Mechanical Formula Derivation in Elementary Geometries *

Shang-Ching Chou and Xiao-Shan Gao†
Department of Computer Sciences
The University of Texas at Austin, Austin, Texas 78712 USA

Abstract

A precise formulation for the relations among certain variables under a set of polynomial equations and a set of polynomial inequations (to exclude certain special cases which cannot be excluded by the selection of parameters alone) is given. Several methods are presented to find such relations. The methods have been implemented and used to find geometry formulas, to discover geometry theorems, and to find geometry locus equations. About 120 non-trivial problems have been solved using the methods.

Keywords:

Elementary geometry, formula derivation, Gröbner bases, Ritt–Wu's method, Heron's formula, Brahmagupta's Formula, locus equations, Peaucellier's linkage.

1. Introduction

In [10], a method for finding geometry formulas was given. The method was used to find several formulas in geometry difficult for humans to derive [10]. However, the method is incomplete, and in many occasions it can lead to some spurious relations (formulas) irrelevant to the original geometry problem. Furthermore, some relations cannot be found by this method. For example, the relation among the variables x_1 and u determined by $\{x_2^2 = 0, x_3x_2 + x_1 - u = 0\}$ is $x_1 - u = 0$, but it cannot be derived by the method in [10]. In [2], another method for formula derivation in geometry was given, but it is also not complete in general cases. In this paper, we give a precise formulation for the relations among certain variables under a set of polynomial equations and a set of polynomial inequations (to exclude certain special cases which cannot be excluded by the selection of parameters alone). Three methods for deriving such relations are given. The first two are based on the Gröbner basis method. The other is based on Ritt-Wu's characteristic method.

Our methods can be used to find geometry formulas and geometry locus equations. About 120 non-trivial problems have been solved by the methods.

2. The Formulation of the Problem

First we use two examples to give the motivation of our formulation of the problem.

Example 2.1. Find the formula for the area of a triangle ABC in terms of its three sides (Heron's Formula, Fig. 1).

Let a, b, and c be the three sides of the triangle, $B = (0, 0)$, $C = (a, 0)$, and $A = (x_1, x_2)$. Then the geometry conditions can be expressed by the following set of polynomial equations HS:

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†On leave from Institute of Systems Science, Academia Sinica, Beijing.

$h_1 = x_2^2 + x_1^2 - 2ax_1 - b^2 + a^2 = 0$ $\qquad b = AC$
$h_2 = x_2^2 + x_1^2 - c^2 = 0$ $\qquad c = AB$
$h_3 = ax_2 - 2k = 0$ $\qquad k =$ the area of ABC.

Here the variables a, b, and c can be considered parameters in the sense that they can generally take any values. Once they are fixed, the values of other variables are determined by the polynomial equations $h_1 = 0, h_2 = 0$, and $h_3 = 0$. Our task is to express the area k in terms of the parameters a, b, and c, i.e., to find a polynomial equation expressing the relationship among a, b, c, and k which can be derived from the above set of polynomial equations. For this example, non-degenerate (exceptional) conditions can be determined solely by the selection of parameters. This is usually the case, especially for geometry theorem proving. Almost all 512 theorems proved in [3] belong to such case (for a theoretical discussion see [6]). But we have also encountered several problems in geometry formula derivation for which some exceptional conditions need to be excluded by inequations. Following is such an example.

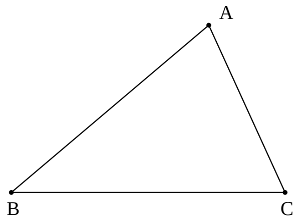

Figure 1: The area of a triangle

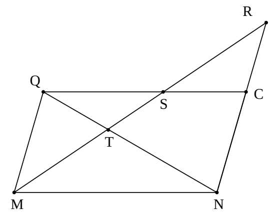

Figure 2: Ratios in a parallelogram

Example 2.2. Let l be a line passing through the vertex of M of a parallelogram $MNPQ$ and intersecting the lines NP, PQ, and NQ in points R, S, and T. Find the relation among MT/MR and MT/MS if there is one (Fig. 2).

Let $M = (0, 0)$, $N = (u_1, 0)$, $P = (u_2, u_3)$, $Q = (x_1, u_3)$, $S = (x_2, u_3)$, $R = (x_3, x_4)$, and $T = (x_5, x_6)$. The geometry conditions can be expressed by the following set of polynomial equations HS:

$h_1 = u_3x_1 + (-u_2 + u_1)u_3 = 0$ $\qquad MQ$ is parallel to NP
$h_2 = (x_1 - u_1)x_6 - u_3x_5 + u_1u_3 = 0$ $\qquad T$ is on QN
$h_3 = (u_2 - u_1)x_4 - u_3x_3 + u_1u_3 = 0$ $\qquad R$ is on NP
$h_4 = x_2x_6 - u_3x_5 = 0$ $\qquad T$ is on MS
$h_5 = x_2x_4 - u_3x_3 = 0$ $\qquad M$ is on RS
$h_6 = x_5 - r_1x_3 = 0$ $\qquad r_1 = MT/MR$
$h_7 = x_5 - r_2x_2 = 0$ $\qquad r_2 = MT/MS$.

However, in specifying $r_1 = MT/MR$ and $r_2 = MT/MS$, we usually have to add the following set of polynomial inequations $DS = \{x_2 \neq 0 \wedge x_3 \neq 0\}$ to exclude certain special cases which sometimes cannot be excluded by the selection of parameters alone. We want to find a relation between MT/MR and MT/MS (if there is one), i.e., between r_1 and r_2. Usually, the above algebraic conditions HS and DS do not imply a polynomial equation between r_1 and r_2 because the dimension (the number of parameters) of the problem is 4. We can select u_1, u_2, u_3 and r_1 as the parameters. Then HS and DS imply (as in this problem) a polynomial equation among u_1, u_2, u_3, r_1 and r_2. If this equation contains r_1 and r_2 only, then problem has a solution. Otherwise, the problem does not have a solution or is not correctly proposed.

Remark. Without $DS = \{x_2 \neq 0, x_3 \neq 0\}$, HS alone does not satisfy Criteria 2.3 below, if we consider u_1, u_2, u_3 and r_1 the parameters. Thus it cannot lead to the result desired. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$, and $D = (x_4, y_4)$ be four points with lines AB and CD being the same or parallel. Then $AB/CD = (x_2 - x_1)/(x_4 - x_3)$ if $x_4 - x_3 \neq 0$. In general, we have to add an inequation $x_4 - x_3 \neq 0$ to exclude that special case which sometimes cannot be excluded by the selection of parameters alone.

We will solve these two problems using the methods presented in Section 4. But we first formulate precisely the problem we want to solve. Let K be a computable field with characteristic zero (in practice, $K = \mathbf{Q}$). Unless stated otherwise, all polynomials mentioned in this paper are over K. Suppose for a geometric problem, after adopting an appropriate coordinate system, the corresponding geometric configuration can be expressed by a set of polynomial equations

$$HS = \{h_1(u_1, ..., u_q, x_1, ..., x_p) = 0 \wedge \cdots \wedge h_s(u_1, ..., u_q, x_1, ..., x_p) = 0\}$$

together with a set of polynomial inequations

$$DS = \{d_1(u_1, ..., u_q, x_1, ..., x_p) \neq 0 \wedge \cdots \wedge d_l(u_1, ..., u_q, x_1, ..., x_p) \neq 0\}.$$

Here we use DS to exclude some special cases in which the problem or specification of the problem becomes invalid. For most cases, DS consists of those inequations that were mentioned in the remark after Example 2.2. Of course, DS can include other non-degenerate conditions which can be excluded by the selection of parameters (for the use of parameters to exclude non-degenerate conditions see [6]). This flexibility can be used to speed up Method 4.6 (see Remark 4.7). Here we divide the variables occurring in HS and DS into two groups: $u_1, ..., u_q$ and $x_1, ..., x_p$ in the sense that in the problem the u can generally take any value and the x can be determined as some functions of the u. We call the u and the x the *parameters* and the *dependent variables* of the corresponding geometric problem, respectively. For a given geometric problem, the selection of parameters is not unique, but is determined by the geometric problem itself. Depending on the context, HS and DS sometimes also denote the polynomial sets $\{h_1, ..., h_s\}$ and $\{d_1, ..., d_l\}$, respectively. Let

$$HD = \{h_1, ..., h_s, z_1 d_1 - 1, ..., z_l d_l - 1\},$$

where $z_1, ..., z_l$ are distinct new variables. A necessary algebraic criteria for $u_1, ..., u_q$ to be a set of parameters is:

Criteria 2.3. (1) The u are algebraically independent wrpt HD, i.e., there is no non-zero polynomial containing the u only in the ideal generated by HD. (2) Each x_i is algebraically dependent on the u wrpt HD, i.e., there is a polynomial containing the u and x_i only in the ideal generated by HD.

Thus we can formulate our problem as follows:

The Formulation of the Problem 2.4. Let HS, DS, the u and the x be the same as before. Furthermore, suppose that the u satisfy Criteria 2.3. Let x_{i_0} be a fixed dependent variable. The *relation set* among the u and x_{i_0} is a set of polynomial equations $r_1(u, x_{i_0}) = 0$, ...,$r_k(u, x_{i_0}) = 0$, all containing x_{i_0}, but not other dependent variables such that: (1) All $r_i(u, x_{i_0})$ are irreducible; (2) There is a non-zero polynomial U containing the u only (*We will call such a polynomial a $u-$polynomial.*) such that $U \cdot r_1(u, x_{i_0}) \cdots r_k(u, x_{i_0})$ is in the radical ideal generated by HD; (3) The set $\{r_1, ..., r_k\}$ is minimal to satisfy (1) and (2), i.e., it is impossible to delete any of its elements while still keeping (1) and (2) valid.

3. The Properties of Relation Sets

We now first prove that the relation set $\{r_1(u, x_{i_0}),...,r_k(u, x_{i_0})\}$ exists and is unique, assuming that the parameters u satisfy Criteria 2.3. Let M be the set of all polynomials in $K[u_1, ..., u_q, x_{i_0}] \cap$

$Ideal(HD)$ with positive degrees in x_{i_0}. Since the u satisfy Criteria 2.3, M is non-empty. A polynomial P in M with minimal $deg(P, x_{i_0})$ is called *a minimal polynomial in* x_{i_0}. The following simple lemma is crucial for our further development.

Lemma 3.1. Let the notations and the conditions be the same as in the previous paragraph, P be a minimal polynomial in x_{i_0}, and Q be another polynomial in M. Then there is a u–polynomial U' such that P divides $U'Q$.

Proof. Pseudo dividing Q by P in variable x_{i_0}, we have

$$U'Q = AP + D$$

where U' is a power of the leading coefficient of P in the variable x_{i_0}, thus is a u–polynomial; D is the pseudo remainder with $deg(D, x_{i_0}) < deg(P, x_{i_0})$. Since $D \in Ideal(HD)$, by the minimal property of $deg(P, x_{i_0})$, $deg(D, x_{i_0}) = 0$. Thus D contains only the u and has to be zero by (1) of 2.3. This proves the lemma. ∎

The Existence and Uniqueness Theorem 3.2. Let the notations be the same as before. Suppose the parameters $u_1, ..., u_q$ satisfy Criteria 2.3. Then the set of the relations $\{r_1, ..., r_k\}$ defined in 2.4 exists and is unique.

Proof. Let P be a minimal polynomial in x_{i_0} and

$$P = U \cdot r_1^{s_1}(u, x_{i_0}) \cdots r_k^{s_k}(u, x_{i_0})$$

where U is a u–polynomial, $deg(r_i, x_{i_0}) \geq 1$ and $s_i \geq 1$ for all $i = 1, ..., k$, and the r_i are distinct irreducible polynomials. Then $R = \{r_1(u, x_{i_0}), ..., r_k(u, x_{i_0})\}$ is a set of polynomials satisfying conditions (1)–(2) in 2.4. From Lemma 3.1, it is clear that R is minimal to satisfy (1)–(2) of 2.4, i.e., it is impossible to delete any of its elements while still making it to satisfy (1) and (2) of 2.4. Thus, R is the relation set among the u and x_{i_0}.

Let $R' = \{r'_1(u, x_{i_0}),...,r'_{k'}(u, x_{i_0})\}$ be another relation set satisfying (1)–(3) of 2.4. We want to show $R = R'$. By Lemma 3.1 and (1)–(2) of 2.4, it is clear that R is a subset of R'. By the minimal property (3) of 2.4 for R', R' cannot contain other elements not in R, thus $R = R'$. This proves the uniqueness property of the relation set specified in 2.4. ∎

Proposition 3.3. Let the notations and conditions be the same as before and F be an extension of the field K. we have

$$\forall xu \in F[(HS \wedge DS \wedge U \neq 0) \rightarrow (r_1 = 0 \vee \cdots \vee r_k = 0)], \tag{3.3.1}$$

where U is the u–polynomial in (2) of 2.4.

Proof. From (2) of 2.4 we have

$$\forall xuz \in F[(HS \wedge d_1 z_1 - 1 = 0 \wedge \cdots \wedge d_l z_l - 1 = 0 \wedge U \neq 0) \rightarrow (r_1 \cdots r_k = 0)]. \tag{3.3.2}$$

Because the z is free in $r_1 \cdots r_k$, the above formula is equivalent to

$$\forall xu \in F[\exists z \in F(HS \wedge d_1 z_1 - 1 = 0 \wedge \cdots \wedge d_l z_l - 1 = 0 \wedge U \neq 0) \rightarrow (r_1 \cdots r_k = 0)].$$

Since $\exists z_i(d_i z_i - 1 = 0)$ are equivalent to $d_i \neq 0$, (3.3.1) is equivalent to (3.3.2). This proves the proposition. ∎

The condition $U \neq 0$ is usually connected with nondegeneracy. Or we can say $r_1 \cdots r_k = 0$ is *generally true* under HS and DS (for a more detailed discussion of the notion of "generally true", see [6]).

Proposition 3.4. Let F be an algebraically closed field containing K and $r(u, x_{i_0})$ be a polynomial containing the parameters u and x_{i_0} only. If there is a u–polynomial U such that

$$\forall xu \in F[(HS \wedge DS \wedge U \neq 0) \rightarrow r = 0], \tag{3.4.1}$$

then $r_1 \cdots r_k$ divides r.

Proof. As we see from the proof of Proposition 3.3, Formula (3.4.1) is equivalent to

$$\forall xuz \in F[(HS \wedge d_1 z_1 - 1 = 0 \wedge \cdots \wedge d_l z_l - 1 = 0 \wedge U \neq 0) \rightarrow r = 0].$$

Since F is algebraically closed, it is equivalent to $Ur \in \text{Radical}(HD)$ by Hilbert Nullstellensatz, i.e., there is some positive integer n, $(Ur)^n \in \text{Ideal}(HD)$. Thus the proposition is clear from Lemma 3.1 and Theorem 3.2. ∎

In the following sections we will give several methods for obtaining such relation set $\{r_1, ...r_k\}$. The methods have been successfully used in solving many geometry problems. Especially, the method based on Ritt–Wu's decomposition (Method 4.6) has solved about 120 geometry problems (see the Collection [5]).

4. Methods for Finding Relation Sets

For simplicity, let $x_{i_0} = x_1$ and we want to find the relation set among the u and x_1 given HS and DS. According to Theorem 3.2, it is enough to find a minimal polynomial in x_1.

Theorem 4.1. Let the notations be the same as before and GB be a Gröbner basis of HD [1] in the polynomial ring $K[u_1, ..., u_q, x_1, ..., x_p, z_1, ..., z_l]$ in a compatible ordering $u < x$, $x_1 < x_i$ for $1 < i$, and $x_1 < z$. Then

(1) The u are algebraically independent wrpt HD iff GB does not contain any u–polynomial.

(2) GB contains a minimal polynomial in x_1 iff x_1 is algebraically dependent on the parameters u under HD.

(3) HD with u algebraically independent satisfies (2) of Criteria 2.3 iff for each $v \in \{x_1, .., x_p, z_1, ..., z_l\}$, GB contains a polynomial whose leading monomial is some positive power of v multiplied by a u–monomial.

Proof. Because of the ordering $u < x$ and $u < z$, GB contains a u–polynomial iff the ideal generated by HD contains a u–polynomial. This proves (1). Also because the ordering $x_1 < x_i$ for $i \neq 1$ and $x_1 < z$, GB contains a polynomial containing the u and x_1 only with a positive degree in x_1 iff x_1 is algebraically dependent on the u. Let P be such a polynomial in GB with with $deg(P, x_1)$ minimal. Since each minimal polynomial in x_1 can be reduced to zero by GB, P must be a minimal polynomial in x_1. This proves (2).

Suppose the u are algebraically independent wrpt HD. By the well known result (Method 6.9 in [1]) the condition in (3) is equivalent to that HD has finitely many solutions for the x and z over $K(u)$, which is in turn equivalent to condition (2) of 2.3. ∎

This theorem immediately gives the following method.

Method 4.2 *For Finding the Relation Set R.*

Step 1. Compute the Gröbner basis GB as stated in Theorem 4.1.

Step 2. If GB contains a u–polynomial, then give the answer: "the parameters u are not algebraically independent."

[1] In this paper we assume the reader is already familiar with the Gröbner basis method. The paper [1] is an excellent review of the subject.

Step 3. Suppose GB does not contain a u–polynomial. If it also does not contain a polynomial containing the u and x_1 only, then give the answer: "x_1 is not algebraically dependent on the parameters u".

Step 4. Otherwise, let $P(u, x_1)$ be one in GB with $deg(P, x_1)$ minimal, then P is a minimal polynomial in x_1. Thus, according to theorem 3.2, the set of distinct irreducible factors of P containing x_1 is a relation set among u and x_1.

Step 5. We can use (3) of theorem 4.1 to check whether Criteria 2.3 is fully satisfied, i.e., whether variables x_i other than x_1 are all dependent on the parameters u. ∎

This method, though simple in theory, is inefficient in practice. The reason is that to compute the corresponding Gröbner bases is slow, and for many problems in practice, the computation is often beyond reasonable time and space limits.

If we work on the polynomial ring $K(u_1, ..., u_q)[x_1, ..., x_p, z_1, .., z_l]$ instead of $K[u, x, z]$, we generally can benefit from the following two facts: (1) The corresponding Gröbner bases generally have fewer elements; (2) Common factors of u–polynomials can be removed, thus polynomials in the computation have less sizes.

Theorem 4.3. Let notations be the same as above and GB the reduced Gröbner basis of HD in the polynomial ring $K(u_1, ..., u_q)[x_1, ..., x_p, z_1, .., z_l]$ in a compatible ordering $x_1 < x_i$ for $1 < i$ and $x_1 < z$ Then

(1) The u are algebraically independent wrpt HD iff GB does not contain 1, i.e., HD does not generate the unit ideal in $K(u)[x, z]$.

(2) The variable x_1 is algebraically dependent on the u iff GB contains a polynomial containing x_1 (and the u) only. Let P be such one with $deg(P, x_i)$ minimal, then $U \cdot P$ is a minimal polynomial in x_1 for some u–polynomial U.

(3) HD with u as parameters satisfies (2) of Criteria 2.3 iff for each $v \in \{x_1, .., x_p, z_1, ..., z_l\}$, GB contains a polynomial whose leading monomial is some positive power of v.

Proof. Let I and I_u be the ideal generated by HD in $K[u, x, z]$ and $K(u)[x, z]$ respectively. We have the following simple fact:

(4.3.1) A polynomial P is in I_u iff there is a u–polynomial U such that $UP \in I$.

As a particular case, $1 \in I_u$ iff there is a u–polynomial U such that $U \cdot 1 \in I$, i.e., I contains a u–polynomial. This proves (1).

(2) Let P' be a minimal polynomial in x_1. Then $deg(P', x_1) \geq deg(P, x_1)$ because P reduces P' to zero. On the other hand, there is a u–polynomial U such that UP is in the ideal of $K[u, x, z]$ generated by HD. Thus, $deg(P, x_1) = deg(UP, x_1) \leq deg(P', x_1)$. Hence $deg(UP, x_1) = deg(P', x_1)$ and UP is a minimal polynomial in x_1.

(3) The proof is similar to that of (3) of Theorem 4.1. ∎

Theorem 4.3 gives the following method.

Method 4.4 *For Finding the Relation Set R.*

Step 1. Compute the Gröbner basis GB as stated in Theorem 4.3.

Step 2. If GB contains 1, then give the answer: "the parameters u are not algebraically independent."

Step 3. Suppose GB does not contain 1. If it also does not contain a polynomial containing the u and x_1 only, then give the answer: "x_1 is not algebraically dependent on the parameters u".

Step 4. Otherwise, let $P(u, x_1)$ be the one in GB with $deg(P, x_1)$ minimal. Thus, according to theorems 3.2 and 4.3, the set of irreducible factors of P in $K[u, x_1]$ containing x_1 is a relation set among u and x_1.

Step 5. We can use (3) of theorem 4.3 to check whether Criteria 2.3 is fully satisfied, i.e., whether variables x_i other than x_1 are all dependent on the parameters u. ∎

For most of our problems, Method 4.4 is much faster than Method 4.2. However, We have also encountered some problems which could not be solved by Method 4.4 within reasonable time and space limits. One reason for this is that for some problems (see Examples 5.2 and 5.6), there are more than one relations in the relation set $\{r_1, ..., r_k\}$, i.e., $k > 1$. Methods 4.2 and 4.4 work on some product of powers of all r_1,...,r_k, which could result in very big polynomials in the intermediate steps. The following method based on Ritt–Wu's decomposition works with each relation r_i separately, thus can solve some problems which were beyond space and time limits of methods 4.2 and 4.4.

According to Ritt–Wu's decomposition algorithm[2] we have the following decomposition in the variable ordering $u < x_1 < x_2 < \cdots < x_p$:

$$\text{Zero}(HS/DS) = \cup_{i=1}^{a}\text{Zero}(PD(ASC_i^*)/DS) \bigcup \cup_{i=1}^{b}\text{Zero}(PD(ASC_i)/DS), \tag{4.5.1}$$

where all ascending chains ASC_i^* and ASC_j are irreducible such that (1) All ASC_i^* does not contain any u–polynomials and all ASC_i contains at least one u–polynomial; (2) $prem(d_k, ASC_i^*) \neq 0$ and $prem(d_k, ASC_j) \neq 0$ for all $d_k \in DS$, i and j. Here we use $PD(ASC)$ to denote

$$PD(ASC) = \{g \mid prem(g; ASC) = 0\}.$$

The zeros in $Zero(HS/DS) = Zero(HS) - Zero(DS)$ are taken from an algebraically closed extension F of K.

Theorem 4.5. Let the notations be the same as in the previous paragraph. Then

(1) The parameters u are algebraically independent wrpt HD iff $a > 0$.

(2) In that case, each x_i appears as a leading variable in each ASC_j^*, iff each x_i is algebraically dependent on the u.

(3) Assume that HD and the u satisfy Criteria 2.3. Let $r_i(u, x_1)$ $(i = 1, ..., k)$ be distinct polynomials appearing as the first elements in all ASC_j^*. Then $\{r_1(u, x_1), ..., r_k(u, x_1)\}$ is the relation set defined by HS and DS.

Proof. First we state the following repeatedly used fact:

For a polynomial P in the u and x, $Zero(HD) \subset Zero(P)$ iff $Zero(HS/DS) \subset Zero(P)$. This can be seen from the proof of 3.3.

(1) Suppose $a = 0$, then according to decomposition (4.5.1), there is a u–polynomial U such that $Zero(HS/DS) \subset Zero(U)$. Thus $Zero(HD) \subset Zero(U)$. Therefore, U is in Rad(HD); hence for some k, U^k, which is also a u–polynomial, is in Ideal(HD). The u are algebraically dependent. Now suppose that the u is algebraically dependent, i.e., Ideal(HD) contains a u–polynomial U. Then $Zero(HD) \subset Zero(U)$, which is equivalent to $Zero(HS/DS) \subset Zero(U)$. Since $Zero(U)$ does not contain each $Zero(PD(ASC_i^*)/DS)$, a must be zero.

(2) Each x_i appears as a leading variable in each ASC_j^* iff $Zero(HS/DS)$ has only finitely many solutions in $K(u)$. This is equivalent to that $Zero(HD)$ has only finitely solutions. This proves (2).

[2]In this paper we assume the reader is already familiar with Ritt-Wu's method. The reader can find the details of the method in [10, 3]. Zero(PS) denotes the common zeros of a polynomial set PS and $prem(g, ASC)$ denotes the succesive pseudo remainder of g by the ascending chain ASC.

(3) From decomposition (4.5.1), there is a u–polynomial U such that $Zero(HS/DS) \subset Zero(U \cdot r_1 \cdots r_k)$. Thus $Zero(HD) \subset Zero(U \cdot r_1 \cdots r_k)$. By Hilbert Nullstellensatz, $U \cdot r_1 \cdots r_k$ is in Rad(HD). If we remove any of $r_1, ..., r_k$, say, r_k, then $Zero(HS/DS)$, hence $Zero(HD)$ is not contained in $Zero(U \cdot r_1 \cdots r_{k-1})$ for any u–polynomial U. Thus $U \cdot r_1 \cdots r_{k-1}$ is not in Rad(HD) for any u–polynomial U. Thus $\{r_1, ..., r_k\}$ is minimal to satisfy (1)–(2) of 2.4, hence is the relation set of HD in the u and x_1. ∎

Method 4.6 *For Finding the Relation Set R.*

Step 1. Use Ritt-Wu's method to decompose $Zero(HS/DS)$ as stated in the paragraph preceding Theorem 4.5.

Step 2. If $a = 0$, then give the answer: "the parameters u are not algebraically independent."

Step 3. Suppose $a > 0$. If the first element of one ASC_i^* does not contain the u and x_1 only, then give the answer "x_1 is not algebraically dependent on the parameters u".

Step 4. Suppose $a > 0$ and the first elements of each ASC_i^* $(i = 1, ..., a)$ contain the u and x_1 only. Then we can use (3) of Theorem 4.5 to obtain the relation set among the u and x_1.

Step 5. We can use (2) of Theorem 4.5 to check whether Criteria 2.3 is fully satisfied, i.e., whether each x_i is algebraically dependent on the u. ∎

Remark 4.7. In the real implementation, we do not have to compute degenerate part

$$\cup_{i=1}^{b} Zero(PD(ASC_i)/DS)$$

explicitly. During the decomposition process, whenever a u–polynomial appears in a polynomial set, we can delete that polynomial set, adding that u–polynomial as a factor of the polynomial U in formulation 2.4. Also adding more degenerate conditions to DS can prevent the growth of number of branches in the decomposition. This leads to the speedup of the process. For Methods 4.2 and 4.4 based on the Gröbner basis method, adding more degenerate conditions to DS generally slow down the process or even lead to exceeding reasonable time limits (see Example 5.7 below).

Remark 4.8. In certain sense, Step 5 of Methods 4.2, 4.4 and 4.6 is not necessary as far as we are only concerned with the relation set among the u and x_1, which is unique even if for some $i > 1$, $u_1, ..., u_q$ and x_i are algebraically independent. In that case, one might add x_i (renaming it to u_{q+1}) to the parameter set $u_1, .., u_q$. Because of the Uniqueness Theorem 3.2, the relation set among $u_1, ..., u_{q+1}$ and x_1 will be the same. Since Criteria 2.3 should be satisfied if we understand the geometric problem and specify the parameters correctly, Step 5 serves at least as a warning to the user of a possible misunderstanding or incorrect algebraic specification of the geometric problem.

5. Applications

We have implemented Methods 4.2, 4.4 and 4.6. The methods have been used in deriving formula, finding theorems and locus equations. Below we give several examples to show how various geometric problems can be solved by our methods.

5.1. Deriving Formulas

Example 5.1. The solution to Example 2.1 (Heron's Formula).

HS is the same as in Example 2.1, DS is empty. Considering a, b and c as the parameters, we want to find the relation set among a, b, c and k.

Using Method 4.2, GB of HS in $\mathbf{Q}[a, b, c, k, x_1, x_2]$ is

$16k^2 + c^4 + (-2b^2 - 2a^2)c^2 + b^4 - 2a^2b^2 + a^4$

$2ax_1 - c^2 + b^2 - a^2$
$ax_2 - 2k$
$(c^2 - b^2)x_2 - 4kx_1 + 2ak$
$8kx_2 + (2c^2 - 2b^2)x_1 - 3ac^2 - ab^2 + a^3$
$x_2^2 + x_1^2 - c^2.$

The first polynomial gives the relations we want, i.e., $k = \pm\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ (Heron's formula).

Using Method 4.4, we find GB of HS in $Q(a,b,c)[k,x_1,x_2]$ is

$16k^2 + c^4 + (-2b^2 - 2a^2)c^2 + b^4 - 2a^2b^2 + a^4$
$2ax_1 - c^2 + b^2 - a^2$
$ax_2 - 2k,$

which gives the same result.

Using method 4.6 (in the ordering $k < x_1 < x_2$), we have found one non-degenerate component of HS with the corresponding ascending chain:

$16k^2 + c^4 + (-2b^2 - 2a^2)c^2 + b^4 - 2a^2b^2 + a^4$
$2ax_1 - c^2 + b^2 - a^2$
$ax_2 - 2k,$

which gives the same result.

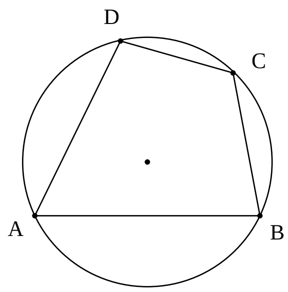

Figure 3: Brahmagupta's Formula

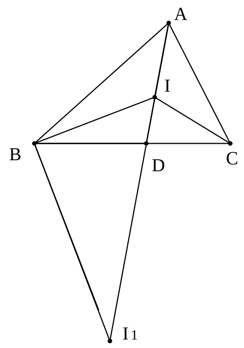

Figure 4: Cross Ratio in a triangle

The following problem is beyond a reasonable time limit using Methods 4.2 or 4.4.

Example 5.2. (Brahmagupta's Formula) $ABCD$ is a cyclic quadrilateral. Determine the signed area of oriented quadrilateral $ABCD$ in terms of its four sides (Fig. 3).

Let $A = (0,0)$, $B = (u_1, 0)$, $C = (x_1, x_2)$, and $D = (x_3, x_4)$. Then the geometry conditions can be expressed by the following set of polynomial equations HS with DS empty:

$h_1 = x_2^2 + x_1^2 - 2u_1x_1 - u_2^2 + u_1^2 = 0$ $\qquad u_2 = BC$
$h_2 = x_4^2 - 2x_2x_4 + x_3^2 - 2x_1x_3 + x_2^2 + x_1^2 - u_3^2 = 0$ $\qquad u_3 = CD$
$h_3 = x_4^2 + x_3^2 - u_4^2 = 0$ $\qquad u_4 = DA$
$h_4 = u_1x_2x_4^2 + (-u_1x_2^2 - u_1x_1^2 + u_1^2x_1)x_4 + u_1x_2x_3^2 - u_1^2x_2x_3 = 0$ $\qquad A, B, C, D$ are cocyclic
$h_5 = x_1x_4 - x_2x_3 + u_1x_2 - 2k = 0$ $\qquad k$ is the sum of the signed areas of ABC and ACD.

Selecting u_1, u_2, u_3, and u_4 to be parameters, we want to find relations among u_1, u_2, u_3, u_4, and k. Using method 4.6 (in the ordering $k < x_1 < x_2 < x_3 < x_4$), we have found two non-degenerate components of HS with the corresponding ascending chains:

$ASC_1^* =$

$$r_1 = 16k^2 + u_4^4 + (-2u_3^2 - 2u_2^2 - 2u_1^2)u_4^2 - 8u_1u_2u_3u_4 + u_3^4 + (-2u_2^2 - 2u_1^2)u_3^2 + u_2^4 - 2u_1^2u_2^2 + u_1^4$$
$$ax_1 + b$$
$$(u_4^2 + u_3^2 - u_2^2 - u_1^2)x_2 - 4kx_1 + 4u_1k$$
$$(2x_2^2 + 2x_1^2)x_3 + (-x_1 - 2u_1)x_2^2 + 4kx_2 - x_1^3 + (-u_4^2 + u_3^2)x_1$$
$$x_1x_4 - x_2x_3 + u_1x_2 - 2k.$$

$ASC_2^* =$

$$r_2 = 16k^2 + u_4^4 + (-2u_3^2 - 2u_2^2 - 2u_1^2)u_4^2 + 8u_1u_2u_3u_4 + u_3^4 + (-2u_2^2 - 2u_1^2)u_3^2 + u_2^4 - 2u_1^2u_2^2 + u_1^4$$
$$ax_1 + b$$
$$(u_4^2 + u_3^2 - u_2^2 - u_1^2)x_2 - 4kx_1 + 4u_1k$$
$$(2x_2^2 + 2x_1^2)x_3 + (-x_1 - 2u_1)x_2^2 + 4kx_2 - x_1^3 + (-u_4^2 + u_3^2)x_1$$
$$x_1x_4 - x_2x_3 + u_1x_2 - 2k.$$

In the above polynomials, a and b are some polynomials in the variables u_1, u_2, u_3, u_4, and k. Thus the relation set is $\{r_1, r_2\}$.

The area k satisfies $r_1 = 0$ or $r_2 = 0$. To decide which one is the real case is generally beyond the scope of our methods. This is typical in the original method developed by Wu for unordered geometry. Actually, we even don't know whether u_1, u_2, u_3, and u_4 are positive or negative. However, for this simple case, we can use a *special example* to solve the problem. Taking $ABCD$ to be a unit square and assuming all u_1, u_2, u_3 and u_4 are positive, we find that r_1 leads to $k^2 - 1 = 0$, while r_2 leads to $k^2 = 0$. Thus r_1 is the real relation we want. It is the well-known Brahmagupta's formula: $k = \pm\sqrt{(s-u_1)(s-u_2)(s-u_3)(s-u_4)}$ where $s = (u_1+u_2+u_3+u_4)/2$. The second relation $r_2 = 0$ leads to $k = \pm\sqrt{s(s-u_1-u_3)(s-u_1-u_2)(s-u_1-u_4)}$ which is a "reflection image" of the first one: when the number of positive variables among the u are even, then r_1 leads to the real result; when the number of positive variables among the u are odd, then r_2 leads to the real result. In either case, the formula is not only valid for the case that $ABCD$ is convex, but also for the cases as shown in Fig. 4 and Fig. 5. In Fig. 4, k is the sum of the signed areas of oriented triangles $\triangle ABO$ and $\triangle CDO$. In Fig. 5, k is the sum of the signed areas of oriented triangles $\triangle BCO$ and $\triangle DAO$.

5.2. Discovering Theorems

One may guess by intuition that there is some relation or property among certain quantities (denoted by variables) for a given geometric problem. If we know the exact relation, we can use theorem provers (based on, e.g., Wu's method or the Gröbner basis method) to prove it. However, if the exact relation is unknown, we might use the methods developed in this paper to derive it.

Example 5.3. Solution to Example 2.2.

Selecting u_1, u_2, u_3, and r_1 to be parameters, we want to find the relation set among u_1, u_2, u_3, r_1 and r_2. Using Method 4.6 (in the ordering $r_2 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$), we have found $Zero(HS/DS)$ has only one non-degenerate component with the corresponding ascending chain $ASC_1^* =$

$$r_2 + r_1 - 1$$
$$u_3x_1 + (-u_2 + u_1)u_3$$
$$r_2x_2 - r_2x_1 + u_1r_2 - u_1$$
$$r_1x_3 - r_2x_2$$
$$(u_2 - u_1)x_4 - u_3x_3 + u_1u_3$$
$$x_5 - r_1x_3$$
$$(x_1 - u_1)x_6 - u_3x_5 + u_1u_3.$$

Thus, $r_2 + r_1 - 1 = 0$ is the relation among r_1 and r_2 (and u_1, u_2, u_3).

Example 5.4. Let I and I_1 be the two tritangent centers of triangle ABC, D be the intersection of AI with BC. Find the cross-ratio (AD, II_1) (Fig. 4).

Let $B = (0,0)$, $C = (u_1, 0)$, $A = (u_2, u_3)$, $I = (x_4, x_5)$, $D = (x_6, 0)$, and $I_1 = (x_7, x_8)$. Then the geometry conditions can be expressed by the following set of polynomial equations HS with DS empty:

$$\begin{array}{lr}
h_1 = u_3^2x_5^2 + 2u_2u_3x_4x_5 - u_3^2x_4^2 = 0 & \angle CBI = \angle IBA \\
h_2 = u_3^2x_5^2 + (((2u_2 - 2u_1)u_3)x_4 + (-2u_1u_2 + 2u_1^2)u_3)x_5 - u_3^2x_4^2 + & \\
\qquad 2u_1u_3^2x_4 - u_1^2u_3^2 = 0 & \angle ABI = \angle IBC \\
h_3 = (x_5 - u_3)x_6 - u_2x_5 + u_3x_4 = 0 & D \text{ is on } AI \\
h_4 = (x_4 - u_2)x_8 + (-x_5 + u_3)x_7 + u_2x_5 - u_3x_4 = 0 & I_1 \text{ is on } AI \\
h_5 = x_5x_8 + x_4x_7 = 0 & BI \perp BI_1 \\
h_6 = (rx_6 + (-r + 1)x_4 - u_2)x_7 + (-x_4 - u_2r + u_2)x_6 + u_2rx_4 = 0 & r = (AD, II_1).
\end{array}$$

Selecting u_1, u_2 and u_3 to be parameters, we want to find relations among u_1, u_2, u_3, and r. Using Method 4.6 (in the ordering $r < x_4 < x_5 < x_6 < x_7 < x_8$) we have found only one non-degenerate component of $Zero(HS)$ with the corresponding ascending chain $ASC_1^* =$

$$\begin{array}{l}
r + 1 \\
4x_4^4 - 8u_1x_4^3 + (-4u_3^2 - 4u_2^2 + 4u_1u_2 + 4u_1^2)x_4^2 + (4u_1u_3^2 + 4u_1u_2^2 - 4u_1^2u_2)x_4 - u_1^2u_3^2 \\
(2x_4 + 2u_2 - 2u_1)x_5 - 2u_3x_4 + u_1u_3 \\
(x_5 - u_3)x_6 - u_2x_5 + u_3x_4 \\
(rx_6 + (-r + 1)x_4 - u_2)x_7 + (-x_4 - u_2r + u_2)x_6 + u_2rx_4 \\
(x_4 - u_2)x_8 + (-x_5 + u_3)x_7 + u_2x_5 - u_3x_4.
\end{array}$$

The relation $r + 1 = 0$ tells us that the two tritangent centers divide the bisector they are located harmonically.

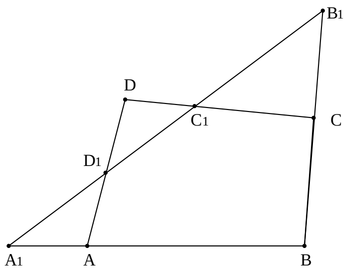

Figure 5: Menellaus' Theorem

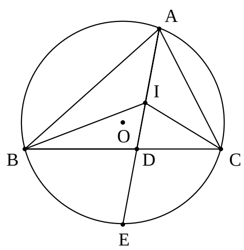

Figure 6: Incenter and circumcenter

Example 5.5. (Menelaus' Theorem for Quadrilaterals) If the sides AB, BC, CD, DA of a quadrilateral $ABCD$ are cut by a transversal in the points A_1, B_1, C_1D_1 respectively, Find the relation among the ratios $AA_1/A_1B, BB_1/B_1C, CC_1/C_1D$, and DD_1/D_1A (Fig. 5).

Let $A = (0,0)$, $B = (u_1, 0)$, $C = (u_2, u_3)$, $D = (u_4, x_2)$, $A_1 = (x_3, 0)$, $B_1 = (x_4, x_5)$, $C_1 = (x_6, x_7)$, and $D_1 = (x_8, x_9)$. Then the geometry conditions can be expressed by the following set of polynomial equations HS with DS empty:

$$\begin{array}{lr}
h_1 = u_4x_9 - x_2x_8 = 0 & D_1 \text{ is on} AD \\
h_2 = (u_2 - u_1)x_5 - u_3x_4 + u_1u_3 = 0 & B_1 \text{ is on } BC \\
h_3 = (u_4 - u_2)x_7 + (-x_2 + u_3)x_6 + u_2x_2 - u_3u_4 = 0 & C_1 \text{ is on } CD \\
h_4 = (x_4 - x_3)x_7 - x_5x_6 + x_3x_5 = 0 & A_1 \text{ is on } B_1C_1 \\
h_5 = (x_4 - x_3)x_9 - x_5x_8 + x_3x_5 = 0 & A_1 \text{ is on } B_1D_1 \\
h_6 = (r_1 + 1)x_3 - u_1r_1 = 0 & r_1 = AA_1/A_1B
\end{array}$$

$h_7 = (r_2 + 1)x_4 - u_2 r_2 - u_1 = 0 \qquad r_2 = BB_1/B_1C$
$h_8 = (r_3 + 1)x_6 - u_4 r_3 - u_2 = 0 \qquad r_3 = CC_1/C_1D$
$h_9 = (r_4 + 1)x_8 - u_4 = 0 \qquad r_4 = DD_1/D_1A.$

Selecting u_1, u_2, u_3, u_4, r_1, r_2, and r_3 to be parameters set, we want to find relations among the parameters and r_4. Using Method 4.6 (in the ordering $r_4 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7 < x_8 < x_9$), we have found only one non-degenerate component of $Zero(HS)$ with the corresponding ascending chain $ASC_1^* =$

$r_1 r_2 r_3 r_4 - 1$
$(((u_2 - u_1)r_1 + u_2)r_2 + u_1)x_2 + u_1 u_3 r_1 r_2 r_4 + ((-u_3 u_4 + u_1 u_3)r_1 - u_3 u_4)r_2$
$(r_1 + 1)x_3 - u_1 r_1$
$(r_2 + 1)x_4 - u_2 r_2 - u_1$
$(u_2 - u_1)x_5 - u_3 x_4 + u_1 u_3$
$(r_3 + 1)x_6 - u_4 r_3 - u_2$
$(u_4 - u_2)x_7 + (-x_2 + u_3)x_6 + u_2 x_2 - u_3 u_4$
$(r_4 + 1)x_8 - u_4$
$u_4 x_9 - x_2 x_8.$

The relation $r_1 r_2 r_3 r_4 - 1 = 0$ is a well-known result. Using Method 4.4, we have found the Gröbner basis of $HS = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$ in $Q(u_1, ..., u_4, r_1, r_2, r_3)[r_4, x_2, ..., x_9]$:

$r_1 r_2 r_3 r_4 - 1$
$(((u_2 - u_1)r_1 + u_2)r_2 + u_1)r_3 x_2 + ((-u_3 u_4 + u_1 u_3)r_1 - u_3 u_4)r_2 r_3 + u_1 u_3$
$(r_1 + 1)x_3 - u_1 r_1$
$(r_2 + 1)x_4 - u_2 r_2 - u_1$
$(r_2 + 1)x_5 - u_3 r_2$
$(r_3 + 1)x_6 - u_4 r_3 - u_2$
$((((u_2 - u_1)r_1 + u_2)r_2 + u_1)r_3 + ((u_2 - u_1)r_1 + u_2)r_2 + u_1)x_7$
$\qquad +(((-u_3 u_4 + u_1 u_3)r_1 - u_3 u_4)r_2)r_3 + (((-u_2 + u_1)u_3)r_1 - u_2 u_3)r_2$
$(r_1 r_2 r_3 + 1)x_8 - u_4 r_1 r_2 r_3$
$((((u_2 - u_1)r_1^2 + u_2 r_1)r_2^2 + u_1 r_1 r_2)r_3 + ((u_2 - u_1)r_1 + u_2)r_2 + u_1)x_9$
$\qquad +(((-u_3 u_4 + u_1 u_3)r_1^2 - u_3 u_4 r_1)r_2^2)r_3 + u_1 u_3 r_1 r_2,$

which gives the same result.

Example 5.6. Let D be the intersection of one of the bisectors of $\angle A$ of triangle ABC with the side BC, E be the intersection of AD with the circumcircle of ABC. Find the relation among AB, AC, AD, and AE (Fig. 6).

Let $A = (0, 0)$, $C = (x_1, x_2)$, $F = (x_1, x_3)$, $B = (x_4, x_5)$, $D = (u_1, 0)$, and $E = (u_2, 0)$. Then the geometry conditions can be expressed by the following polynomial equations HS:

$h_1 = x_3 + x_2 = 0 \qquad F$ and C are symmetric w.r.t the x-axis
$h_2 = x_1 x_5 - x_3 x_4 = 0 \qquad F$ is on AB
$h_3 = (x_1 - u_1)x_5 - x_2 x_4 + u_1 x_2 = 0 \qquad D$ is on BC
$h_4 = u_2 x_2 x_5^2 + (-u_2 x_2^2 - u_2 x_1^2 + u_2^2 x_1)x_5 + u_2 x_2 x_4^2 - u_2^2 x_2 x_4 = 0 \qquad A, B, E, C$ are cyclic
$h_5 = x_5^2 + x_4^2 - u_3^2 = 0 \qquad u_3 = AB$
$h_6 = x_2^2 + x_1^2 - u_4^2 = 0 \qquad u_4 = AC,$

together with the following set of polynomial inequations DS:

$d_1 = x_1 \neq 0 \qquad C$ is not on AB
$d_2 = x_2 \neq 0 \qquad C$ is not on $AB.$

Selecting u_1, u_2, and u_3 to be a parameters of the problem, we want to find relations among u_1,

u_2, u_3 and u_4. Using Method 4.6 (in the ordering $u_4 < x_1 < x_2 < x_3 < x_4 < x_5$), we have found two non-degenerate components of $Zero(HS/DS)$ with the corresponding ascending chains $ASC_1^* =$

$$\begin{aligned}
&r_1 = u_3u_4 - u_1u_2\\
&2u_2x_1 - u_4^2 - u_3u_4\\
&x_2^2 + x_1^2 - u_4^2\\
&x_3 + x_2\\
&u_4x_4 - u_3x_1\\
&x_1x_5 - x_3x_4,
\end{aligned}$$

and $ASC_2^* =$

$$\begin{aligned}
&r_2 = u_3u_4 + u_1u_2\\
&2u_2x_1 - u_4^2 + u_3u_4\\
&x_2^2 + x_1^2 - u_4^2\\
&x_3 + x_2\\
&u_4x_4 + u_3x_1\\
&x_1x_5 - x_3x_4.
\end{aligned}$$

Thus we have the relation set $\{r_1, r_2\}$. As in Example 5.2, r_2 is a "reflection image" of r_1. Assume all u_1, u_2, u_3 and u_4 to be positive, $r_2 \neq 0$, thus the real relation should be $r_1 = 0$.

5.3. Locus Problems

The algorithms described in this paper can also be used to find geometry loci. A locus of a point is actually the relation between the coordinates of this point and some other quantities (coordinates, lengths, etc) which are given (and fixed) in the problem. So if we take one of the coordinate of the locus point and the given quantities as parameters, then the relation set among the parameters and the other coordinate of the locus point found by the methods in Section 3 are the locus equations for that point.

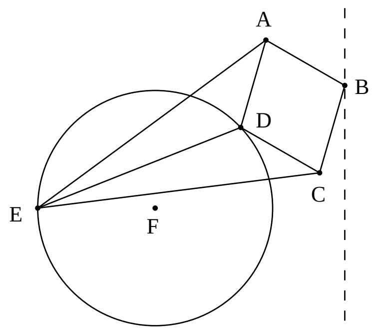

Figure 7: Peaucellier's Linkage

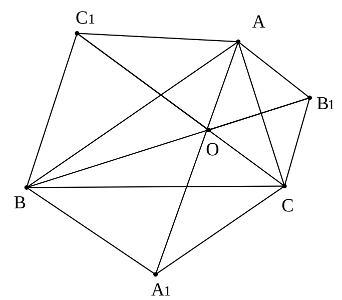

Figure 8: Paterson's problem

Example 5.7. (Peaucellier's Linkage) Links AD, AB, DC and BC have equal length, as do links EA and EC. The length of FD equals the distance from E to F. The locations of joints E and F are fixed points on the plane, but the linkage is allowed to rotate about these points. As it does, what is the traces of the joint B? (Fig. 7)

Let $F = (0, 0)$, $E = (r, 0)$, $C = (x_2, y_2)$, $D = (x_1, y_1)$, and $B = (x, y)$. Then the geometry conditions can be expressed by the following set of polynomial equations HS

$$\begin{aligned}
&h_1 = y_1^2 + x_1^2 - r^2 = 0 && r = FD\\
&h_2 = y_2^2 - 2y_1y_2 + x_2^2 - 2x_1x_2 + y_1^2 + x_1^2 - n^2 - m^2 = 0 && CD = n^2 + m^2\\
&h_3 = y_2^2 - 2yy_2 + x_2^2 - 2xx_2 + x^2 + y^2 - n^2 - m^2 = 0 && CB = n^2 + m^2
\end{aligned}$$

$h_4 = y_2^2 + x_2^2 - 2rx_2 - n^2 - 4rn - m^2 - 3r^2 = 0$ $\quad EC = (n + 2r)^2 + m^2$
$h_5 = (x - r)y_1 - yx_1 + ry = 0$ $\quad E$ is on DB,

together with the following set of polynomial inequations DS:[3] For this example, if we use $\{BD = (x_1 - x)^2 + (y_1 - y)^2 \neq 0\}$ as the set DS, the problem is beyond the time limit using Methods 4.2 and 4.4. But Method 4.6 based on Ritt-Wu's decomposition does not have a similar problem. Actually, the more polynomials in DS, the less (degenerate) components will be in the Ritt–Wu's decomposition process. Hence the less time it takes generally. Thus we can add some non-degenerate conditions, which, though can be excluded by the selection of parameters, are geometrically reasonable, to DS to speed up Method 4.6.

$d_1 = x_1 - x \neq 0$ $\quad B \neq D.$

Selecting m, n, r, and y to be the parameters of the problem, we want to find the relation among m, n, r, y and x. Using Method 4.6 (in the ordering $x < x_1 < y_1 < x_2 < y_2$), we have found $Zero(HS/DS)$ has only one non-degenerate component with the corresponding ascending chain $ASC_1^* =$

$x + 2n + r$
$(x^2 - 2rx + y^2 + r^2)x_1 + rx^2 - 2r^2x - ry^2 + r^3$
$(x - r)y_1 - yx_1 + ry$
$(4x^2 - 8rx + 4y^2 + 4r^2)x_2^2 + (-4x^3 + 4rx^2 + (-4y^2 - 16rn - 12r^2)x - 4ry^2 + 16r^2n + 12r^3)x_2 + x^4 + (2y^2 + 8rn + 6r^2)x^2 + y^4 + (-4n^2 - 8rn - 4m^2 - 6r^2)y^2 + 16r^2n^2 + 24r^3n + 9r^4$
$2yy_2 + (2x - 2r)x_2 - x^2 - y^2 - 4rn - 3r^2.$

The relation $x = -2n - r$ tells us that the locus is a line parallel to the y-axis.

Example 5.8. (M. Paterson's Problem). Three similar isoceles triangles, A_1BC, AB_1C, and ABC_1 are erected on the three respective sides, BC, CA, AB, of a triangle ABC, then AA_1, BB_1, and CC_1 are concurrent. Find the locus of the points of concurrency as the areas of the three similar triangles are varied between 0 and infinity (Fig. 8).

Let $A = (0, 0)$, $B = (u_1, 0)$, $C = (u_2, u_3)$, $O = (x, y)$, $C_1 = (x_2, x_1)$, $B_1 = (x_4, x_3)$, and $A_1 = (x_6, x_5)$. We will find the locus of the intersection points of CC_1 and BB_1. The geometry conditions can be expressed by the following set of polynomial equations HS with DS empty:

$h_1 = (x - u_2)x_1 + (-y + u_3)x_2 + u_2y - u_3x = 0$ $\quad C_1$ is on OC
$h_2 = 2x_2 - u_1 = 0$ $\quad C_1A \equiv C_1B$
$h_3 = (x - u_1)x_3 - yx_4 + u_1y = 0$ $\quad B_1$ is on line OB
$h_4 = 2u_3x_3 + 2u_2x_4 - u_3^2 - u_2^2 = 0$ $\quad B_1A \equiv B_1C$
$h_5 = (u_1u_3x_1 - u_1u_2x_2)x_3 + (u_1u_2x_1 + u_1u_3x_2)x_4 + (-u_1u_3^2 - u_1u_2^2)x_1 = 0$
$\tan(BAC_1) = \tan(ACB_1).$

Selecting u_1, u_2, u_3, and x to be parameters of the problem, we want to find the relation among u_1, u_2, u_3, x, and y. Using Method 4.6 (in the ordering $y < x_2 < x_1 < x_4 < x_3$), we have found one non-degenerate component of $Zero(HS)$ with the corresponding ascending chain: $ASC_1^* =$

$((2u_2 - u_1)u_3)y^2 + ((-2u_3^2 + 2u_2^2 - 2u_1u_2 + 2u_1^2)x + u_1u_3^2 - u_1u_2^2 - u_1^2u_2)y + ((-2u_2 + u_1)u_3)x^2 + ((2u_1u_2 - u_1^2)u_3)x$
$2x_2 - u_1$
$(x - u_2)x_1 + (-y + u_3)x_2 + u_2y - u_3x$
$(2u_3y + 2u_2x - 2u_1u_2)x_4 - 2u_1u_3y + (-u_3^2 - u_2^2)x + u_1u_3^2 + u_1u_2^2$
$2u_3x_3 + 2u_2x_4 - u_3^2 - u_2^2.$

[3] The Gröbner bases method is sensitive with the choice of the set DS.

The locus is a hyperbola.

6. Experimental Results

We have used Methods 4.2, 4.4, and 4.6 to solve the eight problems in Section 5. The timing is shown in the following table.

Examples	Method 4.2	Method 4.4	Method 4.6
5.1	1.450	0.733	3.417
5.2	> 3600	> 3600	**
5.3	34.550	5.517	11.833
5.4	> 3600	> 3600	27.267
5.5	> 3600	17.217	13.517
5.6	> 3600	> 3600	28.217
5.7	> 3600	25.183	45.100
5.8	91.900	6.017	5.583

The time is specified in seconds (on a SUN–3/280). For examples 5.1, 5.3, 5.4, 5.5, 5.7 and 5.8, the three methods gave the same results. Examples 5.2, 5.4, and 5.6 were beyond the time limit using Methods 4.2 and 4.4. With some human interactions, we have solved Example 5.2 using Method 4.6.

We have used Method 4.6 to solve about 120 problems [5], among which four have been solved with certain human interactions; the remaining have been solved automatically by the program. 14 among the 120 problems were beyond the time limit using Method 4.4. Method 4.2[4] is much slower and could solve less problems than Method 4.4. The reader can find more detailed information in the collection [5].

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