polynomial systems are among the most important cases to solve. This paper considers this case only.

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A zero-dimensional triangular system of polynomials has the form  $F_n = \{f_1, \dots, f_n\}$ , where each  $f_i \in \mathbb{Z}[x_1, \dots, x_i]$  ( $i = 1, \dots, n$ ). We are interested in real zeros of  $F_n$ . A real zero of  $F_n$  is  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$  such that  $F_n(\xi) = 0$ , i.e.,

$$f_1(\xi_1) = f_2(\xi_1, \xi_2) = \dots = f_n(\xi_1, \dots, \xi_n) = 0. \quad (1)$$

The standard idea here is to first solve for  $f_1(x_1) = 0$ , and for each solution  $x_1 = \xi_1$  of  $f_1$ , we find the solutions of  $x_2 = \xi_2$  of  $f_2(\xi_1, x_2) = 0$ , etc. This means that the problem can be reduced to solving univariate polynomials of the form

$$f_i(\xi_1, \dots, \xi_{i-1}, x_i) = 0. \quad (2)$$

Such polynomials have algebraic number coefficients. We could isolate roots of such polynomials by using standard root isolation algorithms, but using algebraic number arithmetic. But even for  $n = 2$  or  $3$ , such algorithms are too slow. The numerical approach is to replace the  $\xi_i$ 's by approximations, and thus reduce the problem to isolating roots of such numerical polynomials. The challenge is how to guarantee completeness of such numerical algorithms.

**Results of This Paper.** We will provide a numerical algorithm that solves such triangular systems completely in the following precise sense: given an  $n$ -dimensional box  $R = J_1 \times \dots \times J_n \subseteq \mathbb{R}^n$  where  $J_i$  are intervals, and any precision  $\varepsilon > 0$ , it will isolate the zeros of  $F_n$  in  $R$  to precision  $\varepsilon$ . To isolate the zeros of  $F_n$  in  $R$  means to compute a set of pairwise disjoint  $n$ -dimensional boxes such that each zero of  $F_n$  in  $R$  is contained in one of these boxes, and each box contains just one zero of  $F_n$ . These boxes have diameter bounded by  $\varepsilon$ .

Our solution places no restriction on  $F_n$ . In particular, ours is the first to achieve complete root isolation in the presence of multiple zeros. All the existing algorithms require the system  $F_n$  to be square-free (no multiple zeros) and some require  $F_n$  to be regular<sup>2)</sup> or even irreducible. As is well known, it is expensive to make a triangular polynomial system to be square-free, regular or irreducible.

Many algorithms that seek to provide “exact numerical” solution assume computation over the rational numbers  $\mathbb{Q}$ . But this is much less efficient than using dyadic numbers: let  $\mathbb{D} := \mathbb{Z}[\frac{1}{2}] = \{m2^n : m, n \in \mathbb{Z}\}$  denote the set of dyadic numbers (or bigfloats). Most current fast algorithms for bigfloats can be derived from Brent’s work [5]. In the following, we use the symbol  $\mathbb{F}$  to denote either  $\mathbb{D}$  or  $\mathbb{Q}$ . The only computational assumption about  $\mathbb{F}$  we need are: (1) the ring operations  $(+, -, \times)$  and  $x \mapsto x/2$  (halving) are computed without error, and (2) comparison among the elements of  $\mathbb{F}$  is exact. The algorithms of this paper can be implemented exactly over  $\mathbb{F}$ . We use intervals to isolate real numbers: let  $\square\mathbb{F}$  denote the set of intervals of the form  $[a, b]$  where  $a \leq b \in \mathbb{F}$ . Note that assumptions (1) and (2) are stronger than the axioms in Brent’s model [5]; see [24] for an axiomatic treatment of  $\mathbb{F}$  in real approximations.

Given a polynomial  $f \in \mathbb{R}[X]$  and an interval  $I = [a, b] \in \square\mathbb{F}$ , the basic idea is to construct two polynomials  $f^u, f^d \in \mathbb{F}[X]$  such  $f^u > f > f^d$  holds in  $I$ . We call  $(f^u, f^d)$  a **sleeve** of  $f$  over  $I$ . We show that if the **sleeve bound**  $SB_I(f^u, f^d) := \sup\{f^u(x) - f^d(x) : x \in I\}$  is sufficiently tight, then isolating the roots of  $f^u$  and  $f^d$  can lead to isolation of the roots

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<sup>2)</sup> $F_n$  is **regular** if for each zero  $(\xi_1, \dots, \xi_n)$ , the leading coefficient of the polynomial  $f_i(\xi_1, \xi_2, \dots, \xi_{i-1}, x_i)$  does not vanish.

of  $f$ . Note that the coefficients of  $f^u f^d$  are in  $\mathbb{F}$ , but  $f$  have real coefficients which can be arbitrarily approximated.

Univariate root isolation is a well-developed subject in its own right, with many efficient solutions known (see [10, 12, 13, 14] for some recent work). We can use any of these solutions in our algorithm. The only additional property we require in these univariate solvers is that they handle multiple zeros. It is also easy to classify multiple zeros according to their **parity**: the parity of the root is *even* (resp., *odd*) if the root has even (resp., odd) multiplicity. There are simple ways to modify standard algorithms to satisfy our extra requirements.

The critical idea in this paper is the introduction of **evaluation bounds**. For a differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a subset  $I \subseteq \mathbb{R}$ , let its **evaluation bound** be

$$EB_I(f) := \inf\{|f(x)| : f'(x) = 0, |f(x)| \neq 0, x \in I\}, \quad (3)$$

and its **separation bound** be

$$\Delta_I(f) := \inf\{|x - y| : f(x) = f(y) = 0, x, y \in I, x \neq y\}. \quad (4)$$

By definition, the infimum over an empty set is  $\infty$ . The subscript  $I$  may be omitted when  $I = \mathbb{R}$ . Although separation bounds are well-known tools in the area of root isolation, the use of evaluation bounds appears to be new. It is the ability to compute lower estimates on  $EB_I(f)$  and  $\Delta(f)$  that allows us to detect zeros of even multiplicities. In particular, if the following **sleeve-evaluation inequality**

$$SB_I(f^u, f^d) < EB_I(f) \quad (5)$$

holds, then we show how the isolating intervals of  $f^u f^d$  can be used to define isolating intervals of  $f$ . In order to satisfy this inequality, we need to “refine” our sleeves to yield tighter sleeve bounds. Furthermore, we need to generalize sleeves and (3) to the multivariate case of triangular systems.

A major goal in our algorithmic design is the emphasis on “adaptive” techniques. Informally, adaptivity means that the computational complexity is sensitive to the nature of the input instance, and in typical or nice instances, the complexity is low. Thus, we prefer numerical (iterative) tests which are usually adaptive, over more powerful but non-adaptive algebraic techniques. For instance, in our algorithm below we need determine the sign of a derivative at a point: this could be reduced to detecting a zero of the derivative using a Sturm sequence computation, but we prefer to deploy a numerical iteration whose halting condition is provided by root separation estimates.

**Literature Survey.** The idea of using a sleeve to solve equations was used by [18] and [16]. Lu et al [16] proposed an algorithm to isolate the real roots of triangular polynomial system. Their method could solve many problems in practice. But their algorithm is not complete in the sense that it does not have a termination condition and cannot handle multiple zeros. Collins et al [8] considered the problem with interval arithmetic methods and Descartes’ method using floating point computation. Based on the CAD method, they considered isolating the real roots of a squarefree triangular system. They constructed a bitstream interval for each real coefficient of a univariate polynomial  $f = f_i(\xi_1, \dots, \xi_{i-1}, X)$ . Then they obtain an interval polynomial for  $f$ . The sign determination of  $f_i(\xi_1, \dots, \xi_{i-1}, X)$

can be replaced by determining the sign of the two corresponding endpoints of the interval for each coefficient. In this way, they obtained isolating intervals of the triangular system. They pointed out if a real coefficient is zero (but in some implicit representation), the method will fail. Their system is restricted to be regular. Xia and Yang [20], based on the resultant computation, proposed a method to isolate the real roots of a semi-algebraic set. In fact, they ultimately considered the real root isolation of regular and square-free triangular systems. They mentioned that their method is not complete and will fail in some cases. Our root isolation of real polynomials using sleeves is related to Eigenwillig et al [11] who considered root-isolation for real polynomials with bitstream coefficients. Their algorithm requires  $f$  to be squarefree; but we require algebraic coefficients when  $f$  is non-squarefree. Their algorithm is based on the Descartes method, but ours can be viewed as a generic reduction of the root isolation problem to univariate root isolation in  $\mathbb{F}[X]$ . Our evaluation bound is analogous the curve separation bounds in Yap [23], who used them to provide the first complete subdivision algorithm for detecting tangential intersection of Bezier curves.

**Overview of Paper.** In the next section, we describe the basic technique of using sleeves and evaluation bounds of  $f$ . We next exploit a special property of sleeves called monotonicity. This leads to an effective criteria for isolating zeros of even multiplicity. Using these tools, we provide an algorithm to isolate the real roots of univariate polynomial with real coefficients. In Section 3, we extend the isolation method to the multivariate case. We compute lower estimates on evaluation and separation bounds. We also show how to construct sleeves for  $f_i(\xi_1, \dots, \xi_{i-1}, X)$  and derive upper estimates on the sleeve bound, as a function of the precision of the given isolating box for  $(\xi_1, \dots, \xi_{i-1})$ . This shows convergence of our algorithm. Subalgorithms for refinement of isolating boxes and for verifying zeros are included. Finally, the overall isolation algorithm is presented here. Section 4 describes some experimental work. We conclude in Section 5.

## 2. Root Isolation for Real Univariate Polynomials

In this section, we give a framework for isolating the real roots of a univariate polynomial with real coefficients.

### 2.1. Evaluation and Sleeve Bounds

Let  $\mathbb{Q}$  be the field of rational numbers,  $\mathbb{R}$  the field of real numbers,  $\mathbb{D} := \mathbb{Z}[\frac{1}{2}] = \{m2^n : m, n \in \mathbb{Z}\}$  the set of dyadic numbers, and  $\mathbb{F}$  denote either  $\mathbb{D}$  or  $\mathbb{Q}$ . A real function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is  $C^1$  if it has a continuous derivative  $f'(X) = \frac{\partial f}{\partial X}$ . In this section, we fix  $f, f^u, f^d$  to be  $C^1$  functions, and let  $I = [a, b]$  be an interval. In applications later, we will further assume that  $f \in \mathbb{R}[X]$ ,  $f^u, f^d \in \mathbb{F}[X]$  and  $I \in \square\mathbb{F}$ .

We call  $(I, f^u, f^d)$  a **sleeve** for  $f$  if, for all  $x \in I$ , we have  $f^u(x) > f(x) > f^d(x)$ .

For any real function  $f$ , let  $\mathbf{Zero}_I(f)$  denote the set of distinct real zeros of  $f$  in the interval  $I$ . If  $I = \mathbb{R}$ , then we simply write  $\mathbf{Zero}(f)$ . If  $\mathbf{Zero}_I(f)$  has a single zero, we call  $I$  an **isolating interval** of  $f$ . Sometimes, we need to count the zeros up to the parity (i.e., evenness or oddness) of their multiplicity. Call a zero  $\xi \in \mathbf{Zero}(f)$  an **even zero** if its multiplicity is even, and **odd zero** if its multiplicity is odd. Define the multiset <sup>3)</sup>  $\mathbf{ZERO}_I(f)$

<sup>3)</sup>A multiset  $S$  is a pair  $(x_S, \mu_S)$  where  $x_S$  is a set in the usual sense, and  $\mu_S : x_S \rightarrow \{1, 2, 3, \dots\}$  is a function. We call  $\mu_S(x)$  the **multiplicity** of  $x \in x_S$ , and  $x_S$  the **underlying set** of  $S$ . For simplicity, we

whose underlying set is  $\mathbf{Zero}_I(f)$  and where the multiplicity of  $\xi \in \mathbf{Zero}_I(f)$  is 1 (resp., 2) if  $\xi$  is an odd (resp., even) zero of  $f$ .

To avoid special treatment near the endpoints of an interval, we would like to enforce the following conditions.

$$|f(a)| \geq EB_I(f), \quad f^u(b)f^d(b) > 0. \quad (6)$$

We say that the sleeve  $(I, f^u, f^d)$  is **faithful** for  $f$  if (6) as well as the sleeve-evaluation inequality (5) are both satisfied. We can easily see that  $|f(a)| \geq EB_I(f)$  implies  $f^u(a)f^d(a) > 0$ , using (5). We need a stronger condition at  $X = a$  than at  $X = b$  in (6) because there might be a zero of  $f$  just to the left of  $X = a$  that can cause confusion for our lemmas below: this asymmetry is a consequence of the monotonicity property below. An appendix will treat the case of non-faithful sleeves.

Intuitively,  $f$  is nicely behaved when if we restrict  $f$  to a neighborhood of a zero  $\xi$  where  $|f| < EB(f)$ . This is illustrated in Figure 1.

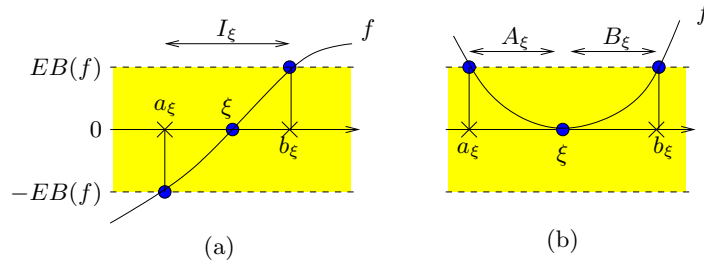


Fig. 1. Neighborhood of zero  $\xi$ :  $I_\xi = A_\xi \cup \{\xi\} \cup B_\xi$ .

Given  $f$  and  $I$ , define the polynomials

$$\hat{f}(X) := f(X) - EB_I(f), \quad \bar{f}(X) := f(X) + EB_I(f).$$

If  $\xi \in \mathbf{Zero}_I(f)$ , we define the points  $a_\xi, b_\xi$  as follows:

$$a_\xi := \max\{a\} \cup \left( \mathbf{Zero}(\hat{f} \cdot \bar{f}) \cap (-\infty, \xi) \right), \quad (7)$$

$$b_\xi := \min\{b\} \cup \left( \mathbf{Zero}(\hat{f} \cdot \bar{f}) \cap (\xi, +\infty) \right). \quad (8)$$

Then define the open intervals (see Figure 1):

$$A_\xi := (a_\xi, \xi), \quad B_\xi := (\xi, b_\xi) \text{ and } I_\xi := (a_\xi, b_\xi). \quad (9)$$

The basic properties of these intervals are captured here:

LEMMA 1. *Let  $(I, f^u, f^d)$  be a faithful sleeve for  $f$ . For all  $\xi, \zeta \in \mathbf{Zero}_I(f)$ , we have:*

(i) *If  $\xi \neq \zeta$  then  $I_\xi$  and  $I_\zeta$  are disjoint.*

write “ $x \in S$ ” instead  $x \in x_S$ . Also, the **size** of  $S$  is defined to be  $|S| := \sum_{x \in X} \mu_S(X)$ .

(ii)  $\text{Zero}_I(f^u f^d) \subseteq \bigcup_{\xi} I_{\xi}$ .

(iii-a)  $A_{\xi} \cap \text{Zero}(f^u)$  is empty iff  $A_{\xi} \cap \text{Zero}(f^d)$  is non-empty.

(iii-b)  $B_{\xi} \cap \text{Zero}(f^u)$  is empty iff  $B_{\xi} \cap \text{Zero}(f^d)$  is non-empty.

(iv) The derivative  $f'$  has a constant non-zero sign in  $A_{\xi}$ , and also in  $B_{\xi}$ .

*Proof.* (i) Suppose  $\xi < \zeta$  are consecutive zeros of  $\text{Zero}_I(f)$ . Then either  $f$  is positive on  $(\xi, \zeta)$  or  $f$  is negative on  $(\xi, \zeta)$ . Wlog,  $f$  is positive on  $(\xi, \zeta)$ . Then the multiset  $\text{ZERO}_I(\widehat{f}) = \text{ZERO}(f - EB_I(f))$  has at least two zeros (they may have the same value) in  $(\xi, \zeta)$ . This proves  $b_{\xi} \leq a_{\zeta}$  and so  $I_{\xi}$  and  $I_{\zeta}$  are disjoint.

(ii) Let  $z \in \text{Zero}_I(f^u f^d)$ . Then (5) implies that  $|f(z)| < EB_I(f)$ . By the definition of evaluation bound, this also means that  $f'(z) \neq 0$ . Thus there are two cases: either  $f(z)f'(z) > 0$  or  $f(z)f'(z) < 0$ . First, suppose  $f(z)f'(z) > 0$ . Then there is a unique largest  $\xi \in \text{Zero}(f)$  that is less than  $z$ , and there is a unique smallest  $b_{\xi} \in \text{Zero}(\widehat{f})$  that is greater than  $z$ . This proves that  $z \in (\xi, b_{\xi})$ . Similarly, if  $f(z)f'(z) < 0$ , we will see that  $z \in (a_{\xi}, \xi)$  for some  $\xi \in \text{Zero}_I(f)$ .

(iii-a) Either  $f(a_{\xi}) > 0$  or  $f(a_{\xi}) < 0$ . If  $f(a_{\xi}) > 0$  then (5) implies  $f^d(a_{\xi}) > 0$ . But  $f^d(\xi) < 0$ , and hence  $A_{\xi} \cap \text{Zero}(f^d)$  is non-empty. Now, since  $f^u$  is positive over  $A_{\xi}$ , we conclude that  $A_{\xi} \cap \text{Zero}(f^u)$  is empty. The other case,  $f(a_{\xi}) < 0$  will similarly imply that  $A_{\xi} \cap \text{Zero}(f^d)$  is empty and  $A_{\xi} \cap \text{Zero}(f^u)$  is non-empty.

(iii-b) This is similar to (iii-a).

(iv) Assume there exist  $s \in A_{\xi}$  (for  $B_{\xi}$ , the proof is similar) such that  $f'(s) = 0$ . We derive a contradiction from the definitions of  $a_{\xi}$  by (7), where  $A_{\xi} = (a_{\xi}, \xi)$ . **Q.E.D.**

If  $s, t \in \text{Zero}_I(f^u f^d)$  such that  $s < t$  and  $(s, t) \cap \text{Zero}_I(f^u f^d)$  is empty, then we call  $(s, t)$  a **sleeve interval** of  $(I, f^u, f^d)$ . The following is immediate from the preceding lemma(iii):

**COROLLARY 2.** *Each zero of  $\text{Zero}_I(f)$  is isolated by some sleeve interval of  $(I, f^u, f^d)$ .*

**LEMMA 3.** *Let  $(I, f^u, f^d)$  be a faithful sleeve. For all  $\xi \in \text{Zero}_I(f)$ , the multiset  $\text{ZERO}_{B_{\xi}}(f^u \cdot f^d)$  has odd size. Similarly, the multiset  $\text{ZERO}_{A_{\xi}}(f^u \cdot f^d)$  has odd size.*

*Proof.* We just prove the result for the multiset  $\text{ZERO}_{B_{\xi}}(f^u \cdot f^d)$ . Wlog, let  $f(b_{\xi}) > 0$  (the case  $f(b_{\xi}) < 0$  is similar). By the sleeve-evaluation inequality,  $f^d(b_{\xi}) > 0$ . Note that when  $b_{\xi} = b$ , the inequality is also true since  $(I, f^u, f^d)$  is faithful. But  $f^d(\xi) < 0$ . Hence  $f^d$  has an odd number of zeros (counting multiplicities) in the interval  $B_{\xi} = (\xi, b_{\xi})$ . Moreover,  $f^u > f$  implies  $f^u$  has no zeros in  $B_{\xi}$ . **Q.E.D.**

It follows from the preceding lemma that for each zero  $\xi$  of  $f$ , the multiset  $\text{ZERO}_{I_{\xi}}(f^u f^d)$  has even size. Hence the multiset  $\text{ZERO}_I(f^u f^d)$  has even size, say  $2m$ . So we may denote the sorted list of zeros of  $\text{ZERO}_I(f^u f^d)$  by

$$(t_0, t_1, \dots, t_{2m-1}). \tag{10}$$

where  $t_0 \leq t_1 \leq \dots \leq t_{2m-1}$ . Note that  $t_i = t_{i+1}$  iff  $t_i$  is an even zero of  $f^u f^d$ . Intervals of the form  $J_i := [t_{2i}, t_{2i+1}]$  where  $t_{2i} < t_{2i+1}$  are called **candidate interval** of the sleeve. We immediately obtain:

**COROLLARY 4.** *Each  $\xi \in \text{Zero}_I(f)$  is contained in some candidate interval of a faithful sleeve  $(I, f^u, f^d)$ .*

*Proof.* We use the notations in (9) and (10), and use  $\xi$  to represent a root of  $f$  in  $I$ . From Lemma 1 (ii), any element of  $\text{ZERO}_I(f^u f^d)$  is in some  $I_\xi$ . From Lemma 3,  $I_\xi \cap \text{ZERO}_I(f^u f^d)$  has even size. Therefore, the smallest element of  $A_\xi \cap \text{ZERO}_I(f^u f^d)$  is of the form  $t_{2k}$ . From Lemma 3,  $A_\xi \cap \text{ZERO}_I(f^u f^d)$  has odd size. Then the largest element of  $A_\xi$  is also of the form  $t_{2s}$  and the smallest element of  $B_\xi$  is  $t_{2s+1}$ . As a consequence,  $\xi$  is in the candidate interval  $(t_{2s}, t_{2s+1})$ . **Q.E.D.**

Which of these candidate intervals actually contain zeros of  $f$ ? To do this, we classify a candidate interval  $[t_{2j}, t_{2j+1}]$  in (10) into two types:

$$\left. \begin{array}{l} \text{(Odd): } t_{2j} \in \text{Zero}(f^d) \text{ if and only if } t_{2j+1} \in \text{Zero}(f^u) \\ \text{(Even): } t_{2j} \in \text{Zero}(f^d) \text{ if and only if } t_{2j+1} \in \text{Zero}(f^d) \end{array} \right\} \quad (11)$$

Thus we call a candidate interval  $J$  an **odd** or **even candidate interval** depending on whether it satisfies (11)(Odd) or (11)(Even). We now treat the easy case of deciding which candidate intervals are isolating intervals of  $f$ :

LEMMA 5 (Odd Zero). *Let  $J$  be a candidate interval. The following are equivalent:*

- (i)  $J$  is an odd candidate interval.
- (ii)  $J$  contains a unique zero  $\xi$  of  $f$ . Moreover  $\xi$  is an odd zero of  $f$ .

*Proof.* Let  $J = [t, t']$ .

(i) implies (ii): Wlog, let  $f^u(t) = 0$  and  $f^d(t') = 0$ . Thus,  $f(t) < 0$  and  $f(t') > 0$ . Thus  $f$  has an odd zero in  $J$ . By Corollary 2, we know that candidate intervals contain at most one distinct zero.

(ii) implies (i): Since  $\xi$  is an odd zero, we see that  $f$  must be monotone over  $J$ . Wlog, assume  $f$  is increasing. This implies  $f^d(t) < 0$  and hence  $f^u(t) = 0$ . Similarly,  $f^u(t') > 0$  and hence  $f^d(t') = 0$ . Hence  $J$  is an odd candidate. **Q.E.D.**

Lemma 5 provides the theoretical basis to isolate zeros of odd multiplicity. Isolate zeros of even multiplicity is more subtle and will be dealt with in the following section. To do this we need to look at the sign of  $\frac{\partial f^u}{\partial X}$  and  $\frac{\partial f^d}{\partial X}$ . We make a first observation along this line:

LEMMA 6. *Let  $t_i \in \text{ZERO}(f^u f^d)$ .*

(a) *If  $t_i$  is a zero of  $f^u$ , then  $i$  is even implies  $\frac{\partial f^u}{\partial X}(t_i) \geq 0$ , and  $i$  is odd implies  $\frac{\partial f^u}{\partial X}(t_i) \leq 0$ .*

(b) *If  $t_i$  is a zero of  $f^d$ , then  $i$  is even implies  $\frac{\partial f^d}{\partial X}(t_i) \leq 0$ . and  $i$  is odd implies  $\frac{\partial f^d}{\partial X}(t_i) \geq 0$ .*

*Proof.* The result is true for  $i = 0$ , using faithfulness. The rest follows by induction based on parity tracking. **Q.E.D.**

## 2.2. Monotonicity Property

We will now exploit a special property of sleeve  $(I, f^u, f^d)$  for  $f$ :

$$\frac{\partial f^u}{\partial X} \geq \frac{\partial f}{\partial X} \geq \frac{\partial f^d}{\partial X} \quad \text{holds in } I \quad (12)$$

We call this the **monotonicity property**. In this subsection, we assume the monotonicity property (12) and as well the faithfulness of the sleeve.

We now strengthen one half of Lemma 3 above.



LEMMA 7. For all  $\xi \in \text{Zero}_I(f)$ , there is a unique zero of odd multiplicity of  $f^u \cdot f^d$  in  $A_\xi = (a_\xi, \xi)$ .

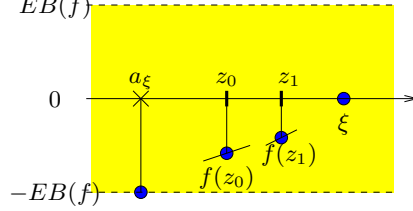


Fig. 2.  $A_\xi$  has a unique zero of  $f^u \cdot f^d$ : CASE of  $f^u(z_0) = f^u(z_1) = 0$ .

*Proof.* Alternatively, this lemma says that the multiset  $\text{ZERO}_{A_\xi}(f^u f^d)$  has size 1.

By way of contradiction, suppose  $z_0 \leq z_1$  are two zeros of  $f^u f^d$  in  $A_\xi = (a_\xi, \xi)$ . Note that we allow the possibility that  $z_0 = z_1$  (in which case  $z_0$  is an even root of  $f^u f^d$ ). From Lemma 1(iii), we know that either  $z_0, z_1 \in \text{ZERO}(f^u)$  or  $z_0, z_1 \in \text{ZERO}(f^d)$  (i.e., it is not possible that one is a zero of  $\text{ZERO}(f^u)$  and the other is a zero of  $\text{ZERO}(f^d)$ ). There are two cases:

(A)  $z_0, z_1$  are roots of  $f^u$ . See Figure 2. By Rolle's theorem, there exists  $z \in [z_0, z_1]$  such that  $\frac{\partial f^u}{\partial X}(z) = 0$ . Therefore, there exist  $z^- < z < z^+$  that are arbitrarily close to  $z$  such that

$$\frac{\partial f^u}{\partial X}(z^-) \cdot \frac{\partial f^u}{\partial X}(z^+) < 0. \quad (13)$$

On the other hand, note that  $f(z_j) < f^u(z_j) = 0$  for  $j = 0, 1$ . Since  $f(\xi) = 0$ , and  $z_j < \xi$ , this means that the interval  $(z_j, \xi)$  contains a point  $z$  with  $f'(z) > 0$ . But  $f'$  has constant sign in  $A_\xi$  from Lemma 1 (iv), and so this sign of  $f'$  is positive. Then by monotonicity (12),

$$\frac{\partial f^u}{\partial X}(z^-) \geq f'(z^-) > 0, \quad \text{and} \quad \frac{\partial f^u}{\partial X}(z^+) \geq f'(z^+) > 0. \quad (14)$$

Now we see that (13) and (14) are contradictory.

(B)  $z_0, z_1$  are roots of  $f^d$ . We similarly derive a contradiction. **Q.E.D.**

**Remark:** It should be observed that this lemma does not hold when  $A_\xi$  is replaced by  $B_\xi$ . This somewhat surprising asymmetry can be seen in the proof of the preceding result.

COROLLARY 8. If  $t_{2j}$  is an even zero of  $f^u f^d$ , then  $J_j = [t_{2j}, t_{2j+1}]$  contains no zero of  $f$ .

*Proof.* If  $J_j$  contains a zero  $\xi$  of  $f$ , then  $t_{2j}$  would be an even zero of  $f^u f^d$  contained in  $A_\xi$ , contradicting Lemma 7. **Q.E.D.**

If  $t_{2j}$  is an even zero we have either  $t_{2j} = t_{2j+1}$  or  $t_{2j} = t_{2j-1}$ . But the former case only give us a trivial candidate interval which clearly has no zeros of  $f$ . The next result is a consequence of monotonicity and faithfulness:

LEMMA 9. The interval  $J_0 = [t_0, t_1]$  is a candidate interval and it isolates a zero of  $f$ .



In Lemma 5, we showed that (11)(Odd) holds iff  $J_j$  isolates an odd zero of  $f$ . The next result shows what condition must be added to (11)(Even) in order to characterize the isolation of even zeros.

LEMMA 10 (Even Zero). *Let  $J_j = [t_{2j}, t_{2j+1}]$  be an even candidate interval. Then  $J_j$  isolates an even zero  $\xi$  of  $f$  iff one of the following conditions hold:*

- (i)  $f^d(t_{2j}) = 0$  and  $\frac{\partial f^u}{\partial X}(t_{2j}) < 0$
- (ii)  $f^u(t_{2j}) = 0$  and  $\frac{\partial f^d}{\partial X}(t_{2j}) > 0$ .

Note: if  $j > 0$  in this lemma, then  $t_{2j-1}$  is a zero of  $f^d$  iff  $t_{2j}$  is a zero of  $f^d$ .

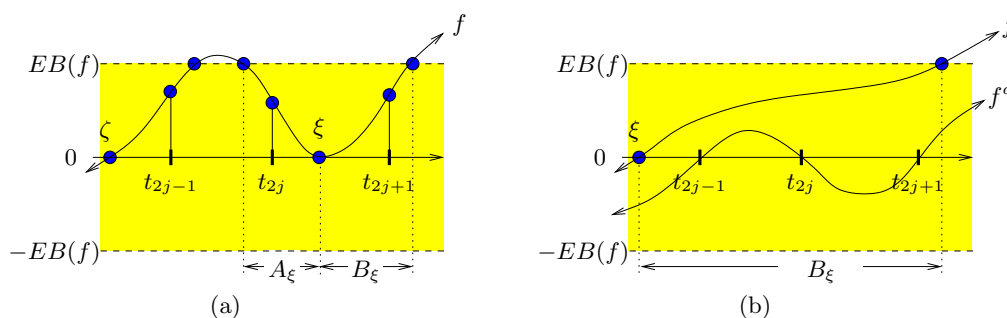


Fig. 3. Detection of even zero when  $t_{2j}, t_{2j+1} \in \text{Zero}_I(f^d)$ : (a) even zero, (b) no zero

*Proof.* Let  $t_{2j}$  be a zero of  $f^d$  (if it is a zero of  $f^u$ , the proof is similar). So  $f^d(t_{2j+1}) = 0$  and by Lemma 6,  $\frac{\partial f^d}{\partial X}(t_{2j+1}) \geq 0$ . Then monotonicity implies  $\frac{\partial f}{\partial X}(t_{2j+1}) \geq 0$ . Next,  $t_{2j+1} \in B_\xi$  for some zero  $\xi$  of  $f$ . This means  $\frac{\partial f}{\partial X}$  is positive in the interval  $(\xi, t_{2j+1})$ . There are two cases: (a)  $t_{2j} < \xi < t_{2j+1}$  or (b)  $\xi < t_{2j} < t_{2j+1}$ . If (a), then since  $f(t_{2j}) > f^d(t_{2j}) = 0$ , we conclude that  $\frac{\partial f}{\partial X}(t_{2j}) < 0$  (see Figure 3(a)). If (b), then  $\frac{\partial f}{\partial X}(t_{2j}) > 0$  since  $\frac{\partial f}{\partial X}$  has constant sign in  $B_\xi$  (see Figure 3(b)). **Q.E.D.**

### 2.3. Effective Root Isolation of $f$

So far, we have been treating the roots  $t_j$  of  $f^u f^d$  exactly. But in our algorithms, we only have isolating intervals  $[a_i, b_i]$  of these  $t_j$ 's. We now want to replace the candidate intervals  $[t_{2i}, t_{2i+1}]$  by their “effective versions” of the form  $[a_{2i}, b_{2i+1}]$ . As usual, we assume that our sleeve  $(I, f^u, f^d)$  is faithful and satisfies the monotonicity property (12). Let  $\text{ZERO}_I(f^u f^d)$  be the sorted list given in (10), and  $[a_i, b_i]$  an isolating interval of  $t_i$ , where any two distinct intervals  $[a_i, b_i]$  and  $[a_j, b_j]$  are disjoint. Let

$$SL_{f,I} = ([a_0, b_0], [a_1, b_1], \dots, [a_{2m-1}, b_{2m-1}]) \quad (15)$$

be the corresponding list of isolating intervals for the roots of  $f^u f^d$  in  $\text{ZERO}_I(f^u f^d)$ . Assume that  $[a_i, b_i] = [a_j, b_j]$  iff  $t_i = t_j$ . Note that  $t_i = t_j$  implies  $|i - j| \leq 1$ . Let

$$K_i := [a_{2i}, b_{2i+1}]. \quad (16)$$

By Corollary 8,  $J_i$  is not an isolating interval if  $t_{2i}$  is an even zero. Hence, we call  $K_i$  an **effective candidate** iff  $t_{2i} < t_{2i+1}$  and  $t_{2i}$  is an odd zero. Thus,  $K_i$  contains the candidate

interval  $J_i = [t_{2i}, t_{2i+1}]$ . Furthermore,  $K_i$  is called an **effective even candidate** (resp., **effective odd candidate**) if  $J_i$  is an even (resp., odd) candidate interval (cf. (11)).

Our next theorem characterizes when  $K_i$  is an isolating interval of  $f$ . This is the “effective version” of Lemma 5 and Lemma 10. But before this theorem, we provide a useful partial criterion in the case when  $K_i$  is an effective even candidate:

LEMMA 11. *Let  $K_i = [a_{2i}, b_{2i+1}]$  be an effective even candidate. Then  $K_i$  isolates an even zero provided one of the following conditions hold:*

(E')<sup>d</sup>:  $t_{2i} \in \text{Zero}(f^d)$  and  $\frac{\partial f^u}{\partial X}$  is negative at  $a_{2i}$  or  $b_{2i}$ ,

(E')<sup>u</sup>:  $t_{2i} \in \text{Zero}(f^u)$  and  $\frac{\partial f^d}{\partial X}$  is positive at  $a_{2i}$  or  $b_{2i}$ .

*Proof.* Say  $t_{2i}$  is a zero of  $f^d$  (the case where  $t_{2i} \in \text{Zero}(f^u)$  is similar). We have  $t_{2i+1} \in B_\xi$  for some  $\xi \in \text{Zero}(f)$ , and we also know that  $f' = \frac{\partial f}{\partial X}$  is positive at  $t_{2i+1}$ . There are just two cases: either (a)  $t_{2i}$  is in  $A_\xi$ , or (b)  $t_{2i}$  is in  $B_\xi$ . If (a) holds, then  $\xi$  is an even zero in  $[c, t_{2i+1}]$ , and our lemma is true.

So assume (b) and (E')<sup>d</sup>. From (E')<sup>d</sup> and the monotonicity (12), we know that  $f'$  is negative at  $c$  where  $c = a_{2i}$  or  $b_{2i}$ . If  $c = b_{2i}$  then we get a contradiction since (b) implies  $f'$  is positive over  $B_\xi \supseteq [t_{2i}, t_{2i+1}] \supseteq [c, t_{2i+1}]$ . If  $c = a_{2i}$ , the argument is more subtle. We know that  $\xi \in [t_{2j}, t_{2j+1}]$  for some  $j < i$  and  $t_{2j+1} < t_{2i}$  (for  $t_{2i}$  is an odd zero). Moreover,  $f'$  has constant sign in  $B_\xi \supseteq [t_{2j+1}, t_{2i+1}] \supseteq [c, t_{2i+1}]$ . Again this yields a contradiction. **Q.E.D.**

This lemma is effective because we have reduced the condition Lemma 10 which evaluating  $\frac{\partial f^u}{\partial X}$  (or  $\frac{\partial f^d}{\partial X}$ ) at an algebraic number  $t_{2i}$  to evaluating  $\frac{\partial f^u}{\partial X}$  (or  $\frac{\partial f^d}{\partial X}$ ) at bigfloats  $a_{2i}$  and  $b_{2i}$ . Let

$$\Delta(\bar{f}) := \min\left\{\Delta\left(\frac{\partial f^u}{\partial X}\right), \Delta\left(\frac{\partial f^d}{\partial X}\right)\right\}. \quad (17)$$

We must next strengthen this to a necessary and sufficient criterion:

THEOREM 12 (Effective Isolation Criteria). *Let  $K_i = [a_{2i}, b_{2i+1}]$  be an effective candidate. If  $K_i$  is an even effective candidate, further assume that  $b_{2i} - a_{2i} < \Delta(\bar{f})$ . Then  $K_i$  is an isolating interval of  $f$  iff one of the following conditions hold:*

(O)  $K_i$  is an effective odd candidate.

(E)<sup>d</sup>:  $K_i$  is an effective even candidate, with  $f^d(t_{2i}) = 0$  and  $\frac{\partial f^u}{\partial X}$  is negative at  $a_{2i}$  or  $b_{2i}$ .

(E)<sup>u</sup>:  $K_i$  is an effective even candidate, with  $f^u(t_{2i}) = 0$  and  $\frac{\partial f^d}{\partial X}$  is positive at  $a_{2i}$  or  $b_{2i}$ .

*Proof.* As a preliminary remark, we note that  $K_i$  contains at most one zero of  $f$ . To see this, since  $K_i = [a_{2i}, t_{2i}] \cup [t_{2i} \cup t_{2i+1}] \cup [t_{2i+1}, b_{2i+1}]$ , and  $[t_{2i}, t_{2i+1}]$  is a candidate interval, it suffices to show that  $[a_{2i}, t_{2i}]$  and  $[t_{2i+1}, b_{2i+1}]$  has no zero of  $f$ . If  $K_i$  is the first (or the last) effective candidate interval, it is clear that there is no root of  $f$  in  $[a_{2i}, t_{2i}]$  (or  $[t_{2i+1}, b_{2i+1}]$ ). Else, we have  $t_{2i-1} < t_{2i}$  (since  $t_{2i}$  is an odd zero), and so  $f$  has no zeros in  $[t_{2i-1}, t_{2i}] \supseteq [a_{2i}, t_{2i}]$  since these are non-candidate intervals. Similarly, if  $t_{2i+1} < t_{2i+2}$  then  $f$  has no zeros in  $[t_{2i+1}, t_{2i+2}] \supseteq [t_{2i+1}, b_{2i+1}]$ . It is possible that  $t_{2i+1} = t_{2i+2}$ , but again  $f$  has no zeros in the non-candidate interval  $[t_{2i+2}, t_{2i+3}] \supseteq [t_{2i+1}, b_{2i+1}]$ . This completes our justification that  $K_i$  has at most one zero.

Suppose  $K_i$  is an effective odd candidate. Then Lemma 5 shows that  $K_i$  is isolating. Suppose  $K_i$  is an effective even candidate. Assume  $f^d(t_{2i}) = 0$  (the case  $f^u(t_{2i}) = 0$  is similar). Then the previous lemma shows if  $\frac{\partial f^u}{\partial X}$  is negative at  $a_{2i}$  or  $b_{2i}$  then  $K_i$  is isolating.

Conversely, suppose  $K_i$  is isolating. We claim that  $\frac{\partial f^u}{\partial X}$  is negative at  $a_{2i}$  or  $b_{2i}$ . Suppose otherwise:  $\frac{\partial f^u}{\partial X}(a_{2i}) \geq 0$  and  $\frac{\partial f^u}{\partial X}(b_{2i}) \geq 0$ . By Lemma 10, that  $\frac{\partial f^u}{\partial X}(t_{2i}) < 0$ . This implies that  $\frac{\partial f^u}{\partial X}(x) = \frac{\partial f^u}{\partial X}(y) = 0$  for some  $x \in [a_{2i}, t_{2i})$  and  $y \in (t_{2i}, b_{2i}]$ . This is a contradiction since  $|x - y| \leq b_{2i} - a_{2i} \leq \Delta(\bar{f})$ . **Q.E.D.**

### 3. Bounds for Triangular Systems

In this section, we generalize the univariate evaluation and sleeves for a univariate polynomial to a triangular polynomial system  $F_n$  where

$$F_n = \{f_1(x_1), f_2(x_1, x_2), \dots, f_n(x_1, \dots, x_n)\} \quad (18)$$

where  $f_i \in \mathbb{Z}[x_1, \dots, x_i]$ . Generalizing our univariate notation, if  $B \subseteq \mathbb{R}^n$ , let  $\mathbf{Zero}_B(F_n)$  denote the set of real zeros of  $F_n$  restricted to  $B$ .

Let  $B = I_1 \times \dots \times I_n$  be a  $n$ -dimensional box,  $I_i = [a_i, b_i]$ , and  $\xi = (\xi_1, \dots, \xi_{n-1}) \in \square \xi = I_1 \times \dots \times I_{n-1}$  be a real zero of  $F_{n-1} = \{f_1, \dots, f_{n-1}\} = 0$ . Consider the polynomial

$$f(X) := f_n(\xi_1, \dots, \xi_{n-1}, X). \quad (19)$$

We have a three-fold goal in this section:

1. Compute lower estimates on the evaluation  $EB_{I_n}(f)$  and separation bounds  $\Delta_{I_n}(f)$ .
2. Compute a sleeve  $(I_n, f^u, f^d)$  for  $f$  that satisfies the monotonicity property.
3. Compute an upper estimate on the sleeve bound  $SB_{I_n}(f^u, f^d)$ .

#### 3.1. Lower Estimate on Evaluation and Separation Bounds

We give two methods to compute lower estimates on the evaluation bound  $EB_{I_n}(f)$ . The first method is based on a general result about multivariate zero bounds in [22]; the other is based on resultant computation. The same ideas apply to estimating the separation bound  $\Delta_{I_n}(f')$ .

Let  $\Sigma = \{p_1, \dots, p_n\} \subseteq \mathbb{Z}[x_1, \dots, x_n]$  be a system of  $n$  polynomials in  $n$  variables. Assume  $\Sigma$  has finitely many complex zeros. Let  $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$  be one of these zeros. Suppose  $d_i = \deg(p_i)$  and  $K := \max\{\sqrt{n+1}, \|p_1\|_2, \dots, \|p_n\|_2\}$  where  $\|p\|_2$  is the 2-norm of  $p$ . Then we have the following result [22, p. 341]:

**PROPOSITION 13.** *Let  $(\xi_1, \dots, \xi_n)$  be a complex zero of  $\Sigma$ . For any  $i = 1, \dots, n$ , if  $|\xi_i| \neq 0$  then*

$$|\xi_i| > MRB(\Sigma) := (2^{3/2}NK)^{-D} 2^{-(n+1)d_1 \dots d_n}. \quad (20)$$

where

$$N := \left(1 + \sum_{i=1}^n d_i\right), \quad D := \left(1 + \sum_{i=1}^n \frac{1}{d_i}\right) \prod_{i=1}^n d_i.$$

Note that this proposition defines a numerical value  $MRB(\Sigma)$  (the **multivariate root bound**) for any zero-dimensional system  $\Sigma \subseteq \mathbb{Z}[x_1, \dots, x_n]$  of  $n$  polynomials. We will now exploit such a value for suitable  $\Sigma$  associated with  $F_n$  as in (18). Consider the set

$$\widehat{F}_n := \left\{f_1, \dots, f_{n-1}, \frac{\partial f_n(x_1, \dots, x_{n-1}, X)}{\partial X}, Y - f_n(x_1, \dots, x_{n-1}, X)\right\} \quad (21)$$

of  $n + 1$  polynomials in  $\mathbb{Z}[x_1, \dots, x_{n-1}, X, Y]$ .

LEMMA 14. *Let  $(\xi_1, \dots, \xi_{n-1})$  be a zero of  $F_{n-1}$ . The evaluation bound of  $f(X) := f_n(\xi_1, \dots, \xi_{n-1}, X) \in \mathbb{R}[X]$  satisfies  $EB_{I_n}(f) > MRB(\widehat{F}_n)$ .*

*Proof.* As  $F_n$  is zero-dimensional, so is  $\widehat{F}_n$ , which is easily seen. If  $(\xi_1, \dots, \xi_n, y)$  is a zero of  $\widehat{F}_n$ , then  $f'(\xi_n) = 0$ . Moreover,  $y = f(\xi_n)$ . By definition of  $EB(f)$ , we have  $EB(f)$  is the minimum of all such non-zero  $|y|$ 's. By Proposition 13,  $EB_{I_n}(f) > MRB(\widehat{F}_n)$ . **Q.E.D.**

Note that our evaluation bound  $EB_{I_n}(f)$  in this lemma is a global one: it does not depend on the interval  $I_n$ . We do not know any general method to exploit this  $I_n$ .

It is instructive to directly define the **evaluation bound** of a triangular system  $F_n$ : for  $B \subseteq \mathbb{R}^n$ , let  $B' = B \times \mathbb{R}$ . Then define

$$EB_B(F_n) := \min\{|y| : (x_1, \dots, x_{n-1}, x, y) \in \mathbf{Zero}_{B'}(\widehat{F}_n), y \neq 0\}. \quad (22)$$

If the set that which we are minimizing over is empty, then  $EB_B(F_n) = \infty$ . Observe that (22) is a generalization of the corresponding univariate evaluation bound (3). Note that for  $i = 2, \dots, n$ , we similarly have evaluation bounds  $EB_{B_i}(F_i)$  for  $F_i$ , where  $F_i = \{f_1, \dots, f_i\}$ .

This multivariate evaluation bound is a lower bound on the univariate one: with  $f$  given by (19), we have

$$EB_{I_n}(f) \geq EB_B(F_n) > MRB(\widehat{F}_n).$$

As  $MRB(\widehat{F}_n)$  is easily computed, our algorithm can use it as a lower estimate on  $EB(F_n)$ . In general, however,  $MRB(\widehat{F}_n)$  is too pessimistic. So we next propose a computational way to derive a lower estimate, via resultants. In Section 4.5, this computational approach sped up the computation of Example 1 (resp., Example 2) by two (resp., over five) orders of magnitude. Consider  $\widehat{F}_n$  defined by (21). Let

$$e_i = \begin{cases} \mathbf{res}_X(Y - f_n, \frac{\partial f_n}{\partial X}) & i = n, \\ \mathbf{res}_{x_i}(e_{i+1}, f_i) & i = n - 1, \dots, 1 \end{cases} \quad (23)$$

where  $\mathbf{res}_x(p, q)$  is the resultant of  $p$  and  $q$  relative to  $x$ . Thus  $e_1 \in \mathbb{F}[Y]$ . If  $e_1 \neq 0$ , define

$$R(F_n) := \min\{|z| : e_1(z) = 0, z \neq 0\}.$$

If  $e_1$  has no real roots, let  $R(F_n) = \infty$ .

LEMMA 15. *If  $e_1 \neq 0$ ,  $EB(F_n) \geq R(F_n)$ , and we can use  $R(F_n)$  as the evaluation bound.*

*Proof.* From properties of the resultant,  $e_1(Y)$  is a linear combination of the polynomials in  $\widehat{F}_n$ . If  $(x_1, \dots, x_{n-1}, x, y) \in \mathbf{Zero}(\widehat{F}_n)$ ,  $y$  must be a zero of  $e_1(Y) = 0$ . From (23), we conclude  $EB(F_n) \geq R(F_n)$ . **Q.E.D.**

Therefore, we may isolate the real roots of  $e_1(Y) = 0$  and take  $\min\{l_1, -r_2\}$  as the evaluation bound for  $F_n$ , where  $(l_1, r_1)$  and  $(l_2, r_2)$  are the isolating intervals for the smallest positive root and the largest negative root of  $e_1(Y) = 0$  respectively.

#### Lower Estimate for Separation Bound.

We similarly need a lower estimate on the separation bound  $\Delta(g)$ , where  $g \in \mathbb{F}[X]$ . Consider the system  $\Xi$  comprising the following 3 polynomials:

$$g(X), g(Y), Z - X + Y. \quad (24)$$

Thus for any zero  $(x, y, z) \in \mathbf{Zero}(\Xi)$ , we have  $x, y$  are zeros of  $g$ . Moreover,  $z = x - y$  and so  $z \neq 0$  implies  $|z| \geq \Delta(g)$ . This proves that  $MRB(\Xi)$  is a lower bound on  $\Delta(g)$ .

Then we can compute  $\Delta(\bar{f}) = \min\{\Delta(\frac{\partial f^u}{\partial X}), \Delta(\frac{\partial f^d}{\partial X})\}$ , where  $f$  is given by (19) and  $(f^u, f^d)$  is its corresponding sleeve.

We also denote  $\Delta(\bar{f})$  as  $\Delta_B(F_n)$  for conveniency.

### 3.2. Construction of a Sleeve

Our construction depend on  $I_n$  only in a very minimal way: we need only to assume a definite sign in  $I_n$ . This means  $0 \notin I_n$ , or equivalently, either  $I_n > 0$  or  $I_n < 0$ . In fact, the construction depends on the signs of each of the intervals  $I_1, \dots, I_{n-1}$ . We will assume that  $I_i > 0$  for  $i = 1, \dots, n$ ; below we indicate how to reduce the general case to this ‘‘positive’’ case.

Given a polynomial  $g \in \mathbb{R}[x_1, \dots, x_n]$ , we may decompose it uniquely as  $g = g^+ - g^-$ , where  $g^+, g^- \in \mathbb{R}[x_1, \dots, x_n]$  each has only positive coefficients, and the support of  $g^+$  and  $g^-$  are both minimum. Here, the support of a polynomial  $g$  is the set of power products with non-zero coefficients in  $g$ .

Given  $f$  as in (19) and an isolating box  $\square\xi \in \square\mathbb{F}^{n-1}$  for  $\xi$ , following [16, 18], we define

$$\begin{aligned} f^u(X) &:= f_n^u(\square\xi; X) = f_n^+(b_1, \dots, b_{n-1}, X) - f_n^-(a_1, \dots, a_{n-1}, X), \\ f^d(X) &:= f_n^d(\square\xi; X) = f_n^+(a_1, \dots, a_{n-1}, X) - f_n^-(b_1, \dots, b_{n-1}, X) \end{aligned} \quad (25)$$

where  $f_n = f_n^+ - f_n^-$  and  $\square\xi = [a_1, b_1] \times \dots \times [a_{n-1}, b_{n-1}]$ .

We briefly indicate two possible solutions when our assumption that  $I_i > 0$  fails. Perhaps the simplest is to shift the origin of  $F_n$  so that the box  $I_1 \times \dots \times I_n$  lies in the first quadrant of  $\mathbb{R}^n$ . E.g., replace  $x_i$  by  $x_i - a_i$  in  $F_n$  and replace  $I_i$  by  $a_i + I_i$ . Alternatively, proceed as follows: for each  $i$ , if  $\xi_i = 0$ , we can replace  $x_i$  in  $f_n(x_1, \dots, x_n)$  by 0. After this, we can split  $I_i$  if necessary so that  $I_i > 0$  or  $I_i < 0$ . For each  $i$  such that  $I_i < 0$ , we replace  $x_i$  in  $f_n(x_1, \dots, x_n)$  by  $-x_i$ . Let  $\bar{f}_n(x_1, \dots, x_n)$  denote the polynomial after these replacements. Now we may carry out the construction of (25)  $\bar{f}_n$  with the box  $B' = I'_1 \times \dots \times I'_n$  where  $I'_i$  is  $-I_i$  iff  $I_i < 0$  and otherwise  $I'_i = I_i$ .

From the construction, it is clear that  $f^u \geq f \geq f^d$ . Moreover, both inequalities are strict if  $a_i = \xi_i = b_i$  does not hold for any  $i = 1, \dots, n-1$ . Hence  $(I_n, f^u(X), f^d(X))$  is a sleeve for  $f(X)$  [16, 18]. We further have:

LEMMA 16. *Over any positive interval  $I_n = [l, r] > 0$ , we have:*

- (i) *(Monotonicity)  $\frac{\partial f^u}{\partial X} \geq \frac{\partial f}{\partial X} \geq \frac{\partial f^d}{\partial X}$ .*
- (ii)  *$f^u(X) - f^d(X)$  is monotonously increasing over  $I_n$ .*

*Proof.* Let  $f(X) = f_n(\xi_1, \dots, \xi_{n-1}, X) = f_n^+(\xi_1, \dots, \xi_{n-1}, X) - f_n^-(\xi_1, \dots, \xi_{n-1}, X)$  and

$$\begin{aligned} T_1(X) &= f^u(X) - f(X) \\ &= (f_n^+(b_1, \dots, b_{n-1}, X) - f_n^+(\xi_1, \dots, \xi_{n-1}, X)) \\ &\quad + (f_n^-(\xi_1, \dots, \xi_{n-1}, X) - f_n^-(a_1, \dots, a_{n-1}, X)), \\ T_2(X) &= f(X) - f^d(X) \\ &= (f_n^+(\xi_1, \dots, \xi_{n-1}, X) - f_n^+(a_1, \dots, a_{n-1}, X)) \\ &\quad + (f_n^-(b_1, \dots, b_{n-1}, X) - f_n^-(\xi_1, \dots, \xi_{n-1}, X)), \\ T_3(X) &= f^u(X) - f^d(X) \\ &= (f_n^+(b_1, \dots, b_{n-1}, X) - f_n^+(a_1, \dots, a_{n-1}, X)) \\ &\quad + (f_n^-(b_1, \dots, b_{n-1}, X) - f_n^-(a_1, \dots, a_{n-1}, X)). \end{aligned}$$

Since  $f_n^+, f_n^-$  are polynomials with positive coefficients and  $0 < a_i \leq \xi_i \leq b_i$  for all  $i$ ,  $f_n^+(b_1, \dots, b_{n-1}, X) - f_n^+(\xi_1, \dots, \xi_{n-1}, X)$ ,  $f_n^-(\xi_1, \dots, \xi_{n-1}, X) - f_n^-(a_1, \dots, a_{n-1}, X)$ , and hence  $T_1(X)$  are polynomials in  $X$  with positive coefficients. Similarly,  $T_2(X)$  and  $T_3(X)$  are polynomials with positive coefficients. For  $x > 0$ , we have  $\frac{\partial T_1(x)}{\partial X} = \frac{\partial f^u(x)}{\partial X} - \frac{\partial f(x)}{\partial X} \geq 0$ . Similarly, we can show that  $\frac{\partial T_2(x)}{\partial X} = \frac{\partial f(x)}{\partial X} - \frac{\partial f^d(x)}{\partial X} \geq 0$ , and  $\frac{\partial T_3(x)}{\partial X} = \frac{\partial f^u(x)}{\partial X} - \frac{\partial f^d(x)}{\partial X} \geq 0$ . Thus  $\frac{\partial f^u}{\partial X} \geq \frac{\partial f}{\partial X} \geq \frac{\partial f^d}{\partial X}$ . As consequence,  $f^u(X) - f^d(X)$  is monotone increasing in  $I_n$ . **Q.E.D.**

As an immediate corollary, we obtain an upper estimate on the sleeve bound:

COROLLARY 17.

$$SB_{I_n}(f^u, f^d) \leq f^u(r) - f^d(r). \quad (26)$$

### 3.3. Upper Estimate on Sleeve Bound

How good is the upper estimate (26)? Our next goal is to give an upper bound on  $f^u(r) - f^d(r)$  as a function of

$$b := \max\{b_1, \dots, b_n\}, \quad w := \max\{w_1, \dots, w_n\}$$

where  $w_i = b_i - a_i$ . Also let  $\mathbf{w} = (w_1, \dots, w_n)$ . For  $f \in \mathbb{R}[x_1, \dots, x_n]$ , write  $f = \sum_{\alpha} c_{\alpha} p_{\alpha}(x_1, \dots, x_n)$  where  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , and  $p_{\alpha}(x_1, \dots, x_n)$  denotes the monomial  $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ . Let  $\|f\|_1 := \max_{\alpha} |c_{\alpha}|$  denote its 1-norm. The inner product of two vectors, say  $\mathbf{w}$  and  $\alpha$ , is denoted  $\langle \mathbf{w}, \alpha \rangle = \sum_{i=1}^n w_i \alpha_i$ .

LEMMA 18. *Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  where  $m = \sum_{i=1}^n \alpha_i \geq 1$ . Then*

$$p_{\alpha}(b_1, \dots, b_n) - p_{\alpha}(a_1, \dots, a_n) \leq b^{m-1} \langle \alpha, \mathbf{w} \rangle \leq w m b^{m-1}.$$

*Proof.* We have

$$X^m - Y^m = (X - Y)(X^{m-1} + X^{m-2}Y + \cdots + Y^{m-1}) \leq (X - Y)mX^{m-1}, \quad (27)$$

provided  $X \geq Y \geq 0$  and  $m \geq 1$ . Then, assuming each  $\alpha_i \geq 1$ ,

$$\begin{aligned} p_{\alpha}(b_1, \dots, b_n) - p_{\alpha}(a_1, \dots, a_n) &= \sum_{i=1}^n \left\{ \left( \prod_{j=1}^{i-1} a_j^{\alpha_j} \right) (b_i^{\alpha_i} - a_i^{\alpha_i}) \left( \prod_{k=i+1}^n b_k^{\alpha_k} \right) \right\} \\ &\leq \sum_{i=1}^n w_i \alpha_i \left\{ \left( \prod_{j=1}^{i-1} a_j^{\alpha_j} \right) (b_i^{\alpha_i-1}) \left( \prod_{k=i+1}^n b_k^{\alpha_k} \right) \right\} \quad (\text{by (27)}) \\ &\leq \sum_{i=1}^n w_i \alpha_i \left\{ \left( \prod_{j=1}^{i-1} b_j^{\alpha_j} \right) (b_i^{\alpha_i-1}) \left( \prod_{k=i+1}^n b_k^{\alpha_k} \right) \right\} \\ &= b^{m-1} \sum_{i=1}^n w_i \alpha_i. \end{aligned}$$

In general, if any  $\alpha_i = 0$ , the corresponding term in the summation could be omitted in the above derivation, and the proof remains valid. **Q.E.D.**

For example, if each  $\alpha_i = m/n$  then  $\sum_{i=1}^n w_i \alpha_i \leq mw/n$ .

COROLLARY 19. *Let  $f = \sum_{\alpha} c_{\alpha} p_{\alpha}(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$ . If each coefficient  $c_{\alpha}$  is positive and  $m = \deg(f) \geq 1$ , then*

$$f(b_1, \dots, b_n) - f(a_1, \dots, a_n) \leq b^{m-1} \sum_{\alpha} |c_{\alpha}| \langle \mathbf{w}, \alpha \rangle \leq w m b^{m-1} \|f\|_1.$$

*Proof.* The above lemma extends linearly to a polynomial  $f$ :

$$\begin{aligned} f(b_1, \dots, b_n) - f(a_1, \dots, a_n) &= \sum_{\alpha} c_{\alpha} (p_{\alpha}(b_1, \dots, b_n) - p_{\alpha}(a_1, \dots, a_n)) \\ &\leq \sum_{\alpha} |c_{\alpha}| \langle \alpha, \mathbf{w} \rangle \\ &\leq wmb^{m-1} \|f\|_1. \end{aligned}$$

**Q.E.D.**

**THEOREM 20.** *Let  $(I_n, f^u, f^d)$  be a sleeve as in (25), and  $\square_{n-1}\xi = I_1 \times \dots \times I_{n-1}$  an isolating box for  $\xi \in \mathbb{R}^{n-1}$ , where  $I_i = [a_i, b_i] > 0$ ,  $I_n = [l, r] > 0$ , and  $w = \max_{i=1}^{n-1} \{b_i - a_i\}$ . Then*

$$SB_I(f^u, f^d) \leq wm \|f_n\|_1 b^{m-1},$$

where  $m = \deg(f_n)$ ,  $b = \max\{b_1, \dots, b_{n-1}, r\}$ .

*Proof.* Let  $f(X) = \sum_{i=0}^m C_i(\xi_1, \dots, \xi_{n-1})X^i$  where  $C_i \in \mathbb{Z}[x_1, \dots, x_{n-1}]$  has degree  $\leq m - i$ ,  $C_i = C_i^+ - C_i^-$ ,  $\mathbf{a} = (a_1, \dots, a_{n-1})$ , and  $\mathbf{b} = (b_1, \dots, b_{n-1})$ . We have  $f^u(X) = \sum_{i=0}^m (C_i^+(\mathbf{b}) - C_i^-(\mathbf{a}))X^i$ ,  $f^d(X) = \sum_{i=0}^m (C_i^+(\mathbf{a}) - C_i^-(\mathbf{b}))X^i$ . For  $x \in I_n$ , we have

$$\begin{aligned} f^u(x) - f^d(x) &= \sum_{i=0}^m (C_i^+(\mathbf{b}) - C_i^+(\mathbf{a}) + C_i^-(\mathbf{b}) - C_i^-(\mathbf{a}))x^i \\ &\leq \sum_{i=0}^m w(m-i)b^{m-i-1} (\|C_i^+\|_1 + \|C_i^-\|_1)b^i \\ &< wmb^{m-1} \sum_{i=0}^m \|C_i\|_1 = wmb^{m-1} \|f_n\|_1. \end{aligned}$$

**Q.E.D.**

We give two corollaries to the above theorem.

**COROLLARY 21.** *For a fixed  $F_n$  and  $I_n$ , when  $w \rightarrow 0$ ,  $SB_{I_n}(f^u, f^d) \rightarrow 0$ .*

So when  $w \rightarrow 0$ ,  $f^u \rightarrow f$  and  $f^d \rightarrow f$ . The correctness of our algorithm follows from the fact with sufficient refinement, the sleeve-evaluation inequality (5) will eventually hold. The next corollary gives an explicit condition to guarantee this:

**COROLLARY 22.** *The sleeve-evaluation inequality (5) holds provided*

$$w < \frac{EB_{I_n}(f)}{m \|f_n\|_1 b^{m-1}}. \quad (28)$$

#### 4. The Main Algorithm

In this section, we present our root isolation algorithm for a triangular system: given  $F_n$  as in (18), to isolate the real zeros of  $F_n$  in a given  $n$ -dimensional box  $B = I_1 \times \dots \times I_n$ . But first, we outline the method for the case  $n = 2$ . Most of the issues in the general algorithm already appear in this case, but the notations are more transparent. We also give two subalgorithms for root refinement and an effective method for verifying zeros.



#### 4.1. Bivariate Isolation Algorithm

Suppose we are given  $f(X), g(X, Y) \in \mathbb{Z}[X, Y]$ , and we want to compute the zeros of  $f = g = 0$  in a box  $B = I \times J$  ( $I, J \in \square\mathbb{F}$ ) to within precision  $\epsilon$ . The algorithm proceeds as follows:

INPUT:  $\Sigma = \{f(X), g(X, Y)\}$ ,  $B = I \times J$ ,  $\epsilon$ .  
 OUTPUT: Isolating boxes of  $\Sigma$  to precision  $\epsilon$ .

1. Compute an evaluation bound  $EB = EB_B(f, g)$  as defined in (22).
2. Isolate all the zeros of  $f$  in  $I$ .
3. For each  $[a, b]$  that isolates some zero  $\xi \in \text{Zero}_I(f)$ , we perform the following:
  - 3.1 Initialize sleeve bound,  $SB \leftarrow \infty$ .
  - 3.2 while ( $EB \leq SB$ ) do
    - 3.2.1 Refine the isolating interval  $K = [a, b]$  of  $\xi$ .
    - 3.2.2 Compute a sleeve  $(J, g^u, g^d)$  for  $g(Y) = g(\xi, Y)$ .
    - 3.2.3 Compute the sleeve bound  $SB := SB(J, g^u, g^d)$ .
  - 3.3 Isolate the zeros of  $g^u g^d$ .
    - 3.3.1 Compute parity of each zero of  $g^u g^d$ .
    - 3.3.2 Form the sorted list  $SL_{g, J}$ .
    - 3.3.3 From this, we have the effective candidate intervals  $K_i$  for various  $i$ .
  - 3.4 for each effective candidate  $K_i$ , apply Theorem 12 to check if  $K_i$  is an isolating interval.
  - 3.5 Refine the isolating intervals to the given precision  $\epsilon$ .

We note various operations assumed in the above algorithm:

- Step 1: Compute an evaluation bound  $EB_B(f, g)$ . This is given in Section 3.1..
- Steps 2 and 3.3: Isolate the zeros of polynomials in  $\mathbb{F}[X]$ .
- Steps 3.2.1, 3.4, and 3.6: We need to refine an isolating interval  $[a, b]$ , which will be discussed in Section 4.2..
- Step 3.2.2: Given  $[a, b]$  we need to compute a sleeve  $g^u, g^d$ . This is given in Section 3.3..
- Step 3.2.3: Given a sleeve,  $(J, g^u, g^d)$ , compute a sleeve bound. This is given in Section 3.3..
- Step 3.3.1: We need to compute the parity. If  $[c, d]$  is an isolating interval of a real root  $\xi$  of  $h = g^u g^d = 0$ . Assuming  $c < \xi < d$ . Then  $\xi$  is an odd root of  $h = 0$  iff  $h(c)h(d) < 0$ ;  $\xi$  is an even root of  $h = 0$  iff  $h(c)h(d) > 0$ .
- Step 3.4: According to Theorem 12, we need to detect the sign of  $\frac{\partial f^u}{\partial X}$  (or  $\frac{\partial f^d}{\partial X}$ ) at  $a_{2i}$  or  $b_{2i}$  when  $K_i = [a_{2i}, b_{2i+1}]$  is an even effective candidate interval under  $b_{2i} - a_{2i} < \Delta(\bar{f})$  (see (17)). In fact, when the condition in Lemma 11 holds, we need not to ensure that  $b_{2i} - a_{2i} < \Delta(f)$ .

## 4.2. Refinement of Isolating Box

Since refining an isolating box is an essential subroutine in our algorithm, we now provide more details of this subalgorithm. Let  $\square_n \xi = \square_{n-1} \xi \times [c, d] > 0$  be an isolating box for a zero  $\xi = (\xi_1, \dots, \xi_n)$  of  $F_n$ . With  $f(X) = f_n(\xi_1, \dots, \xi_{n-1}, X)$  as usual, we construct the sleeve  $([c, d], f^d, f^u)$  associated with  $\square_n \xi$  satisfying the sleeve-evaluation inequality (5) and the monotonicity property (12). Suppose  $\square'_{n-1} \xi$  is a **proper refinement** of  $\square_{n-1} \xi$ , i.e.,  $\square'_{n-1} \xi \subsetneq \square_{n-1} \xi$  (proper subset). We obtain the corresponding sleeve functions:

$$\begin{aligned}\bar{f}^u(X) &= f_n^u(\square'_{n-1} \xi, X) \text{ (see definition in (25))}, \\ \bar{f}^d(X) &= f_n^d(\square'_{n-1} \xi, X).\end{aligned}$$

LEMMA 23. *Let  $t_0 < t_1$  be distinct zeros of  $f^u f^d$  in  $[c, d]$ , and  $t'_0 < t'_1$  be the two smallest zeros of  $\bar{f}^u \bar{f}^d$  in  $[c, d]$ . If  $\square'_{n-1}$  is not a point (i.e.,  $\square \neq (\xi_1, \dots, \xi_{n-1})$ ), then  $\xi \in \square'_{n-1} \xi \times [t'_0, t'_1] \subseteq \square_n \xi$ .*

*Proof.* From  $\square'_{n-1} \xi \subsetneq \square_{n-1} \xi$ ,  $\square'_{n-1} \neq [\xi_1, \xi_1] \times \dots \times [\xi_{n-1}, \xi_{n-1}]$ , and (25), we have

$$f^d(x) < \bar{f}^d(x) < f(x) < \bar{f}^u(x) < f^u(x), \forall x \in [c, d].$$

It is not difficult to check that sleeve-evaluation inequality (5) and the monotonicity property (12) hold for the sleeve  $([c, d], \bar{f}^u, \bar{f}^d)$ . Wlog, we assume  $f^u(t_0) = 0, f^d(t_1) = 0$ . The proofs for other cases are similar. We have  $\bar{f}^u(t_0) < f^u(t_0) = 0$  and  $\bar{f}^u(\xi_n) > f(\xi_n) = 0$ . Then  $\bar{f}^u$  has at least one root in  $(t_0, \xi_n)$ . Since  $(t_0, \xi_n) \subset A_{\xi_n}$ , by Lemma 7,  $\bar{f}^u(x)$  has a unique real root in  $(t_0, \xi_n)$ . Let  $t'_0$  be this root. Then,  $t'_0 > t_0$ . Since  $\bar{f}^u(x) < f^u(x) < 0$ ,  $\bar{f}^u$  has no real roots in  $[c, t_0]$  and  $t'_0$  is the smallest root of  $\bar{f}^u \bar{f}^d = 0$  in  $[c, d]$ . Similarly, we could show that  $\bar{f}^d(x) = 0$  has at least one root in  $(\xi_n, t_1)$ . Let  $t'_1$  be the smallest of these roots. Then  $t'_0$  and  $t'_1$  are the two smallest roots of  $\bar{f}^u \bar{f}^d = 0$  in  $[c, d]$  and  $\xi_n \in (t'_0, t'_1) \subset [t_0, t_1]$ . **Q.E.D.**

The lemma tells us how to refine the isolating box of a triangular system without checking which of the subdivided intervals is the isolating interval with Theorem 12.

Refine( $F_n, K, \epsilon$ )

**Input:**  $K = I_1 \times \dots \times I_n$  (an isolating box of the triangular system  $F_n$ ) and  $\epsilon$  (a given precision).

**Output:** A refined isolating box  $\hat{K} = \hat{I}_1 \times \dots \times \hat{I}_n$  of  $K$  such that  $w = \max_{j=1}^n \{|\hat{I}_j|\} \leq \epsilon$ .

1. If  $n = 1$ , subdivide  $I_n$  by half until  $|I_n| < \epsilon$  and return  $I_n$ .
2. Let  $K_{n-1} = I_1 \times \dots \times I_{n-1}$ .  
 $w = \max_{j=1}^n \{|I_j|\}$ .  
 If  $w \leq \epsilon$ , return  $K$ .  
 $\delta = \epsilon$ .
3. while  $w > \epsilon$ , do
  - 3.1.  $\delta = \delta/2$ .
  - 3.2.  $K_{n-1} \leftarrow \text{Refine}(F_{n-1}, K_{n-1}, \delta)$ .
  - 3.3. If  $K_{n-1}$  is a point,  $f(X) \leftarrow f_n(\xi_1, \dots, \xi_{n-1}, X)$  is a univariate polynomial with rational coefficients. Subdivide  $I_n$  by half until  $|I_n| < \epsilon$  and return  $I_n$ .
  - 3.4. Compute the sleeve:  $f^u(X) \leftarrow f_n^u(K_{n-1}, X), f^d(X) \leftarrow f_n^d(K_{n-1}, X)$ .
  - 3.5. Isolate the real roots of  $f^u f^d$  in  $I_n$  with precision  $\delta$ .
  - 3.6. Denote the first two intervals as  $[c_1, d_1], [c_2, d_2]$ .
  - 3.7.  $w \leftarrow d_2 - c_1$ .
4. Return  $\hat{K} \leftarrow K_{n-1} \times [c_1, d_2]$ .

**Remark:** In step 3.3, when  $K_{n-1}$  is a point,  $K_{n-1} = [\xi_1, \xi_1] \times \cdots \times [\xi_{n-1}, \xi_{n-1}] \in \mathbb{F}^{n-1}$ , where  $(\xi_1, \dots, \xi_{n-1})$  is the real root of  $F_{n-1}$  in  $I_1 \times \cdots \times I_{n-1}$ .

*Proof of Correctness.* By Lemma 23, we need only select the first two isolating intervals of  $f^d f^u = 0$ . By Corollary 21, when  $|K_{n-1}| \rightarrow 0$ ,  $f^u \rightarrow f$  and  $f^d \rightarrow f$ . Since we isolate the real roots of  $f^u f^d$  in  $I_n$  with precision  $\delta$ , after enough subdivision,  $w$  will be smaller than  $\epsilon$  and the algorithm will terminate. ■

### 4.3. Verifying Zeros

We are given a box  $B = I_1 \times \cdots \times I_k$ , a triangular system  $\Sigma = \{h_1(x_1), \dots, h_k(x_1, \dots, x_k)\}$  and a polynomial  $g(x_1, \dots, x_k) \in \mathbb{Z}[x_1, \dots, x_k]$ . If  $B$  is the isolating box of a zero  $\xi = (\xi_1, \dots, \xi_k) \in \mathbf{Zero}(\Sigma)$ , we will provide a subroutine to verify whether  $g(\xi_1, \dots, \xi_k) = 0$ . Consider the polynomial system:

$$\Sigma_g = \{h_1(x_1), h_2(x_1, x_2), \dots, h_k(x_1, \dots, x_k), Y - g(x_1, \dots, x_k)\}. \quad (29)$$

We define a **generalized evaluation bound**,

$$EB_B(g; \Sigma) := \inf\{|y| : (x_1, \dots, x_k, y) \in \mathbf{Zero}_B(\Sigma_g), y \neq 0\}.$$

Note that the evaluation bound definition (22) can be reduced to this case. It is easily seen that if  $y_0 := g(\xi_1, \dots, \xi_k) \neq 0$  then  $|y_0| \geq EB_B(g; \Sigma)$ . By Proposition 13, we obtain the lower estimate  $EB_B(g; \Sigma) > MRB(\Sigma_g)$ . We again provide a resultant-based computational estimate, as follows: define the sequence  $r_{k+1}, r_k, \dots, r_1$  of polynomials where  $r_{k+1} := Y - g(x_1, \dots, x_k)$  and  $r_i := \mathbf{res}_{x_i}(h_i(x_1, \dots, x_i), r_{i+1}(x_1, \dots, x_i, Y))$  for  $i = k, \dots, 1$ . Then  $r_1 = r_1(Y)$ . If  $r_1 \not\equiv 0$ , let  $\xi_+$  (resp.,  $\xi_-$ ) be the smallest positive zero of  $r_1(Y)$  (resp.,  $r_1(-Y)$ ). If  $y_0 = g(\xi_1, \dots, \xi_k) \neq 0$  then  $|y_0| \geq \min\{\xi_+, \xi_-\}$ ; this is because  $y_0$  is a zero of  $r_1(Y) = 0$ . We can compute lower bounds on  $\xi_+, \xi_-$  via root isolation. Note that the second method is only complete for regular triangular systems (when  $r_1 \not\equiv 0$ ).

We give the following algorithm.

ZeroTest( $F_n, K = I_1 \times \cdots \times I_n, g(x_1, \dots, x_n)$ )  
**Input:** an isolating box  $K$  of a zero  $\xi$  of triangular system  $F_n$ ,  $g \in \mathbb{F}[x_1, \dots, x_n]$ ,  $I_i = [a_i, b_i] > 0$ .  
**Output:** TRUE iff  $g(\xi) = 0$ .

1.  $\delta = \max_{j=1}^n \{|I_j|\}$ .
2. Compute bounds similar to a sleeve of  $g$ :  
 $g^u = g^+(b_1, \dots, b_n) - g^-(a_1, \dots, a_n)$ ,  
 $g^d = g^+(a_1, \dots, a_n) - g^-(b_1, \dots, b_n)$ .
3. If  $g^d = g^u$ , then  $g = g^d = g^u$ . If  $g^d = 0$  return TRUE; otherwise return FALSE.
4. If  $g^u g^d \geq 0$ , then  $g \neq 0$  and return FALSE.
5. Compute the zero bound  $\rho$  if it does not exist.
6. If  $|g^u| < \rho$ , and  $|g^d| < \rho$ , then  $g < \rho$  and hence  $g = 0$  and return TRUE.
7.  $\delta = \delta/2$ ,  $K = \mathbf{Refine}(F_n, K, \delta)$ , and goto step 2.

*Proof of Correctness.* From the construction, we have  $g^d \leq g \leq g^u$ . If  $g^d = g^u$ , then  $g = g^d = g^u$  and  $g = 0$  iff  $g^d = 0$ . If  $g^u g^d \geq 0$ , then  $g \neq 0$ . Note that  $g^d < g < g^u$  in this case. The sign of  $g$  is the same as the sign of  $\mathbf{sign}(g^u)$  or  $\mathbf{sign}(g^d)$ . In the two cases, we need not to compute zero bound of  $g$ . If  $g^u g^d < 0$ , we need to compute the zero bound  $\rho$ . If  $|g^u| < \rho$  and  $|g^d| < \rho$ , then  $g < \rho$  and hence  $g = 0$  by the definition of the zero bound. It is obvious that the algorithm will terminate since  $g^u$  and  $g^d$  approach  $g$  when  $|K| \rightarrow 0$ . ■

#### 4.4. Isolation Algorithm

We now give the main algorithm of this paper. Note that our algorithm can detect whether the input triangular system is positive dimensional.

RootIsol( $F_n, B_n, \epsilon$ )  
**Input:**  $F_n \subseteq \mathbb{Z}[x_1, \dots, x_n]$ ,  $B_n = \prod_{i=1}^n I_i \subset \square \mathbb{F}^n$  with  $I_i > 0$  and  $\epsilon > 0$ .  
**Output:** An isolating set for  $\square \text{Zero}_{B_n}(F_n)$ , or else “ $F_n$  is positive dimensional”.

1. Compute an isolating interval set  $\square \text{Zero}_{B_1}(F_1)$  for  $F_1$  to precision  $\epsilon$ .  
*Result* :=  $\square \text{Zero}_{B_1}(F_1)$ . *NewResult*  $\leftarrow \emptyset$ .  
 If *Result* =  $\emptyset$ , return *Result*.
2. For  $i$  from 2 to  $n$ , do
  - 2.1. Compute an evaluation bound  $EB_i := EB(F_i)$  for  $F_i$ .
  - 2.2.  $\delta \leftarrow \epsilon$ .
  - 2.3. while *Result*  $\neq \emptyset$ , do
    - 2.3.1. Choose an element  $\square_{i-1}\xi$  from *Result*.  
*Result*  $\leftarrow \text{Result} \setminus \{\square_{i-1}\xi\}$ .
    - 2.3.2. Let  $f(X) \leftarrow f_i(\xi_1, \dots, \xi_{i-1}, X) = \sum_k c_k(\xi_1, \dots, \xi_{i-1})X^k$ .  
 If ZeroTest( $F_{i-1}, \square_{i-1}\xi, c_k(x_1, \dots, x_{i-1})$ ) = TRUE for all  $k$  then  $f_i \equiv 0$ , then  
 return(“ $F_n$  is positive dimensional”).
    - 2.3.3. Compute the sleeve:  $f^u(X) \leftarrow f_i^u(\square_{i-1}\xi; X)$ ,  $f^d(X) \leftarrow f_i^d(\square_{i-1}\xi; X)$ .
    - 2.3.4. While  $f^u(b_i) - f^d(b_i) \geq EB_i$ ,  
 $\delta \leftarrow \delta/2$  and  $\square_{i-1}\xi \leftarrow \text{Refine}(F_{i-1}, \square_{i-1}\xi, \delta)$ .  
 $f^u(X) \leftarrow f_i^u(\square_{i-1}\xi; X)$ ,  $f^d(X) \leftarrow f_i^d(\square_{i-1}\xi; X)$ .
    - 2.3.5. Isolate the real roots of  $f^u f^d$  in  $I_i$ .
    - 2.3.6. Compute the parity of each root of  $f^u f^d$  in  $I_i$ .
    - 2.3.7. Construct the effective candidate intervals.
    - 2.3.8. for each effective candidate interval  $K$ ,
      - 2.3.8.1. Apply Theorem 12 to decide whether  $K$  is isolating.  
 $K$  is an effective odd candidate interval,  $K$  is isolating;  
 $K$  is an effective even candidate interval:  
 If  $K$  satisfies the condition in Lemma 11,  $K$  is isolating;  
 Else, compute a lower estimate  $\Delta_i = \Delta_{B_n}(F_i) = \Delta(\bar{f})$  by (17),  
 refine  $K$  to the precision  $\Delta_i$ .  
 $K$  is isolating iff it satisfies the condition of Lemma 11.
      - 2.3.8.2. If  $K$  is isolating, then  
 $K \leftarrow \text{Refine}(F_i, K, \epsilon)$ .  
 $\text{NewResult} \leftarrow \text{NewResult} \cup \{\square_{i-1}\xi \times K\}$ .
  - 2.4. If *NewResult* =  $\emptyset$ , return *NewResult*.
  - 2.5. *Result*  $\leftarrow \text{NewResult}$ . *NewResult*  $\leftarrow \emptyset$ .
3. return *Result*.

#### Remarks:

1. In Step 2.1, when the system is not regular, we need to compute the evaluation bound by Proposition 13. When the system is not zero-dimensional, we still use Proposition 13 to compute the evaluation bound. Though the evaluation bound is not right, the system can be detected to be positive dimensional. The algorithm is correct in global sense.
2. Algorithm RootIsol can be improved in several ways.

- In the Section 3.3., we give two methods to compute the sleeve bound. Note that the

algorithm based on (26) is more adaptive. If we use (28) to give the sleeve bound, the Sleeve-Evaluation Inequality holds automatically.

- Theorem 12 gives a criterion to isolate roots in an open interval. We thus need to check whether a rational number  $r$  is a zero of  $f(X)$ . In other words, we need to check whether  $f(r) = f_i(\xi_1, \dots, \xi_{i-1}, r) = 0$ ; this can be done using ZeroTest.
- We will show that the assumption  $B_n > 0$  is reasonable. If we want to obtain the real roots of  $f$  in the interval  $I = (a, b) < 0$ , we may consider  $g(X) = f(-X)$  in the interval  $(-b, -a)$ . If  $0 \in (a, b)$ , we can consider the two parts,  $(a, 0)$  and  $(0, b)$  respectively, since we can check if 0 is a zero of  $f(X)$ .
- If we want to find all the real roots of  $f$ , we first isolate the real roots of  $f$  in  $(0, 1)$ , then isolate the real roots of  $g(X) = X^n * f(1/X)$  in  $(0, 1)$ , and check whether 1 is a root of  $f$ . As a result, we can find all the zeros of  $f(X)$  in  $(0, +\infty)$ . We can find the zeros of  $f(X)$  in  $(-\infty, 0)$  by isolating the zeros of  $f(-X)$  in  $(0, +\infty)$ . Finally, we check whether 0 is a zero of  $f(X)$ .
- Theorem 12 assumes that the sleeves are faithful (see (6)). We will show how to isolate the real roots of  $f$  when the sleeve-evaluation inequality (5) holds but  $|f(a)| < EB_I(f)$  or  $f^u(b)f^d(b) \leq 0$ . In fact, if we replace  $EB_I(f)$  with

$$ET_I(f) := \min\{|f(z)| : z \in \text{Zero}_I(f') \cup \{a\} \setminus \text{Zero}_I(f)\}, \quad (30)$$

then almost all the sleeve  $(I, f^u, f^d)$  is faithful except for  $f(a) = 0$  or  $f(b) = 0$ . If  $f(a) = 0$  or  $f(b) = 0$ , we can ignore the first or last element in  $SL_{f,I}$  to form effective candidate intervals of  $f$ . When  $f(a) = 0$ , the first effective candidate interval may or may not be the isolating interval of  $f$ , we need to check it by Theorem 12. And we need to use the first isolating interval in  $SL_{f,I}$  to decide whether the first effective candidate interval is isolating if the first three elements in  $SL_{f,I}$  are all isolating intervals of  $f^u$  (or  $f^d$ ).

Although we can simply solve the non-faithful problem as mentioned above, when  $f(a)$  or  $f(b)$  is very close but not equal to 0,  $ET_I(f)$  is very small. It is expensive to construct  $(I, f^u, f^d)$  in order to satisfy the sleeve-evaluation inequality (5). In order to avoid this case, we just use  $EB_I(f)$  directly and deal with the non-faithful sleeve case as in the Appendix.

#### 4.5. Worked Examples

We provide some worked examples with multiple zeros. Note that all the rational numbers in our examples are dyadics,  $\mathbb{D}$ .

**Example 1:** Consider the system  $F_2 = \{f_1, f_2\}$  where

$$\begin{aligned} f_1 &= x^4 - 3x^2 - x^3 + 2x + 2, \\ f_2 &= y^4 + xy^3 + 3y^2 - 6x^2y^2 + 4xy + 2xy^2 - 4x^2y + 4x + 2. \end{aligned}$$

We isolate all the real roots of the system to precision  $2^{-4}$  with algorithm RootIsol. Isolating the real roots of  $f_1$  to precision  $2^{-4}$ , we obtain the following isolating intervals:  $[[\frac{-23}{16}, \frac{-11}{8}], [\frac{-5}{8}, \frac{-9}{16}], [\frac{11}{8}, \frac{23}{16}], [\frac{25}{16}, \frac{13}{8}]]$ . Next consider the first positive isolating interval

$\square_1\xi = [\frac{11}{8}, \frac{23}{16}]$ , where  $\xi$  satisfies  $f_1(\xi) = 0$  and  $\xi \in [\frac{11}{8}, \frac{23}{16}]$ . We will isolate the real roots of  $f_2(\xi, y) = 0$  in  $[0, 2]$ .

Computing the evaluation bound with the resultant method introduced in Section 3.1., we have  $EB_2 = \frac{1}{2}$ . The sleeve computed using the interval  $\square_1\xi$  is

$$\begin{aligned} f^u(y) &= -\frac{175}{32}y^2 - \frac{29}{16}y + y^4 + \frac{23}{16}y^3 + \frac{31}{4}, \\ f^d(y) &= -\frac{851}{128}y^2 - \frac{177}{64}y + y^4 + \frac{11}{8}y^3 + \frac{15}{2}. \end{aligned}$$

The sleeve bound of  $([0, 2], f^u, f^d)$  is  $SB = f^u(2) - f^d(2) = \frac{59}{8}$ . Since the sleeve-evaluation inequality (5) does not hold, we refine  $\square_1\xi$ . Let  $\square_1\xi = \text{Refine}(f_1, \square_1\xi, \frac{1}{28}) = [\frac{181}{128}, \frac{363}{256}]$ . We have the new sleeve

$$\begin{aligned} f^u(y) &= -\frac{50475}{8192}y^2 - \frac{9529}{4096}y + y^4 + \frac{363}{256}y^3 + \frac{491}{64}, \\ f^d(y) &= -\frac{204331}{32768}y^2 - \frac{39097}{16384}y + y^4 + \frac{181}{128}y^3 + \frac{245}{32} \end{aligned}$$

with sleeve bound  $SB = f^u(2) - f^d(2) = \frac{949}{2048} < \frac{1}{2} = EB_2$ . The sleeve  $([0, 2], f^u, f^d)$  is faithful (6) since  $f^u(0) = \frac{491}{64} > \frac{1}{2}$ ,  $f^d(0) = \frac{245}{32} > \frac{1}{2}$ ,  $f^u(2) = \frac{2927}{512} > \frac{1}{2}$ ,  $f^d(2) = \frac{10759}{2048} > \frac{1}{2}$ . Isolating  $f^u f^d$  in  $[0, 2]$  to precision  $2^{-8}$ , we obtain  $SL_{f_2, [0, 2]}$ :  $[[\frac{165}{128}, \frac{331}{256}], [\frac{395}{256}, \frac{99}{64}]]$  both with parities 1. The two isolating intervals are both isolating intervals of  $f^d$ . It is an isolating interval of  $f_2(\xi, y)$  by Lemma 9. So there is an even root of  $f_2(\xi, y)$  in  $[0, 2]$  by Theorem 12. It is in  $[\frac{165}{128}, \frac{99}{64}]$ . So  $[\frac{11}{8}, \frac{23}{16}] \times [\frac{165}{128}, \frac{99}{64}]$  is an isolating box of triangular system  $F_2$ .

The isolating box does not satisfy our output precision requirement. Refine the isolating box with Refine, we obtain  $[\frac{181}{128}, \frac{5793}{4096}] \times [\frac{1423}{1024}, \frac{2947}{2048}]$ .

Eventually, we obtain all the isolating boxes for  $F_2 = 0$  in 0.141s with RootIsol. If we use Lemma 14 to compute  $MRB(F_2)$ , we have  $\frac{1}{2^{289}} < MRB(F_2) < \frac{1}{2^{288}}$ , and the time to isolate the roots is 9.282s, about 100 times slower.

**Example 2:** Consider the system  $F_3 = \{f_1, f_2, f_3\}$  where

$$\begin{aligned} f_1 &= x^3 - 2x^2 + 8, \\ f_2 &= 4y^4 + (4x^3 - 8x^2 - 32)y^2 + x^6 - 4x^5 + 4x^4 + 16x^3 - 32x^2 + 64, \\ f_3 &= (2z^2 + 2y^2 + x^3 - 2x^2 - 8)^2 + 32x^3 - 64x^2. \end{aligned}$$

Here  $f_3 = 0$  is a surface in  $\mathbb{R}^3$  discussed in [4] and [7]. We isolate  $F_3$  in the box  $[-5, 5] \times [-5, 5] \times [-5, 5]$  with RootIsol. We deal with the non-faithful case in the appendix of the paper.

At first, isolating  $f_1$  to precision  $2^{-10}$ , we have one isolating interval for the real root  $\xi$  of  $f_1 = 0$ :  $[-\frac{773}{512}, -\frac{1545}{1024}]$ .

Computing the evaluation bound  $EB_2$  of  $F_2$  by resultant computation, we have  $EB_2 = 256$ .

Let  $\bar{f}_2(x, y) := f_2(-x, y)$ . Then we consider  $\bar{f}_2(-\xi, y)$  in  $[0, 5]$  with  $-\xi \in [\frac{1545}{1024}, \frac{773}{512}]$ . Computing its sleeve, we have

$$\begin{aligned} f_2^u &= 4y^4 - \frac{17166515417}{268435456}y^2 + \frac{3622826437025545}{18014398509481984}, \\ f_2^d &= 4y^4 - \frac{2147501437}{33554432}y^2 - \frac{231682548623839247}{1152921504606846976}. \end{aligned}$$

So

$$f_2^u(5) - f_2^d(5) = \frac{1912673888824283727}{1152921504606846976} < EB_2.$$

We can check:  $f_2^u(5) f_2^d(5) > 0$ , but  $f_2^u(0) f_2^d(0) < 0$ . Isolating  $f_2^u f_2^d$  in  $[0, 5]$ , we obtain  $\text{ZERO}_{[0,5]}(f_2^u f_2^d)$ :

$$\left[ \left[ \frac{57}{1024}, \frac{29}{512} \right], \left[ \frac{2047}{512}, \frac{4095}{1024} \right], \left[ 4, \frac{4097}{1024} \right] \right].$$

Since  $\text{ZeroTest}(f_1, [-\frac{773}{512}, -\frac{1545}{1024}], f_2(x, 0)) = \text{TRUE}$ , 0 is a root of  $f_2(\xi, y) = 0$ . Then the first element in  $SL_{f_2, [0,5]}$  should be ignored. And the last two elements form an effective candidate interval  $K$  of  $\bar{f}_2(-\xi, y)$ . Since one element is the isolating interval of  $f_2^u$  and the other is the isolating interval of  $f_2^d$ ,  $K = [\frac{2047}{512}, \frac{4097}{1024}]$  is an isolating interval of  $f_2(-\xi, y)$  by Theorem 12 (O). Then the isolating interval of  $f_2(-\xi, y)$  in  $[0, 5]$  is

$$\left[ [0, 0], \left[ \frac{2047}{512}, \frac{4097}{1024} \right] \right].$$

In the end, we can find all the isolating boxes of  $F_2 = \{f_1, f_2\}$  in  $[-5, 5] \times [-5, 5]$ :

$$\left[ \left[ -\frac{773}{512}, -\frac{1545}{1024} \right] \times \left[ -\frac{4097}{1024}, -\frac{2047}{512} \right], \left[ -\frac{773}{512}, -\frac{1545}{1024} \right] \times [0, 0], \left[ -\frac{773}{512}, -\frac{1545}{1024} \right] \times \left[ \frac{2047}{512}, \frac{4097}{1024} \right] \right].$$

Computing the evaluation bound of  $F_3$  with resultant computation, we obtain  $EB_3 = 256$ . Consider the isolating box  $[-\frac{773}{512}, -\frac{1545}{1024}] \times [-\frac{4097}{1024}, -\frac{2047}{512}]$ . Assume that it is an isolating box of root  $(\xi, \eta)$ . Let  $\bar{f}_3(x, y, z) = f_3(-x, -y, z)$ . So  $\bar{f}_3(-\xi, -\eta, z) = f_3(\xi, \eta, z)$ . We will isolate  $\bar{f}_3(-\xi, -\eta, z)$  in  $(0, 5)$ . Computing its sleeve, we have

$$\begin{aligned} f_3^u &= 4z^4 + \frac{17210002215}{268435456}z^2 + \frac{53978322667881993}{18014398509481984}, \\ f_3^d &= 4z^4 + \frac{2143272579}{33554432}z^2 - \frac{3122152760668065807}{1152921504606846976}. \end{aligned}$$

We derive  $f_3^u(5) - f_3^d(5) = \frac{13429555705511252559}{1152921504606846976} < 256 = EB_3$ . It is easy to check that  $f_3^u(5) f_3^d(5) > 0$  and  $f_3^u(0) f_3^d(0) < 0$ . Isolating  $f_3^u f_3^d$  in  $[0, 5]$ , we obtain:  $\left[ \left[ \frac{105}{512}, \frac{211}{1024} \right] \right]$ , which means that  $f_3(\xi, \eta, z) = 0$  has no roots in  $(0, 5)$ . With  $\text{ZeroTest}$ , we may check that 0 is a root of  $f_3(\xi, \eta, z) = 0$ . That is, the real roots of  $\bar{f}_3(-\xi, -\eta, z)$  in  $[0, 5]$  is  $\left[ [0, 0] \right]$ . In a similar way, we can show that the real roots of  $f_3(\xi, \eta, z)$  in  $[-5, 5]$  is  $\left[ [0, 0] \right]$ .

We can compute all the isolating boxes for  $F_3$  in  $[-5, 5] \times [-5, 5] \times [-5, 5]$ :  $\left[ \left[ -\frac{773}{512}, -\frac{1545}{1024} \right] \times \left[ -\frac{4097}{1024}, -\frac{2047}{512} \right] \times [0, 0], \left[ -\frac{773}{512}, -\frac{1545}{1024} \right] \times [0, 0] \times \left[ -\frac{4097}{1024}, -\frac{2047}{512} \right], \left[ -\frac{773}{512}, -\frac{1545}{1024} \right] \times [0, 0] \times [0, 0], \left[ -\frac{773}{512}, -\frac{1545}{1024} \right] \times [0, 0] \times \left[ \frac{2047}{512}, \frac{4097}{1024} \right], \left[ -\frac{773}{512}, -\frac{1545}{1024} \right] \times \left[ \frac{2047}{512}, \frac{4097}{1024} \right] \times [0, 0] \right]$ . Using Lemma 15 over Lemma 14 improves the time from  $> 40,000s$  to  $0.235s$ .

#### 4.6. Experimental Results

In order to evaluate the effectiveness of our algorithms, we implemented  $\text{RootIsol}$  in Maple 10 and did extensive tests on randomly generated triangular systems. In our implementation, we lower estimate the evaluation bound with the resultant computation method described in Section 3.1.. The most time-consuming parts are the computation of the evaluation bounds for the system and the refinement for the isolating boxes.

We tested our program with three sets of examples. The coefficients of the tested polynomials are between  $-100$  and  $100$ . The precision is set to  $2^{-10}$ . We use the method mentioned



in the **Remarks** for RootIsol to compute *all the real solutions* for the triangular systems. The timings are collected on a PC with a 3.2G CPU and 512M memory.

The first set of examples are sparse polynomials and the results are given in Table 1. We use the Maple command `randpoly({ $x_1, \dots, x_n$ }, degree= $d$ , terms= $t$ )` to generate polynomials with given degree and given number of terms. The *type* of a triangular system  $F_n = \{f_1, \dots, f_n\}$  is a list  $(d_1, \dots, d_n)$  where  $d_i$  is the degree of  $f_i$  in  $x_i$ . The tested triangular systems have types indicated under the TYPE column. TIME is the average running time for each triangular system in seconds. NS is the average number of real solutions for each triangular system. NT is the number of tested triangular systems. NE is the average number of terms in each polynomial.

TYPE	TIME	NS	NT	NE
(3, 3)	0.04862	2.04	100	(4, 10)
(9, 7)	0.52717	3.99	100	(10, 10)
(21, 21)	108.9115	5.45	20	(10, 10)
(3, 3, 3)	0.15783	3.48	100	(4, 10, 10)
(9, 7, 5)	16.20573	8.36	100	(10, 10, 10)
(3, 3, 3, 3)	1.69115	5.64	100	(4, 10, 10, 10)
(3, 3, 3, 3, 3)	159.1199	8.0	10	(4, 10, 10, 10, 10)

Table 1. Timings for solving sparse triangular systems

The second set of examples are dense polynomials and the results are given in Table 2. A triangular system  $F_n = \{f_1, \dots, f_n\}$  of type  $(d_1, \dots, d_n)$  is called *dense* if  $f_i = \sum_{k=0}^{d_i} c_k x_i^k$  and  $\deg(c_k, x_j) = d_j - 1$  for all  $k$  and  $i > j$ .

TYPE	TIME	NS	NT	NE
(3, 3)	0.05355	1.91	100	(3.99, 8.02)
(9, 8)	1.87486	4.26	100	(9.94, 43.98)
(11, 11)	8.78255	4.5	80	(11.975, 72.5)
(16, 14)	50.22294	6.0	100	(16.9, 127.13)
(21, 15)	164.23443	6.22	100	(21.91, 176.8)
(3, 3, 3)	0.38702	2.91	100	(3.99, 7.77, 13.01)
(5, 4, 4)	2.97011	4.88	100	(5.99, 14.72, 24.24)
(5, 5, 5)	33.225275	5.6125	80	(5.9625, 17.775, 42.1375)
(8, 7, 6)	592.1848	7.6	10	(8.9, 36.0, 79.8)
(3, 3, 3, 3)	119.94042	6.96	50	(4.0, 8.12, 12.82, 20.92)
(5, 5, 5, 3)	551.4401	3.4	10	(6.0, 32.1, 42.3, 21.5)

Table 2. Timings for solving dense triangular systems

The third set of test examples are triangular systems with multiple roots and the results are given in Table 3. A triangular system of type  $(d_1, \dots, d_n)$  is generated as follows:  $f_1$  is a random polynomial in  $x_1$  and with degree  $d_1$  in  $x_1$  and  $f_i = a_i^2(b_i x_i + c_i)^{\lfloor \frac{d_i+1}{2} \rfloor - \lfloor \frac{d_i}{2} \rfloor}$  for  $i = 2, \dots, n$ , where  $a_i$  is a random polynomial in  $x_1, \dots, x_i$  of degree  $\lfloor d_i/2 \rfloor$  in  $x_i$ , and  $b_i, c_i$  are random polynomials in  $x_1, \dots, x_{i-1}$ . The column NM gives the average number of multiple roots for the tested triangular systems.

From the above experimental results, we could conclude that our algorithm is capable of handling quite large triangular systems.

## 5. Conclusion

This paper provides a complete algorithm of isolating the real roots for arbitrary zero-

TYPE	TIME	NS	NM	NT	NE
(5, 5)	0.71251	3.71	1.57	100	(5.97, 34.47)
(9, 8)	0.60408	3.1	3.1	100	(9.94, 18.92)
(13, 11)	32.44376	6.55	3.92	100	(13.94, 107.68)
(23, 21)	466.0289	6.15	3.75	20	(24.0, 183.4)
(3, 3, 3)	3.21342	5.59	3.24	100	(3.99, 13.08, 31.71)
(9, 7, 5)	425.95055	12.95	8.15	20	(9.95, 60.85, 100.35)
(3, 3, 3, 3)	130.617	11.15	6.1	20	(4.0, 12.2, 33.7, 62.95)

Table 3. Timings for solving dense triangular systems

dimensional triangular polynomial systems. The key idea is to use a sleeve satisfying the sleeve-evaluation inequality to isolate the roots for a univariate polynomial with algebraic number in its coefficients. We further introduce the new tools of evaluation and separation bounds, as well as methods to estimate them. Even with our current simple implementation, the algorithm is shown to be quite effective for modest size problems. To solve larger problems, the bottle neck of the algorithm is the estimation of evaluation and separation bounds. An important research problem is to derive “local bounds”, i.e., bounds that exploit the box  $B$  in  $\Delta_B(F_n)$  or  $EB_B(F_n)$ . Furthermore, current bounds cannot distinguish among the different components of a zero of  $F_n$ , but we only want to bound the last component.

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**Appendix A. Non-faithful  $(I, f^u, f^d)$  treatment.** If we want to use  $EB_I(f)$  directly, we need to deal with the special cases near the endpoints of  $I$ .

Before solving these cases, we will show a useful lemma. We will define the similar open intervals for any real root  $\xi$  of  $f$  in the whole interval  $(-\infty, +\infty)$  as (9):

$$L_\xi := (l_\xi, \xi), R_\xi := (\xi, r_\xi) \text{ and } M_\xi := (l_\xi, r_\xi), \quad (31)$$

where  $l_\xi := \max\{z : z \in \mathbf{Zero}(\widehat{f} \cdot \bar{f}) \cap (-\infty, \xi)\}$ ,  $r_\xi := \min\{z : z \in \mathbf{Zero}(\widehat{f} \cdot \bar{f}) \cap (\xi, +\infty)\}$ .

LEMMA 24. *Assuming the sleeve-evaluation inequality (5). If  $t \in \mathbb{F}$  and  $|f(t)| < EB_I(f)$ , then we have*

- (i) If  $f(t) > 0$  and  $f'(t) > 0$ , then  $t \in R_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ ;
- (ii) If  $f(t) > 0$  and  $f'(t) < 0$ , then  $t \in L_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ ;
- (iii) If  $f(t) < 0$  and  $f'(t) > 0$ , then  $t \in L_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ ;
- (iv) If  $f(t) < 0$  and  $f'(t) < 0$ , then  $t \in R_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ .

Note that all the isolating intervals are strictly inside  $I$  and  $[a, a]$ ,  $[b, b]$  are not included in  $\mathbb{S}_{\cup, \mathbb{I}}$  even if  $f^u(t)f^d(t) = 0$ , where  $t = a, b$ . If  $[c, d]$  is an isolating interval of  $g$  in an interval  $J$ , we labeled as  $[c, d] \rightsquigarrow \mathbf{Zero}_J(g)$ .

When  $|f(a)| < EB_I(f)$ , there are three cases may happen:  $f^u(a)f^d(a) > 0$ ,  $f^u(a)f^d(a) < 0$  or  $f^u(a)f^d(a) = 0$ . We analyze all the cases and solve them below. Note that we need only to check whether or not there is a root of  $f$  in the first effective candidate interval or between  $a$  and the first root of  $f^u f^d$ .

1. If  $f^u(a)f^d(a) > 0$ , of course,  $f(a) \neq 0$ . In this case, the first effective candidate interval of  $f$  may or may not be the isolating interval of  $f$ . We need to check it. Denote the first two isolating intervals in  $\mathbb{S}_{\cup, \mathbb{I}}$  as  $[a_0, b_0]$ ,  $[a_1, b_1]$ , if they exist.

(a)  $f(a) > 0$ ,

- i.  $\frac{\partial f(a)}{\partial X} > 0$ . We know  $a$  is in some  $R_\xi$  by Lemma 24 (i), where  $\xi \in \mathbf{Zero}(f)$ .

A.  $[a_0, b_0]$ ,  $[a_1, b_1]$  both exist.

- When  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^d)$  iff  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^u)$ , it is clear that  $[a_0, b_1]$  is an isolating interval of  $f$  by Theorem 12(O).
- When both of them are the isolating intervals of  $f^d$ ,  $[a_0, b_1]$  may not be the isolating interval of  $f$  as mentioned in Theorem 12. But we can check it by Theorem 12 (E). We can define  $b_{-1} := a$  (note that  $\frac{\partial f^u(a)}{\partial X} \geq \frac{\partial f(a)}{\partial X} > 0$ ) and refine  $[a_1, b_1]$ , if needed, such that  $\frac{\partial f^u(X)}{\partial X}$  is positive in  $[a_1, b_1]$ . Then check whether  $\frac{\partial f^u(X)}{\partial X}$  has real zeros in  $[b_{-1}, b_1]$ . If so,  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ . If not,  $[a_0, b_1]$  is not an isolating interval of  $f$ .

B.  $\mathbb{S}_{\cup, \mathbb{I}} = \emptyset$ , it is clear that  $\mathbf{Zero}_I(f) = \emptyset$ . In fact, if there is a root of  $f$  in  $I$ , then there must at least exist a root of  $f^d$  in  $I$ .

C.  $\mathbb{S}_{f, I} = \{[a_0, b_0]\}$ . It is clear that  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^d)$ .

- $f(b) > 0$ ,  $\mathbf{Zero}_I(f) = \emptyset$ .
- $f(b) < 0$ , we can easily find that  $[a_0, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- $f(b) = 0$ , it is clear that  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .

- ii.  $\frac{\partial f(a)}{\partial X} < 0$ . We can derive that  $a$  is in some  $L_\xi$  by Lemma 24 (ii), where  $\xi \in \mathbf{Zero}(f)$ .
- A.  $[a_0, b_0], [a_1, b_1]$  both exist. It is clear that  $\xi \in I$  and  $A_\xi = (a, \xi)$ . We can derive that  $[a_0, b_0] \subset A_\xi$  and  $[a_1, b_1] \subset B_\xi$  by Lemma 7. So  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ .
  - B.  $\mathbb{S}_{f,I} = \emptyset$ . It is clear that  $\mathbf{Zero}_I(f) = \emptyset$ .
  - C.  $\mathbb{S}_{f,I} = \{[a_0, b_0]\}$ . Then  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^d)$ .
    - $f(b) > 0$ ,  $\mathbf{Zero}_I(f) = \emptyset$ .
    - $f(b) < 0$ , we can easily find that  $[a_0, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
    - $f(b) = 0$ , it is clear that  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- (b)  $f(a) < 0$ ,
- i.  $\frac{\partial f(a)}{\partial X} > 0$ . It is clear that  $a \in L_\xi$  by Lemma 24 (iii). This case is similar as 1-(a)-(ii).
    - A.  $[a_0, b_0], [a_1, b_1]$  both exist. It is clear that  $\xi \in I$  and  $A_\xi = (a, \xi)$ . We can derive that  $[a_0, b_0] \subset A_\xi$  and  $[a_1, b_1] \subset B_\xi$  by Lemma 7. So  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ .
    - B.  $\mathbb{S}_{\cup, \mathbb{I}} = \emptyset$ . It is clear that  $\mathbf{Zero}_I(f) = \emptyset$ .
    - C.  $\mathbb{S}_{f,I} = \{[a_0, b_0]\}$ . Then  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^u)$ .
      - $f(b) > 0$ , we can easily find that  $[a_0, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
      - $f(b) < 0$ ,  $\mathbf{Zero}_I(f) = \emptyset$ .
      - $f(b) = 0$ , it is clear that  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
  - ii.  $\frac{\partial f(a)}{\partial X} < 0$ . Then  $a \in R_\xi$  for some  $\xi \in \mathbf{Zero}(f)$  by Lemma 24 (iv). This case is similar to 1-(a)-(i).
    - A.  $[a_0, b_0], [a_1, b_1]$  both exist.
      - When  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^u)$  iff  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^d)$ , it is clear that  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$  by Theorem 12(O).
      - When both of them are the isolating intervals of  $f^u$ ,  $[a_0, b_1]$  may not be the isolating interval of  $f$  as mentioned in Theorem 12. But we can check it by Theorem 12 (E). We can define  $b_{-1} := a$  (note that  $\frac{\partial f^d(a)}{\partial X} \leq \frac{\partial f(a)}{\partial X} < 0$ ) and refine  $[a_1, b_1]$ , if needed, such that  $\frac{\partial f^d(X)}{\partial X}$  is negative in  $[a_1, b_1]$ . Then check whether  $\frac{\partial f^d(X)}{\partial X}$  has real zeros in  $[b_{-1}, b_1]$ . If so,  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ . Else,  $[a_0, b_1]$  is not an isolating interval of  $f$ .
    - B.  $\mathbb{S}_{f,I} = \emptyset$ , it is clear that  $\mathbf{Zero}_I(f) = \emptyset$ .
    - C.  $\mathbb{S}_{f,I} = \{[a_0, b_0]\}$ . It is clear that  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^u)$ .
      - $f(b) > 0$ , we can easily find that  $[a_0, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
      - $f(b) < 0$ ,  $\mathbf{Zero}_I(f) = \emptyset$ .
      - $f(b) = 0$ , it is clear that  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
2. If  $f^u(a)f^d(a) < 0$ , from Lemma 3, we need to ignore the first isolating interval, denoted it as  $[a_{-1}, b_{-1}]$ , if it exists, in  $\mathbb{S}_{\cup, \mathbb{I}}$  in order to form candidate interval of  $f$ . There may or

may not exist real roots of  $f$  in  $[a, b_{-1}]$ . We need to check it as below. Let  $[a_0, b_0], [a_1, b_1]$  be the second and third isolating intervals in  $\mathbb{S}_{\cup, \mathbb{I}}$ , if they exist.

- (a)  $f(a) > 0$ , we can derive that  $f^d(a) < 0$  since  $f^d(a) < f(a) < f^u(a)$ .
- i.  $\frac{\partial f(a)}{\partial X} > 0$ . Then  $a \in R_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . We can derive that  $[a_{-1}, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f^d)$  and  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^d)$  if they exist.
    - A.  $[a_{-1}, b_{-1}], [a_0, b_0], [a_1, b_1]$  all exist.
      - If  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^u)$ , it is clear that  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ .
      - If  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^d)$ , we can refine  $[a_{-1}, b_{-1}]$  and  $[a_1, b_1]$ , if needed, such that  $\frac{\partial f^u(X)}{\partial X}$  is positive in the two intervals. If  $\frac{\partial f^u(X)}{\partial X}$  has real root in  $[b_{-1}, b_1]$ , we can derive that  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ . Else, there is no real root of  $f$  in  $[a_0, b_1]$ .
    - B.  $\mathbb{S}_{f,I} = \{[a_{-1}, b_{-1}], [a_0, b_0]\}$ .
      - $f(b) > 0$ , we can find there is no real root of  $f$  in  $I$ .
      - $f(b) < 0$ , we can derive that  $[a_0, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
      - $f(b) = 0$ , it is clear that  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
    - C.  $\mathbb{S}_{f,I} = \{[a_{-1}, b_{-1}]\}$  or  $\mathbb{S}_{f,I} = \emptyset$ . We can find that there is on real root of  $f$  in  $I$ .
  - ii.  $\frac{\partial f(a)}{\partial X} < 0$ . Then  $a \in L_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . We can derive that  $[a_{-1}, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f^u)$  and  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^u)$  if they exist.
    - A.  $[a_{-1}, b_{-1}], [a_0, b_0], [a_1, b_1]$  all exist. Then  $\xi \in I$ . So  $A_\xi = (a, \xi)$ . We can find that there is no real root of  $f^u f^d$  in  $A_\xi$ . So  $[a, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f)$ .
      - If  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^d)$ , it is clear that  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ .
      - If  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^u)$ , we can refine  $[a_{-1}, b_{-1}]$  and  $[a_1, b_1]$ , if needed, such that  $\frac{\partial f^d(X)}{\partial X}$  is negative in the two intervals. If  $\frac{\partial f^d(X)}{\partial X}$  has real root in  $[b_{-1}, b_1]$ , we can derive that  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ . Else, there is no real root of  $f$  in  $[a_0, b_1]$ .
    - B.  $\mathbb{S}_{f,I} = \{[a_{-1}, b_{-1}], [a_0, b_0]\}$ . At first, we can derive that  $[a, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f)$ .
      - $f(b) > 0$ , we can find that  $[a_0, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
      - $f(b) < 0$ , we can derive that there is no real root of  $f$  in  $[a_0, b]$ .
      - $f(b) = 0$ , it is clear that  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
    - C.  $\mathbb{S}_{f,I} = \{[a_{-1}, b_{-1}]\}$ . Similarly,  $[a, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f)$ . Note that  $f(b) < 0$  in this case. Otherwise, there must exist at least another isolating interval of  $f^u$  in  $I$ .
    - D.  $\mathbb{S}_{\cup, \mathbb{I}} = \emptyset$ .
      - $f(b) > 0$ .  
If  $\frac{\partial f(b)}{\partial X} > 0$ , then  $b \in R_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . So we have  $[a, b] \rightsquigarrow \mathbf{Zero}_I(f)$ . And the root is an even root of  $f$ .  
If  $\frac{\partial f(b)}{\partial X} < 0$ , then  $b \in L_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . It is clear that there is no real root of  $f$  in  $I$ .
      - $f(b) < 0$ , it is clear that  $[a, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
      - $f(b) = 0$ , then  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .

(b)  $f(a) < 0$ , we have  $f^u(a) > 0$ .

i.  $\frac{\partial f(a)}{\partial X} > 0$ . Then  $a \in L_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . We can derive that  $[a_{-1}, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f^d)$  and  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^d)$  if they exist.

A.  $[a_{-1}, b_{-1}], [a_0, b_0], [a_1, b_1]$  all exist. Then  $\xi \in I$ . So  $A_\xi = (a, \xi)$ . We can find that there is no real root of  $f^u f^d$  in  $A_\xi$ . So  $[a, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f)$ .

- If  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^u)$ , it is clear that  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- If  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^d)$ , we can refine  $[a_{-1}, b_{-1}]$  and  $[a_1, b_1]$ , if needed, such that  $\frac{\partial f^u(X)}{\partial X}$  is positive in the two intervals. If  $\frac{\partial f^u(X)}{\partial X}$  has real root in  $[b_{-1}, b_1]$ , we can derive that  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ . Else, there is no real root of  $f$  in  $[a_0, b_1]$ .

B.  $\mathbb{S}_{f,I} = \{[a_{-1}, b_{-1}], [a_0, b_0]\}$ . At first, we can derive that  $[a, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f)$ .

- $f(b) > 0$ , we can derive that there is no real root of  $f$  in  $[a_0, b]$ .
- $f(b) < 0$ , we can find that  $[a_0, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- $f(b) = 0$ , it is clear that  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .

C.  $\mathbb{S}_{f,I} = \{[a_{-1}, b_{-1}]\}$ . Similarly,  $[a, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f)$ . Note that  $f(b) > 0$  in this case. Otherwise, there must exist at least another isolating interval of  $f^d$  in  $I$ .

D.  $\mathbb{S}_{f,I} = \emptyset$ .

- $f(b) > 0$ , it is clear that  $[a, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- $f(b) < 0$ .  
If  $\frac{\partial f(b)}{\partial X} > 0$ , then  $b \in L_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . It is clear that there is no real root of  $f$  in  $I$ .  
If  $\frac{\partial f(b)}{\partial X} < 0$ , then  $b \in R_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . So we have  $[a, b] \rightsquigarrow \mathbf{Zero}_I(f)$ . And the root is an even root of  $f$ .
- $f(b) = 0$ , then  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .

ii.  $\frac{\partial f(a)}{\partial X} < 0$ . Then  $a \in R_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . We can derive that  $[a_{-1}, b_{-1}] \rightsquigarrow \mathbf{Zero}_I(f^u)$  and  $[a_0, b_0] \rightsquigarrow \mathbf{Zero}_I(f^u)$  if they exist.

A.  $[a_{-1}, b_{-1}], [a_0, b_0], [a_1, b_1]$  all exist.

- If  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^d)$ , it is clear that  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- If  $[a_1, b_1] \rightsquigarrow \mathbf{Zero}_I(f^u)$ , we can refine  $[a_{-1}, b_{-1}]$  and  $[a_1, b_1]$ , if needed, such that  $\frac{\partial f^d(X)}{\partial X}$  is negative in the two intervals. If  $\frac{\partial f^d(X)}{\partial X}$  has real root in  $[b_{-1}, b_1]$ , we can derive that  $[a_0, b_1] \rightsquigarrow \mathbf{Zero}_I(f)$ . Else, there is no real root of  $f$  in  $[a_0, b_1]$ .

B.  $\mathbb{S}_{f,I} = \{[a_{-1}, b_{-1}], [a_0, b_0]\}$ .

- $f(b) > 0$ , we can derive that  $[a_0, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- $f(b) < 0$ , we can find there is no real root of  $f$  in  $I$ .
- $f(b) = 0$ , it is clear that  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .

C.  $\mathbb{S}_{f,I} = \{[a_{-1}, b_{-1}]\}$  or  $\mathbb{S}_{f,I} = \emptyset$ . We can find that there is on real root of  $f$  in  $I$ .

(c)  $f(a) = 0$ .

i.  $\mathbb{S}_{\cup, \mathbb{I}} = \emptyset$ . In this case  $f \equiv 0$  may happen.



- A. If  $f(a^+) \neq 0 (a < a^+ < b)$ ,  $[a, a] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- B. If  $f(a^+) = 0 (a < a^+ < b)$ ,  $f \equiv 0$ . Otherwise, there is a root  $t$  of  $f'$  such that  $f(t) \neq 0$  and  $f'(t) = 0$ , so there must exist real roots of  $f^u f^d$  in  $[a, a^+]$  since the sleeve-evaluation inequality (5) holds.
- ii.  $\mathbb{S}_{\cup, \mathbb{I}} \neq \emptyset$ . Of course,  $[a, a] \rightsquigarrow \mathbf{Zero}_I(f)$  in this case. Denote the first three elements in  $\mathbb{S}_{\cup, \mathbb{I}}$  as  $[a_{-1}, b_{-1}]$ ,  $[a_0, b_0]$ ,  $[a_1, b_1]$ , if they exist.
- A.  $[a_{-1}, b_{-1}]$ ,  $[a_0, b_0]$ ,  $[a_1, b_1]$  all exist. We can use the same method to check whether there is a root of  $f$  in  $[a_0, b_1]$ .
- B.  $\mathbb{S}_{f, I} = \{[a_{-1}, b_{-1}], [a_0, b_0]\}$ . Let  $t$  be the first nonzero value in the sorted list  $\{f'(a), f''(a), \dots, f^{(k)}(a)\}$ . In practice, we can let  $t = f(a^+)$ , where  $a^+$  is close to  $a$  on its right side. When  $f(b) = 0$ , it is clear that  $[b, b] \rightsquigarrow \mathbf{Zero}_I(f)$  and there are no real roots of  $f$  in  $(a, b)$ . So we just consider the case  $f(b) \neq 0$ .
- $t f(b) > 0$ , there are no real roots of  $f$  in  $(a, b)$ .
  - $t f(b) < 0$ ,  $[a_0, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- C.  $\mathbb{S}_{f, I} = \{[a_{-1}, b_{-1}]\}$ . There are no real roots of  $f$  in  $(a, b)$ .
3. If  $f^u(a)f^d(a) = 0$ , then  $f^u(a) = 0$  or  $f^d(a) = 0$ . Note that just one of the two equations holds. We need to mention that in this case  $[a, a]$  is an isolating interval of  $f^u f^d$ , but we have ignored it from  $\mathbb{S}_{\cup, \mathbb{I}}$ .
- (a)  $f^u(a) = 0$ . That means  $f^d(a) < 0$ . We can choose a point which is very close to  $a$  and on the right side of  $a$ , denoted as  $a^+$ . In fact, we can compute the derivative of  $f^d$  at the point  $a$  to replace  $f^u(a^+)$ . And if  $\frac{\partial^i f^u(a)}{\partial X^i} = 0, i = 0, 1, \dots, j$  and  $\frac{\partial^{j+1} f^u(a)}{\partial X^{j+1}} \neq 0$ , we can use  $\frac{\partial^{j+1} f^u(a)}{\partial X^{j+1}}$  to replace  $f^u(a^+)$ . If  $f^u(a^+)f^d(a) > 0$ , we can deal with it by the method discussed above in 1-(b). Else,  $f^u(a^+)f^d(a) < 0$ , we can deal with it by the method discussed above in 2-(b).
- (b)  $f^d(a) = 0$ . We can deal with this case similar to 3-(a). If  $f^d(a^+)f^u(a) > 0$ , we can deal with it by the method discussed above in 1-(a). Else,  $f^d(a^+)f^u(a) < 0$ , we can deal with it by the method discussed above in 2-(a).

Now we need to consider the other endpoint of the non-faithful sleeve  $(I, f^u, f^d)$ . When  $f^u(b)f^d(b) \leq 0$ , we point out all the cases and solve them below. Note that  $|f(b)| < EB_I(f)$  in this case. Otherwise, it can not happen. Since we have considered the special cases, there are less than four isolating intervals in  $\mathbb{S}_{\cup, \mathbb{I}}$ , when we discuss the case  $|f(a)| < EB_I(f)$ , now we always assume there are more than three isolating intervals in  $\mathbb{S}_{\cup, \mathbb{I}}$ . We need only to check whether or not there is a real root of  $f$  between the last root in  $\mathbf{Zero}_I(f^u f^d)$  and  $b$ .

1. If  $f^u(b)f^d(b) < 0$ , it is similar to the case  $f^u(a)f^d(a) < 0$ . The last isolating interval in  $\mathbb{S}_{\cup, \mathbb{I}}$ , denoted as  $[l_1, l_2]$ , need to be ignored when forming effective candidate intervals.
- (a)  $f(b) > 0$ . Then  $f^d(b) < 0$ .
- i.  $\frac{\partial f(b)}{\partial X} > 0$ . Then  $b \in R_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . It is clear that  $\xi \in I$ . So  $B_\xi = (\xi, b)$ .

- A. If  $[l_1, l_2] \rightsquigarrow \mathbf{Zero}_I(f^u)$ , then there is no real root of  $f^u f^d$  in  $B_\xi$ . We can derive that  $[l_1, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- B. If  $[l_1, l_2] \rightsquigarrow \mathbf{Zero}_I(f^d)$ , we can not conclude directly whether there is a real root of  $f$  in  $[l_1, b]$  or not since the last root in  $\mathbf{ZERO}_I(f^u f^d)$  may be in  $A_\xi$  or in  $B_\xi$ . But we can check whether there is a root of  $f$  in  $[l_1, b]$  by Theorem 12 (E). Note that the isolating interval in  $\mathbb{S}_{\mathbb{U}, \mathbb{I}}$  adjacent to  $[l_1, l_2]$ , denoted as  $[l'_1, l'_2]$ , is clearly an isolating interval of  $f^d$ . Refine  $[l'_1, l'_2]$ , if needed, such that  $\frac{\partial f^u(X)}{\partial X}$  is positive in it. Note that  $\frac{\partial f^u(b)}{\partial X} > \frac{\partial f(b)}{\partial X} > 0$ . We can check whether  $\frac{\partial f^u(X)}{\partial X}$  has real root in  $[l'_2, b]$ . If it has, then  $[l_1, b] \rightsquigarrow \mathbf{Zero}_I(f)$ . Else, there is no real root in  $[l_1, b]$ .
- ii.  $\frac{\partial f(b)}{\partial X} < 0$ . Then  $b \in L_\xi$  for some  $\xi \in \mathbf{Zero}(f)$ . It is clear that  $\xi \notin I$ . We can find that  $[l_1, l_2] \rightsquigarrow \mathbf{Zero}_I f^d$ . And there is no real root of  $f$  in  $[l_1, b]$ .
- (b)  $f(b) < 0$ . We can find  $f^u(b) > 0$ .
- i.  $\frac{\partial f(b)}{\partial X} > 0$ . Then  $b \in L_\xi$  for some  $\xi \in \mathbf{Zero}(f)$  and  $\xi \notin I$ . We can derive that  $[l_1, l_2] \rightsquigarrow \mathbf{Zero}_I(f^u)$ . This case similar to 1-(a)-(ii). There is no real root of  $f$  in  $[l_1, b]$ .
- ii.  $\frac{\partial f(b)}{\partial X} < 0$ . Then  $b \in R_\xi$  for some  $\xi \in \mathbf{Zero}_I(f)$ . Of course,  $\xi \in I$ .
- A. If  $[l_1, l_2] \rightsquigarrow \mathbf{Zero}_I(f^d)$ , then there is no real root of  $f^u f^d$  in  $B_\xi$ . We can derive that  $[l_1, b] \rightsquigarrow \mathbf{Zero}_I(f)$ .
- B. If  $[l_1, l_2] \rightsquigarrow \mathbf{Zero}_I(f^u)$ , we can not conclude directly whether there is a real root of  $f$  in  $[l_1, b]$  or not since the last root in  $\mathbf{ZERO}_I(f^u f^d)$  may be in  $A_\xi$  or in  $B_\xi$ . But we can check whether there is a root of  $f$  in  $[l_1, b]$  by Theorem 12 (E). Note that the isolating interval in  $\mathbb{S}_{\mathbb{U}, \mathbb{I}}$  adjacent to  $[l_1, l_2]$ , denoted as  $[l'_1, l'_2]$ , is clearly an isolating interval of  $f^d$ . Refine  $[l'_1, l'_2]$ , if needed, such that  $\frac{\partial f^d(X)}{\partial X}$  is negative in it. Note that  $\frac{\partial f^d(b)}{\partial X} \leq \frac{\partial f(b)}{\partial X} < 0$ . We can check whether  $\frac{\partial f^d(X)}{\partial X}$  has real root in  $[l'_2, b]$ . If it has, then  $[l_1, b] \rightsquigarrow \mathbf{Zero}_I(f)$ . Else, there is no real root in  $[l_1, b]$ .
2.  $f^u(b)f^d(b) = 0$ . We can deal with this case similar to the case  $f^u(a)f^d(a)$ .
- (a)  $f^u(b) = 0$ . We can find a point  $b^-$ , which is close to  $b$  and smaller than  $b$ . In fact, if  $\frac{\partial^i f^u(b)}{\partial X^i} = 0, i = 0, 1, \dots, j$  and  $\frac{\partial^{j+1} f^u(b)}{\partial X^{j+1}} \neq 0$ , we can use  $-\frac{\partial^{j+1} f^u(b)}{\partial X^{j+1}}$  to replace  $f^u(b^-)$ . If  $f^u(b^-)f^d(b) < 0$ , we can deal with it in the same way as 1-(b). Else, we can treat it as a good case. Note that when  $f^u(b)f^d(b) > 0$  is faithful for this endpoint.
- (b)  $f^d(b) = 0$ . The treatment of this case is similar to 2-(a). If  $f^u(b)f^d(b^-) < 0$ , we can deal with it in the same way as 1-(a). Else, we can treat it as a good case.

**Remark:** From the discussion above, we need to solve the following problems: How to check whether  $|f(a)| < EB_I(f)$ ?  $f(t) = 0$ ,  $f(t) > 0$  or  $f(t) < 0$ ? And  $\frac{\partial f(t)}{\partial X} > 0$  or  $\frac{\partial f(t)}{\partial X} < 0$ ? Here  $t = a$  or  $t = b$ . In fact, it is easy to find that these problems all can be transform to Problem 2 in Section 2.3, which we have mentioned above. Note that when  $c_1 \leq g(t) \leq c_2$  and  $c_1 c_2 > 0$ , we can decide the sign of  $g(t)$  by  $c_1$  or  $c_2$ , where  $g(X)$  can be  $f(X)$  or  $\frac{\partial f(X)}{\partial X}$  and  $c_1, c_2 \in \mathbb{F}$  are used to approximate  $g(t)$ .