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An analogue of the Harer-Zagier formula for unicellular maps on general surfaces

Olivier Bernardi (MIT)

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A unicellular map is the embedding of a connected graph in a surface in such a way that the complement of the graph is simply connected. In a famous article, Harer and Zagier established a formula for the generating function of unicellular maps counted according to the number of vertices and edges. The keystone of their approach is a counting formula for unicellular maps on orientable surfaces with n edges, and with vertices colored using every color in $[q]$ (adjacent vertices are authorized to have the same color). We give an analogue of this formula for general (locally orientable) surfaces. Our approach is bijective and is inspired by Lass's proof of the Harer-Zagier formula. We first revisit Lass's proof and twist it into a bijection between unicellular maps on orientable surfaces with vertices colored using every color in $[q]$, and maps with vertex set $[q]$ on orientable surfaces \emph{with a marked spanning tree}. The bijection immediately implies Harer-Zagier's formula and a formula by Jackson concerning bipartite unicellular maps. It also shed a new light on constructions by Goulden and Nica, Schaeffer and Vassilieva, and Morales and Vassilieva. We then extend the bijection to general surfaces and obtain a correspondence between unicellular maps on general surfaces with vertices colored using every color in $[q]$, and maps on orientable surfaces with vertex set $[q]$ \emph{with a marked planar submap}. This correspondence gives an analogue of the Harer-Zagier formula for general surfaces. We also show that this formula implies a recursion formula due to Ledoux for the numbers of unicellular maps with given numbers of vertices and edges.

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