On the Probability of the Number of Solutions for the P4P Problem*

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Abstract

This paper studies the multi-solution phenomenon for the perspective four point (P4P) problem from geometric and algebraic aspects. We give a pure geometric proof that the P4P problem could have up to five solutions. We also give a clear picture on how these five solutions could be realized. We prove that with probability one, the P4P problem has a unique solution which can be represented by a set of rational functions in the parameters. The simulant experiments show that to solve the P4P problem with the rational functions is stable and accurate.

Keywords: Camera calibration, pose determination, perspective four point problem, P4P, probability, number of solutions.

1 Introduction

One of the fundamental goals of computer vision is to discover properties that are intrinsic to a scene by one or several images of this scene[3, 7, 14]. Within this paradigm, an essential process is the determination of the position and orientation of the sensing device (the camera) with respect to objects in the scene [2]. This problem is known as the perspective n point (PnP) problem and has many applications in automation, image

analysis, automated cartography, photogrammetry, robotics and model based machine vision system. Fishler and Bolles [3] summarized the problem as follows:

> "Given the relative spatial locations of n control points, and given the angle to every pair of control points from an additional point called the Center of Perspective (C_P), find the lengths of the line segments joining C_P to each of the control points."

One of the major concerns of the PnP problem is its multi-solution phenomenon. The reason is that if the solution is not unique then we need further determine which solution is the one we want. Unfortunately, all PnP problems for $n \leq 5$ have multiple solutions. It is well-known that the P3P problem could have one to four essentially different solutions [1, 3, 5, 12, 21, 20, 18].

In order to obtain a unique solution, one natural way is to add one more control point to consider a P4P problem. In [15, 16], the P4P problem was reduced to a system of polynomial equations and then a set of solutions was found under the assumption that there exists one solution for the P4P problem in normal circumstances. Algorithms for solving the P4P problem were proposed in [9, 17] without considering the number of solutions. When the control points are coplanar, the P4P problem has a unique solution [1]. If the control points are not coplanar, the P4P problem could have up to five solutions [10]. It is proved that the P5P problem could have two solutions [11]. For $n \geq 6$, the PnP problem has one solution and can be solved with the DLT method [4].

For the P4P problem, we first give a pure geometric proof that the P4P problem has up to five solutions. This proof gives a clear picture on how the five solutions could be achieved. Previous work makes the assumption that in the general case the P4P problem has a unique solution [10, 15, 16]. In [10], this fact is also supported by the computation of a large set of data. In this paper, we give a proof that the probability for the P4P problem to have a unique solution is one. This result provides a theoretical support for the algorithms for solving P4P problems, which are based on the assumption that the P4P problem generally has one solution. To show this property, it is not enough to say that adding more constraints to the P3P problem will lead to constraints among the parameters. Geometrically, we need to prove the following fact. Let N be the set of parameters where the P4P problem has solutions, and M the set of parameters where the P4P problem has more than one solutions. Then M must be of less dimension than that of N. In certain sense, this is to say that M is a subset of N with zero "volume". We further show that this unique solution could be represented by a set of rational functions in the parameters, and to solve the P4P problem with the rational functions is stable and accurate. Similar results are also true for the P5P problem.

The rest of the paper is organized as follows. In Section 2, give a geometric proof that the P4P problem has up to five solutions. In Section 3, we prove the main result about the P4P problem. In Section 4, conclusions are presented. The proof for the main theorem is given in the appendix.

2   Number of solutions for P4P problem

Lemma 2.1 *If the four control points of the P4P problem are not on the same plane then it could only have a finite number of solutions.*

Proof. Let the four control points be A, B, C and D. The P3P problem with center of perspective P and control points A, B, C has an infinite number of solutions if and only if P is on the circumscribed circle of triangle ABC [5]. Thus, the P4P problem has an infinite number of solutions if and only if point P is on the circumscribed circles of triangles ABC, ABD, ACD, BCD, which could happen if and only if points A, B, C, D and P are on the same circle. This contradicts to the fact that A, B, C, D are not co-planar. ▮

Lemma 2.2 *Let P and Q be solutions of the P4P problem with non-planar control points A, B, C and D. If P and Q are symmetric with respect to plane ABC, then D must be on line PQ.*

Proof. Since the angles between line PD and the lines PA, PB and PC are given, PD is the intersection line of three cones with PA, PB and PC as the central axes respectively. PD is the unique intersection line of the three cones since points A, B, C are not collinear. Similarly, QD is also the unique intersection line of three cones Q_A, Q_B, Q_C with QA, QB, QC as central axes. Suppose that D is not on line PQ. Let line PD meet plane ABC at point $\bar{D}$. Since A, B, C, D are not co-planar, $\bar{D} \neq D$. Since D is not on the line PQ, QD and $Q\bar{D}$ are different lines. Because P and Q are symmetric with plane ABC, we have $\angle AQ\bar{D} = \angle AP\bar{D} = \angle APD = \angle AQD$. Similarly, $\angle BQ\bar{D} = \angle BQD$, $\angle CQ\bar{D} = \angle CQD$. In other words, $Q\bar{D}$ is also the intersection of the three cones Q_A, Q_B, Q_C. This contradicts to the uniqueness of the intersection line. Therefore, D must be on line PQ. ▮

Theorem 2.3 *The P4P problem with non co-planar control points A, B, C, D could have up to five solutions. If it has five solutions, then there exists one pair of solutions which are symmetric with one of the planes ABC, ABD, ACD or BCS and among the other three solutions there exist no two which are symmetric with this plane.*

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Proof. By Lemma 2.1, the P4P problem could only have a finite number of solutions. There are at most eight solutions for the P3P problem with control points A, B and C, of which four solutions are above the plane ABC and four are under the plane ABC symmetrically. The solutions of the P4P problem must be from these eight solutions. By Lemma 2.2, if two solutions P and Q, symmetric with respect to plane ABC, are both solutions of the P4P problem, then D is on line PQ. It is clear that no other solutions can be on line PQ. It is impossible for another pair of solutions symmetric with plane ABC of the P3P problem to be the solutions of the P4P problem, since point D is already on line PQ. Then the P4P problem could have at most one pair of solutions symmetric with plane ABC. As a consequence, the P4P problem has at most five solutions. In the following table, we give a concrete P4P problem which has five solutions (Fig. 1 and Fig. 2). Therefore, the P4P problem could have up to five solutions. ∎

Parameter	$\lvert AB \rvert$	$\lvert AC \rvert$	$\lvert BC \rvert$	$\lvert AD \rvert$	$\lvert CD \rvert$	$\lvert BD \rvert$
value	1	1	1	$\frac{\sqrt{13}}{6}$	$\frac{\sqrt{13}}{6}$	$\frac{\sqrt{13}}{6}$
Parameter	$\cos\angle BPC$	$\cos\angle APC$	$\cos\angle APB$	$\cos\angle CPD$	$\cos\angle APD$	$\cos\angle BPD$
value	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

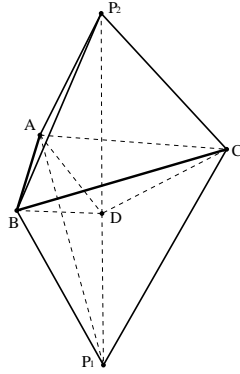Table 1. Parametric values for which the P4P problem has 5 solutions
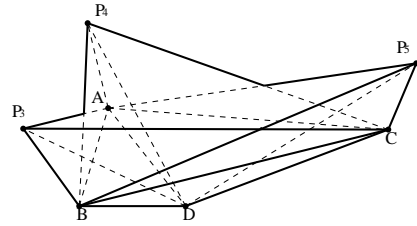

Figure 1: Two solutions symmetric with plane ABC

Figure 2: Three other solutions

3 Solving the P4P problem

3.1 Probabilities for the number of solutions

Let P be the center of perspective, and A, B, C, D the control points. Let $\lvert PA \rvert = X, \lvert PB \rvert = Y, \lvert PC \rvert = L, \lvert PD \rvert = Z, l_1 = \lvert AB \rvert^2, l_2 = \lvert AC \rvert^2, l_3 = \lvert BC \rvert^2, l_4 = \lvert AD \rvert^2, l_5 = \lvert CD \rvert^2, l_6 = \lvert BD \rvert^2, p = 2\cos\angle(BPC), q = 2\cos\angle(APC), r = 2\cos\angle(APB),$

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$s = 2\cos\angle(CPD), t = 2\angle(APD), u = 2\cos\angle(BPD)$. The *P4P equation system* is:

$$\begin{cases} p_1 = X^2 + Y^2 - XYr - l_1 = 0 \\ p_2 = X^2 + L^2 - XLq - l_2 = 0 \\ p_3 = Y^2 + L^2 - YLp - l_3 = 0 \\ p_4 = X^2 + Z^2 - XZs - l_4 = 0 \\ p_5 = Z^2 + L^2 - ZLt - l_5 = 0 \\ p_6 = Y^2 + Z^2 - YZu - l_6 = 0 \end{cases} \tag{1}$$

We will explain that the solutions of the P4P problem for four non-coplanar points A, B, C, D correspond to the positive solutions of equation system (1) for X, Y, Z, and L. It is clear that the solutions of the P4P problem must be solutions of the P3P problem with points A, B, C. As shown by Theorem 2.3, if two of the solutions P_1 and P_2 are on different sides of plane ABC, then points P_1, P_2, D are collinear. Since A, B, C, D are not co-planar, this guarantees that P_1 and P_2 are different solutions of equation system (1). Therefore, we did not miss any solution with equation system (1).

Let $l_1 = awL^2$, $l_2 = bwL^2$, $l_3 = wL^2$, $l_4 = cwL^2$, $l_5 = dwL^2$, $l_6 = ewL^2$, $X = xL$, $Y = yL$, $Z = zL$. Equation system (1) becomes the following equation system

$$\begin{cases} q_1 = x^2 + y^2 - xyr - aw = 0 \\ q_2 = y^2 + 1 - yp - bw = 0 \\ q_3 = x^2 + 1 - xq - w = 0 \\ q_4 = x^2 + z^2 - xzs - cw = 0 \\ q_5 = z^2 + 1 - zt - dw = 0 \\ q_6 = y^2 + z^2 - yzu - ew = 0 \end{cases} \tag{2}$$

We assume the "reality conditions" (See [5]) for the P3P problems with perspective center P and control points A, B, C; A, B, D; A, C, D; and B, C, D respectively. Basically speaking, we assume that each of the above triangles is proper and P is not on the planes determined by these triangles.

Lemma 3.1 *For a set of values of the parameters* $\mathcal{V} = \{l_1, l_2, l_3, l_4, l_5, l_6, p, q, r, s, t, u\}$, *the number of positive solutions of equation system (1) is the same as that of (2).*

Proof. Since $|q| = |2\cos(\angle APC)| < 2$, we have $w = x^2 + 1 - xq = (x - q/2)^2 + 1 - q^2/4 > 0$. We thus may establish the following one to one correspondence between the positive solutions of (1) and (2)

$$(X, Y, L, Z) \xrightarrow{x=X/L, y=Y/L, z=Z/L, w=l_3/L^2} (x, y, z, w) \xrightarrow{L=\sqrt{l_3/w}, X=xL, Y=yL, Z=zL} (X, Y, L, Z). \blacksquare$$

Definition 3.2 *A set of parametric values for* $\mathcal{U} = \{a, b, c, d, e, p, q, r, s, t, u\}$ *is called physical if equation system (2) has at least one set of positive solution.*

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Let $\mathbf{P}$ be the set of physical parametric values for parameters $\mathcal{U}$. For $\mathcal{U}_0 \in \mathbf{P}$, let $\mathbf{Z}_{\mathcal{U}_0}$ be the set of positive solutions of equation system (2) for parametric values $\mathcal{U}_0$.

Theorem 3.3 *Use the notations introduced above.*

1. *For all $\mathcal{U}_0 \in \mathbf{P}$, the probability for $|\mathbf{Z}_{U_0}| = 1$ is one. In other words, for physical parameters, the probability for the P4P problem to have a unique solution is one.*

2. *We may find rational functions P, Q, R in the parameters such that with a probability one for all physical parameters, the solutions for the P4P problem can be obtained as follows $x = P, y = Q, z = R$.*

To explain the above theorem, we need some concepts which may be found in [8, 19]. Here we use examples to illustrate the concepts. For a set of polynomials PS and a polynomial D, we use $\mathrm{Zero}(PS)$ to denote the set of real solutions for equations $PS = 0$ and $\mathrm{Zero}(PS/D) = \mathrm{Zero}(PS) - \mathrm{Zero}(D)$. $\mathrm{Zero}(PS/D)$ is called a *quasi-variety*. We may add inequalities to a quasi-variety to obtain *semi-algebraic sets*. The *dimension* of a quasi variety or a semi algebraic set is the number of variables that can take arbitrary values. For instance, the half unit sphere $H = \mathrm{Zero}(\{x^2 + y^2 + z^2 - 1\}) \cap \{(x, y, z) \,|\, z \geq 0\}$ is a semi-algebraic set of dimension two; the unit circle $C = \mathrm{Zero}(\{x^2 + y^2 - 1, z\})$ is a quasi-variety of dimension one. Since the dimension for C is one, its area is zero. While the area for H is 2π. Then the probability for a random point $P \in H$ belongs to C is zero and the probability for a point $P \in H$ belongs to $H - C$ is one.

For the P4P problem, let $\mathbf{P}$ be the set of physical parametric values for parameters $\mathcal{U}$ and $\mathbf{T}$ a set of parametric values such that the P4P has one solution in $\mathbf{P} - \mathbf{T}$. We will prove that $\mathbf{P}$ is of dimension nine and $\mathbf{T}$ is of lower dimension than that of $\mathbf{P}$. Since $\mathbf{P}$ and $\mathbf{T}$ are quasi-algebraic sets, inside any sufficiently large finite box B, $B \cap \mathbf{P}$ has a positive volume as a nine-dimensional manifold and $B \cap \mathbf{T}$ has a zero volume. Since B can be of any size, we may conclude that the probability for a random parametric point from $\mathbf{P}$ lies in $\mathbf{P} - \mathbf{T}$ is one. With these concepts, Theorem 3.3 is a consequence of the following result.

Theorem 3.4 *The physical parameter set $\mathbf{P}$ for the P4P problem is a semi-algebraic set of dimension nine in $\mathbb{R}^{11}$. We may find a subset $\mathbf{T}$ of $\mathbf{P}$ with dimension less than nine and rational functions P, Q, R in the parameters $\mathcal{U}$ such that for $\mathcal{U}_0 \in \mathbf{P} - \mathbf{T}$, the equation system (2) has one solution which can be given by $x = P, y = Q, z = R$.*

The proof can be found in the appendix. In what below, we will show how to compute the three rational functions P, Q, R and give a sketch of the proof.

For polynomials f and g, let $r = \text{Res}(f, g, x)$ be the resultant of f and g with variable x [13]. Eliminating w from (2) with $q_3 = 0$, we have the following equivalent equations:

$$\begin{cases} h_1 = x^2 + y^2 - xyr - a(x^2 + 1 - xq) = 0 \\ h_2 = y^2 + 1 - yp - b(x^2 + 1 - xq) = 0 \\ h_3 = x^2 + z^2 - xzs - c(x^2 + 1 - xq) = 0 \\ h_4 = z^2 + 1 - zt - d(x^2 + 1 - xq) = 0 \\ h_5 = y^2 + z^2 - yzu - e(x^2 + 1 - xq) = 0 \end{cases} \qquad (3)$$

Let $f_1 = h_1 - h_2$, $f_2 = \text{Res}(f_1, h_1, y)$, $f_3 = h_3 - h_4$, $f_4 = \text{Res}(f_3, h_3, z)$. Then

$$\begin{cases} f_1 = (p - rx)y + (1 - a + b)x^2 + (qa - bq)x - a + b - 1 \\ f_2 = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 \\ f_3 = (t - sx)z + (d - c + 1)x^2 + (cq - dq)x - c + d - 1 \\ f_4 = b_0 x^4 + b_1 x^3 + b_2 x^2 + b_3 x + b_4 \end{cases} \qquad (4)$$

where

$$\begin{cases} a_0 = b^2 + a^2 - br^2 - 2ab + 2b + 1 - 2a \\ a_1 = -2a^2 q + (4bq + pr + 2q)a - pr + prb - 2b^2 q - 2bq + bqr^2 \\ a_2 = (2 + q^2)a^2 + (-prq - p^2 - 4b - 2q^2 b)a - 2 + 2b^2 - br^2 + b^2 q^2 + r^2 - prbq + p^2 \\ a_3 = -2a^2 q + (p^2 q + pr - 2q + 4bq)a - pr - 2b^2 q + 2bq + prb \\ a_4 = 1 + 2a - 2b + b^2 + a^2 - 2ab - p^2 a \\ b_0 = d^2 + c^2 - 2cd + 2d - ds^2 - 2c + 1 \\ b_1 = -2d^2 q + (-2q + qs^2 + 4cq + ts)d + 2cq - 2c^2 q - ts + tsc \\ b_2 = (2 + q^2)d^2 + (-tsq - 2cq^2 - s^2 - 4c)d + 2c^2 + c^2 q^2 + s^2 - t^2 c - 2 + t^2 - tscq \\ b_3 = -2d^2 q + (ts + 2q + 4cq)d + t^2 cq - 2cq - 2c^2 q - ts + tsc \\ b_4 = 1 - t^2 c - 2d + 2c + c^2 - 2cd + d^2 \end{cases} \qquad (5)$$

Compute the subresultant sequence[13] for f_2 and f_4 with respect to x. Let the final two polynomials in the sequence be f_5 and f_6 respectively. We have

$$f_5 = i_5 x - u_5, \quad f_6 = i_6 a^8 - u_6 \qquad (6)$$

where i_5, u_5, i_6, u_6 are polynomials in the parameters $\mathcal{U}$ and u_6 is of degree less than eight in a; These polynomials can be found at http://www.mmrc.iss.ac.cn/˜xgao/paper/gao-pnpa.pdf. Substituting x by u_5/i_5 in f_1 and f_3 and clear denominators, we have

$$f_7 = i_7 y - u_7, \quad f_8 = i_8 z - u_8 \qquad (7)$$

where $u_7 = (1 - a + b)u_5^2 + (qa - bq)u_5 i_5 + i_5^2(-a + b - 1)$, $i_7 = ru_5 i_5 - pi_5^2$, $u_8 = (d - c + 1)u_5^2 + (cq - dq)u_5 i_5 + (-c + d - 1)i_5^2$, $i_8 = su_5 i_5 - ti_5^2$. From $f_5 = f_7 = f_8 = 0$, we may represent x, y and z as rational functions in the parameters $\mathcal{U}$.

$$x = \frac{u_5}{i_5}, y = \frac{u_7}{i_7}, z = \frac{u_8}{i_8}. \qquad (8)$$

Substituting x, y, z by $\frac{u_5}{i_5}, \frac{u_7}{i_7}, \frac{u_8}{i_8}$ in h_5 and clear the denominators, we have

$$f_9 = i_9 e - u_9 \qquad (9)$$

where $u_9 = -i_5^2 i_3^2 u_2^2 - i_5^2 i_2^2 u_3^2 + u i_5^2 i_2 i_3 u_2 u_3$ and $i_9 = -i_2^2 i_3^2 u_5^2 - i_5^2 i_2^2 i_3^2 + q i_5 i_2^2 i_3^2 u_5$.

Lemma 3.5 *For a set of parametric values, if $f_6 = 0, f_9 = 0, i_5 i_6 i_7 i_8 i_9 \neq 0$, then equation system (2) has one set of solutions which can be computed with (8) and $w = x^2 + 1 - xq$.*

The proof will be given in the Appendix.

The following is a sketch of the proof for Theorem 3.4. From the computation process, the physical parameters must satisfy two equations $f_6 = 0, f_9 = 0$. There are eleven parameters in $\mathcal{U}$ and equations $f_6 = 0, f_9 = 0$ will reduce the number of free parameters by two. Intuitively speaking, the set of physical parameters $\mathbf{P}$ should be of dimension nine. Let $\mathbf{I} = \mathrm{Zero}(i_5 i_6 i_7 i_8 i_9)$. For a $\mathcal{U}_0 \in \mathbf{P} - \mathbf{I}$, x, y, z can be solved with (8). We need only to show that the dimension of $\mathbf{P} \cap \mathbf{I}$ is less than that of $\mathbf{P}$.

3.2 Experimental Result

We give the following procedure to solve the P4P problem. By Lemma 3.5, if the parameters are from real observations, that is, the P4P problem has at least one solution for these parameters, then the procedure will solve the P4P problem with probability one.

- Compute the l_1, l_2, l_3, l_4, l_5, l_6 and p, q, r, s, t, u from the control points, the image points and the camera calibration matrix.

- Compute the a, b, c, d, e as follows. $a = \frac{l_1}{l_3}, b = \frac{l_2}{l_3}, c = \frac{l_4}{l_3}, d = \frac{l_5}{l_3}, e = \frac{l_6}{l_3}$.

- Compute (x, y, z, w) as follows. $x = \frac{u_5}{i_5}, y = \frac{u_7}{i_7}, z = \frac{u_8}{i_8}, w = \frac{u_5^2 + i_5^2 - q u_5 i_5}{i_5^2}$ where $i_5, u_5, i_7, i_8, u_7, u_8$ may be computed with the formulas given after equations (6) and (7). If $i_5 i_7 i_8 = 0$, the computation fails.

- Compute X, Y, L, Z as follows: $L = \sqrt{l_3/w}$, $X = xL, Y = yL$, $Z = zL$. By Lemma 3.5, X, Y, L, Z are a set of solutions to the P4P problem.

We will discuss the accuracy and the stability of the algorithm. The following experiments are done with Maple.

The optical center is located at the origin and the matrix of camera's intrinsic parameters is assumed to be the identity matrix. At each trial, four non-coplanar controls points are generated at random within a cube centered at $(0, 0, 50)$ and of dimension $60 \times 60 \times 60$.

In the first experiment, we test the accuracy of our algorithm with a large set of data. For a set of solutions obtained with the algorithm, we substitute them into (2) and check whether the substituted values are zero or not. We take 1000 sets of parameters randomly. The maximal substituted value into (2) is $0.1 * 10^{-9}$, which is satisfactory.

In the second experiment, we test the stability of our algorithm with a larger set of data. One hundred trials are carried out and one hundred sets of parameters are computed for each trial. For each set of parametric values, two sets of solutions are computed: one with the original control points denoted by $S = (X, Y, L, Z)$; the other with the control points perturbed by random noises denoted by $\hat{S} = (\hat{X}, \hat{Y}, \hat{L}, \hat{Z})$. Similar to [10], if $\max(|X - \hat{X}|, |Y - \hat{Y}|, |L - \hat{L}|, |Z - \hat{Z}|) < 1.5$, then S and $\hat{S}$ are considered as the same solution. In trial i, let n_i be the number of the parametric values such that the two results are the same. Let $\tilde{n} = (\sum_{i=1}^{100} n_i)/100$, $\frac{\|100 - \tilde{n}\|}{100}$ the absolute error, and $\frac{\sum_{i=1}^{100} \widetilde{X}_i}{100}$ the relative error where $\widetilde{X}_i$ is the maximal value of $\max(\frac{|X-\hat{X}|}{|X|}, \frac{|Y-\hat{Y}|}{|Y|}, \frac{|L-\hat{L}|}{|L|}, \frac{|Z-\hat{Z}|}{|Z|})$ for the one hundred sets of parametric values in the i-th trial. The experimental results are listed in the following table, from which we can observe that the algorithm yields very graceful degradation with increasing noises and is, therefore very stable.

Noise Level	0.001	0.01	0.1	0.5	1.0	2.0	3.0	4.0
Absolute error	0	0	0	0	0.09	0.55	0.81	0.91
Relative error	$1.2 * 10^{-5}$	$1.2 * 10^{-4}$	0.0012	0.0061	0.012	0.025	0.035	0.046

Table 2. Error for the Algorithm

4 Conclusion

For the P4P problem, we proved that the probability for them to have a unique solution is one. We further show that the unique solution to the P4P problem could be represented by a set of rational functions in the parameters and to solve these problems with the rational functions is quite stable and accurate.

Theoretically, we may obtain the explicit conditions for the P4P problem to have one, two, ... solutions. But, these conditions will be quite complicated and are computationally very difficult to get. On the other hand, the probabilities obtained in this paper give clear information on the distribution of the solutions for the P4P problems and may be used as a guidance on the number of solutions during the solving process.

5 Appendix: Proof for Theorem 3.4

We need several concepts from algebraic geometry which may be found in [8, 13, 19].

Proof for Lemma 3.5. Continue with the computation in Section 3.2. Note that f_6 in (6) is free of e, then the following is an irreducible ascending chain [13, 19] under the

variable order $b < c < d < p < q < r < s < t < u < a < e < x < y < z < w$.

$$\begin{cases} f_6 = i_6 a^8 - u_6, f_9 = i_9 e - u_9 \\ f_5 = i_5 x - u_5, f_7 = i_7 y - u_7, f_8 = i_8 z - u_8, q_2 = -w + x^2 + 1 - xq \end{cases} \tag{10}$$

We may check that the pseudo-remainders of the polynomials in (2) with respect to (10) are zero. Hence, we have the following remainder formulas (see [19, 13] for definition)

$$J^{s_i} q_i = b_{i,1} f_6 + b_{i,2} f_9 + b_{i,3} f_5 + b_{i,4} f_7 + b_{i,5} f_8 + b_{i,6} q_2, i = 1, \dots, 6 \tag{11}$$

where $J = i_5 i_6 i_7 i_8 i_9$, $s_i, i = 1, \dots, 6$ are non-negative integers, and $b_{i,j}$ are polynomials. From equations (11), we have

$$\text{Zero}((2)) = \text{Zero}((10)/J) \cup \text{Zero}((2) \cup \{J\}). \tag{12}$$

Let $S_0 = \text{Zero}(\{f_6, f_9\}/J)$. Then S_0 is a quasi-variety for parameters $\mathcal{U}$ with dimension nine. For a set of values $\eta \in S_0$, since $J(\eta) \neq 0$, we may solve x, y, z, w with $f_5 = f_7 = f_8 = q_2 = 0$ in (10). Let the solutions be x_0, y_0, z_0, w_0. Substitute η, x_0, y_0, z_0, w_0 into (11). Since $J(\eta) \neq 0$, we have $q_i(\eta, x_0, y_0, z_0, w_0) = 0, i = 1, \dots, 6$. Then η, x_0, y_0, z_0, w_0 consist of a set of solutions to (2). By the computation procedure, the polynomials in (10) are linear combinations of q_i. Then this is the only solution to (2). $\blacksquare$

Proof for Theorem 3.4. Use the notations introduced in the proof of Lemma 3.5. Since $w = x^2 + 1 - xq > 0$, we have $\text{Zero}((2)) = \text{Zero}((2)/w)$. Under the variable order $x < y < z < p < q < r < u < t < w < a < b < c < d < e$, (2) is an irreducible ascending chain [19, 13]. The initial of the polynomials in (2) is w. Then $\text{Zero}((2)/w)$ is an irreducible quasi variety whose dimension is the number of the variables minus the number of polynomials in (2), which is $15 - 6 = 9$. Since the pseudo-remainder of J with respect to (2) is not zero, J does not vanish on $\text{Zero}((2)/w)$. By the affine dimension theorem [8], $Z_0 = \text{Zero}((2) \cup \{J\}/w)$ is of lower dimension than nine. Applying Wu-Ritt Zero decomposition [13, 19], we have

$$\text{Zero}((2) \cup \{J\}) = \cup_{j=1}^d \text{Zero}(C_i/J_j)$$

where C_j are irreducible ascending chains [13, 19] and J_j are the products of initials of the polynomials in C_j. Since Z_0 is of dimension less that nine, we have $|C_j| > 6$. Let $A_j = C_j \cap Q[\mathcal{U}]$. By Lemma 2.1, the P4P problem has a finite number of solutions. Then each C_j contains four polynomials with leading variables x, y, z, w. As a consequence, $|A_j| > 2$. We may assume that J_j are polynomials free of x, y, z, w [19]. Let $S_j = \text{Zero}(A_j/J_j), j = 1, \dots, d$. Since $|A_j| > 2$, S_j is of dimension less than nine. Then **S** is the set of the parametric values for which the x, y, z, w have complex solutions. S_0 is of dimension nine and $S_j, j > 0$ are of dimension lower than nine.

The physical parametric set could be obtained as follows

$$\mathbf{P} = \cup_{j=0}^{d} S_i'$$

where $\mathbf{S}_j'$ is a semi-algebraic set obtained by adding some inequalities to $\mathbf{S}_i$ so that x, y, z, w have positive solutions [5]. Let

$$\mathbf{T} = \cup_{j=1}^{d} S_i'.$$

Since $\mathbf{S}_i' \subset \mathbf{S}_i$, the dimension of $\mathbf{T}$ is less than nine. By Lemma 3.5, for parametric values in $\mathbf{P} - \mathbf{T}$, we may solve x, y, z with (8). We now need only to show that the dimension of $\mathbf{P}$ is nine. Let $\mathbf{N}$ be the semi-algebraic set of dimension nine defined in $\mathbb{R}^9$ by $x > 0, y > 0, z > 0, |p| < 2, |q| < 2, |r| < 2, |s| < 2, |t| < 2, |u| < 2$. We know that $w = x^2 + 1 - xq > 0$. Since $|r| < 2$, $a = \frac{(x-y)^2 + (2-r)xy}{w} > 0$ for values from $\mathbf{N}$. Similarly, b, c, d, e are also positive. Then we may define a rational map from $\mathbf{N}$ to $\mathbf{P}$ as follows:

$$
\begin{aligned}
&p = p, q = q, r = r, s = s, t = t, u = u, \\
&a = \tfrac{x^2+y^2-xyr}{x^2+1-xq}, b = \tfrac{y^2+1-yp}{x^2+1-xq}, c = \tfrac{x^2+z^2-xzs}{x^2+1-xq}, d = \tfrac{z^2+1-zt}{x^2+1-xq}, e = \tfrac{y^2+z^2-yzu}{x^2+1-xq}
\end{aligned}
\tag{13}
$$

We may also define a rational map from $\mathbf{P}$ to $\mathbf{N}$ as follows.

$$p = p, q = q, r = r, s = s, t = t, u = u, x = \tfrac{u_5}{i_5}, y = \tfrac{u_7}{i_7}, z = \tfrac{u_8}{i_8} \tag{14}$$

For $J \neq 0$, map (14) has meanings. We may prove that (13) and (14) are inversion maps and thus are biratiobal maps between $\mathbf{N}$ and $\mathbf{P}$. Therefore, $\mathbf{P}$ is of dimension nine. ∎

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