A class of variational functionals in conformal geometry

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Abstract

We derive a class of variational functionals which arise naturally in conformal geometry. In the special case when the Riemannian manifold is locally conformal flat, the functional coincides with the well studied functional which is the integration over the manifold of the k-symmetric function of the Schouten tensor of the metric on the manifold.

1 Introduction

The purpose of this article is to derive a class of variational functionals which arise naturally in conformal geometry. Recall on a Riemannian manifold (M^n, g), $n \geq 3$, the full Riemannian curvature tensor Rm decomposes as

$$Rm = W \ \oplus \ P \owedge g,$$

where W denotes the Weyl tensor,

$$P \ = \ \frac{1}{n-2}(Ric - \frac{R}{2(n-1)}g)$$

denotes the Schouten tensor, and $\owedge$ is the Kulkarni-Nomizu wedge product (see [Be87, pp. 110]). Under a conformal change of metrics

$g_w = e^{2w}g$, where w is a smooth function over the manifold, the Weyl curvature changes pointwisely as $W_{g_w} = e^{2w}W_g$. Thus, all the information of the Riemannian tensor under a conformal change of metrics is reflected by the change of the Schouten tensor, and it is natural to study the elementary symmetric function $\sigma_k(g^{-1}P_g)$ (which we later denote as $\sigma_k(g)$) of the eigenvalues of the Schouten under the conformal change of metrics. For example when $k = 1$, $\sigma_1(g) = \frac{1}{2(n-1)}R_g$, where R_g denotes the scalar curvature of g. The study of the equation $\sigma_1(g) = $ constant under conformal change of metrics is the classical Yamabe problem. In [V00], Viaclovsky proved the following statements: Consider the functional

$$F_k(g) := \int_{M^n} \sigma_k(g)dv_g, \qquad (1)$$

(i): When $k = 1$ *or* 2, and $2k < n$, F_k is variational in the conformal class of metrics $g_w \in [g]$ with fixed volume one, i.e. the extremal metric for the functional in this class of metrics, when achieved, satisfies the equation

$$\sigma_k(g_w) = \text{constant}.$$

(ii) When $k \geq 3$ and $2k < n$, assertion in (i) only holds when the manifold M^n is locally conformally flat.

(iii) When $k = 2$, $n = 4$, $F_2(g)$ is conformally invariant; while for $k = \frac{n}{2}$ and $k \geq 3$, $F_k(g)$ is conformally invariant only when the manifold (M^n, g) is locally conformally flat.

We remark that in [BG06], Branson and Gover have also proved that the metric being locally comformally flat is also a necessary condition for the statement of (ii) above.

In this article, we generalize the role played by the curvature polynomial $\sigma_k(g)$ to a new class of curvature invariant, $v^{(2k)}(g)$, so that: $v^{(2k)}(g)$ agrees with $\sigma_k(g)$ when (M^n, g) is locally conformally flat; the functional

$$\mathcal{F}_k(g) = \int_{M^n} v^{(2k)}(g)dv_g \qquad (2)$$

satisfies the variational property in the statement (ii) above for all $2k < n$; $\mathcal{F}_k(g)$ is conformally invariant when $2k = n$ for all Riemannian manifolds (M^n, g).

2

Our construction is closely related to the recent construction for the Q curvature of R. Graham and Juhl [GJ06]. To state our result, we first recall some definitions and basic results.

Definition 1 *Given (X^{n+1}, M^n, g^+) with smooth boundary $\partial X = M^n$. Let r be a defining function for M^n in X^{n+1} so that $r > 0$ in X while $r = 0$ on M and $dr|_M \neq 0$. We say g^+ is conformally compact, if there exists such r so that $(X^{n+1}, r^2 g^+)$ is a compact manifold. We say (X^{n+1}, M^n, g^+) is conformally compact Einstein if g^+ is Einstein (i.e. $Ric_{g^+} = cg^+$ for some constant c). We call g^+ Poincare metric if $Ric_{g^+} = -ng^+$.*

Fefferman-Graham [FG85] proved that for any given (M^n, g_0), there is an extension, g^+, which is "asymptotically Poincare Einstein" in a neighborhood of M^n, i.e. on $[0, \epsilon) \times M^n$ for some positive ϵ.

For c sufficiently small, we have $X_c = \{r \leq c\} \subset X$ is diffeomorphic to $[0, c] \times M$. Hence, given a local coordinate chart $(x^1, \cdots, x^n)$ on M, $(r, x^1, \cdots, x^n)$ is a coordinate chart on X_c. Thus, we can write

$$g_+ = \frac{1}{r^2}(dr^2 + h_{ij}(r, x)dx^i dx^j), \tag{3}$$

where $h(r, \cdot)$ is a metric defined for $M_c := r = c \subset X$. We remark that in this expression, $(X, r^2 g_+)$ is a compact manifold.

$$dvol_h(r, x) = \sqrt{\det h_{ij}(r, x)}dx^1 \cdots dx^n.$$

We now expand the quantity $\sqrt{\frac{\det h_{ij}(r,x)}{\det h_{ij}(0,x)}}$ in an expansion near $r = 0$ as

$$\sqrt{\frac{\det h_{ij}(r, x)}{\det h_{ij}(0, x)}} = \sum_{k=0}^{\infty} v^{(k)}(x, h_0)r^k, \tag{4}$$

where $v^{(k)}(x, h_0)$ is a curvature invariant of the metric $h_0 := h_{ij}(0, \cdot)$.

In [G00], R. Graham asserted that $v^{(k)}$ vanished for k odd and $2k < n$; furthermore, he has established that when n is even, the quantity $\int_M v^{(n)} dvol_{h_0}$ is conformally invariant over the conformal class of metrics of $[h_0]$. This quantity is related to the conformal invariant term in the expansion of "renormalized volume" in the study of conformal field theory. In the later papers, [GZ03], [FG02] and [GJ06], the authors have also established that the quantity is the same of the integral of the Q curvature and is an important global conformally invariant term.

Another fact which has been pointed out in [GJ06], based on a result in [SS00], is that when h_0 is locally conformally flat, then

$$v^{(2l)}(h_0) = (-2)^{-l}\sigma_l(h_0). \tag{5}$$

In this article, we prove the following

Theorem 2 *For any metric h on M and $2k \leq n = \dim M$, define the functional*

$$\mathcal{F}_k(h) = \int_M v^{(2k)}(h)dvol_h / (\int_N dvol_h)^{\frac{n-2k}{n}}, \tag{6}$$

then $\mathcal{F}_k$ is variational within the conformal class when $2k < n$; i.e., the critical metric in $[h]$ satisfies the equation

$$v^{(2k)} = \text{constant}.$$

For $n = 2k$, $F_{\frac{n}{2}}(h)$ is constant in the conformal class $[h]$.

Remark 3 *As we have mentioned before, the case $n = 2k$ in the theorem above has been established earlier in [G00].*

For $k = 1$, 2 cases, the new curvature invariant $v^{(2k)}$ turns out to agree, up to a scale, with the well-studied curvature polynomial $\sigma_k(g)$. Actually we have

$$v^{(2)}(g) = -\frac{1}{2}\sigma_1(g),$$

$$v^{(4)} = \frac{1}{4}\sigma_2(g).$$

For $k = 3$, Graham and Juhl ([GJ06], page 5) have also listed the following formula for $v^{(6)}$

$$v^{(6)}(g) = -\frac{1}{8}\left[\sigma_3(g) + \frac{1}{3(n-4)}(P_g)^{ij}(B_g)_{ij}\right],$$

where $(B_g)_{ij} := \frac{1}{n-3}\nabla^k\nabla^l W_{likj} + \frac{1}{n-2}R^{kl}W_{likj}$ is the Bach tensor of the metric.

In this article, we carry out the computation for $v^{(2)}$ and $v^{(4)}$. As the computation indicates, a straight forward computation of $v^{(6)}$ is quite complicated. Instead, we derive the variational properties of $v^{(6)}$ under conformal change of metrics directly, which is another verification of our main theorem in this special case. Another purpose of the derivation is to derive the variational formulas of $v^{(6)}$

under conformal change of metrics. We believe the study of the PDE $v^{(6)}(g_w) = constant$ will be of interest to problems in conformal geometry. Another interesting question is the "uniqueness" problem of curvature invariants which are extensions of $\sigma_k(g)$ invariants in the locally conformally flat case and satisfy the properties as $v^{(2k)}$ in Theorem 2 above. We hope to address these two problems in a future work.

This article is organized as follows: In Section 2, we prove Theorem 2; In Section 3, we explicitly compute $v^{(2)}$ and $v^{(4)}$ for all dimension n; In Section 4, we show the variational property of $v^{(6)}$ and discuss some properties of this curvature invariants under conformal change of metrics.

2 Proof of Theorem 2

Suppose (X^{n+1}, M^n, g_+) is a conformally compact Einstein manifold. Let r be an arbitrary smooth defining function for $M = \partial X$ defined near M and set $\bar{g} = r^2 g+$. We now recall some basic properties of the metric g_+ in this setting with respect to the changing of defining functions.

Lemma 4 *([G00], Lemma 2.1 and 2.2):*

(a) A metric on M in the conformal infinity of g^+ determines a unique defining function r in a neighborhood of M such that $\bar{g}|_{TM}$ is the prescribed boundary metric with $|dr|_{\bar{g}}^2 = 1$.
(b)

$$g_+ = r^{-2}(dr^2 + g_r) \tag{7}$$

on $[0, \epsilon) \times M$ for some $\epsilon > 0$. Furthermore,

$$g_r = g + r^2 g^{(2)} + r^4 g^{(4)} + .. + r^n g^{(n)} + \tilde{h} r^n log r + \cdots \tag{8}$$

when n is even, with $g = r^2 g^+|_M$, and symmetric tensors $g^{(2)}$, $g^{(4)}$, ... up to $g^{2(n-1)}$ and $Tr_g g^{(n)}$ are determined by g, and $Tr_g \tilde{h} = 0$.
(c) Let r and $\hat{r}$ be two special defining functions as in (a) associated with two different conformal representatives in the conformal class of metrics in $[g]$; then

$$\hat{r} = re^w \tag{9}$$

for a function w on $[0, \epsilon) \times M$ satisfying

$$w_r + r(w_r^2 + |d_M w|^2) = 0. \tag{10}$$

Furthermore, the power series expansion of w at $r = 0$ consists only of even power of r up through and including the r^{n+1} term.

Now for a fixed smooth function ϕ defined on M, we consider a family of conformal metrics $g_t = e^{2t\phi}g$ on M. By Lemma 4 there exist functions r_t on a neighborhood of M in X such that

$$g_+ = \frac{1}{r_t^2}(dr_t^2 + h_t(r_t, \cdot)), \tag{11}$$

where $h_t(c, \cdot)$ is a metric defined on $M_{t,c} = \{r_t = c\} \subset X$. Furthermore, we have the following asymptotic expansion

$$h_t = g_t + r_t^2 g_t^{(2)} + \cdots. \tag{12}$$

For a point $p \in X$, define

$$w(t, p) = \log(\frac{r_t(p)}{r(p)}).$$

On the boundary, we have that $w(t, \cdot)|_M = t\phi(\cdot)$. Thus, $w(t, p) = w(t, r, x)$ is a smooth extension of $t\phi(x)$ to $X_{t,c} := \{r_t < c\} \subset X$ for some proper c. We then get the following from Lemma 4:

Corollary 5 *For $(t, x, r) \in [0, 1] \times [0, \epsilon) \times M$, we have*

$$\frac{\partial}{\partial r}|_{r=0}w(t, r, x) = 0. \tag{13}$$

$$\frac{\partial^2}{\partial r^2}|_{r=0}w(t, r, x) = -\frac{1}{2}t^2|d_M\phi(x)|_g^2. \tag{14}$$

$$\frac{\partial^k}{\partial r^k}|_{r=0}w(t, r, x) = 0, \text{for k odd.} \tag{15}$$

$$\frac{\partial^k}{\partial r^k}|_{r=0}w(t, r, x) = O(t^2), \text{for k even, } 0 < k < n. \tag{16}$$

In particular, we have

$$w(t, r, x) = t\phi(x) - \frac{1}{4}t^2|\nabla_M\phi(x)|^2 r^2 + O(t^2 r^4). \tag{17}$$

and

$$\frac{d}{dt}|_{t=0}w(t, r, x) = \phi(x), \tag{18}$$

independent of the choice of the defining function r.

For future use, we define a useful vector field associated to the conformal metric variation.

First we notice that, fix an ϵ small enough, for a given point $p \in [0, \epsilon) \times M = X_{0,\epsilon}$, for each t, we can assign a local coordinate chart $p = (r_t, x_t) \in X_{t,\epsilon}$ for $t \in [0, 1]$, with $r_0 = r, x_0 = x$ and with x_t defined as $pr_t(p)$, the projection image of p onto M under the metric $r_t^2 g_+$.

For each fixed $c < \epsilon$, denote $M_{c,t} = \{p \in X | r_t = c\}$, then by Lemma 4, the set $M_{c,t}$ is diffeomorphic to M via the projection with respect to the metric $r_t^2 g_+$. Hence, they are diffeomorphic to each other. For a fixed level set $M_c := M_{c,0}$, since the projectoin pr_t is a small perturbation of pr_0 when t is small, the map $p \to (c, x_t)$ gives arise to a diffeomorphism of M_c; hence it introduces a vector field

$$F_c := F_{c,0} = \frac{d}{dt}|_{t=0}(c, x_t) \tag{19}$$

on TM_c. Since M_c is naturally diffeomorphic to M, without confusion, we also denote the pushed-forward vector field on M as F_c.

It is worth pointing out that: the vector field is induced from the one parameter family of points x_t, which depends on the original point p; hence the induced vector field depends on the choice of c.

Since we have a family of diffeomorphisms to identify a neighborhood of M in X with $[0, \epsilon) \times M$, given a local coordinate chart $(x^1, \cdots, x^n)$ on M, $(r_t, x^1, \cdots, x^n)$ is a coordinate chart on $M_{c,t}$, for each t. Thus, a given point $p \in X$ near M can be represented in these coordinate system as (r_t, x_t), respectively.

We now consider the volume of g_+ at a given point $p \in M$. For future convenience, we will omit the foot index 0, and denote $h = h_0$, $g = g_0 = h_0|_{TM}$, $r = r_0$, $x = x_0$ etc. Thus, $p = (r, x)$ and the metric $r^2 g_+$ is compact.

Recall $r_t(p) = re^{w(t,p)}$ near M. For each t, by (11) and (12),

$$dvol_{g^+}(p) = r_t^{-n-1} dr_t dvol_{h_t}(r_t, x_t) = r_t^{-n-1} \sqrt{\frac{\det h_t(r_t, x_t)}{\det g_t(x_t)}} dr_t dvol_{g_t}(x_t), \tag{20}$$

and

$$\sqrt{\frac{\det h_t(r_t, x_t)}{\det g_t(x_t)}} = \sum_k v^{(k)}(x_t, g_t) r_t^k.$$

Notice that, via diffeomorphism, we can view $dvol_{g_t}(x_t)$ as a n-form on $M_{r_t, t}$.

We now proceed by take the time derivative of Equation (20). For notational convenience, we define the following linear operator

$$D = \frac{d}{dt}|_{t=0}.$$

We prove a technical lemma.

Lemma 6 *At the point $p = (r, x)$, the following formula hold:*
   (a)
$$D(dr_t)(r, x) \wedge dvol_g(x) = \phi(x)dr \wedge dvol_g(x).$$

   (b)
$$D[v^{(k)}(x_t, g_t)] = (F_r v^{(k)})(x, g) + D[v^{(k)}(x, g_t)],$$

where the definition of F_r is given in (19) and the remarks following (19);
   (c)

$$D[dvol_{g_t}(x_t)] = [\mathcal{L}_{F_r}(dvol_g)](x) + n\phi(x)dvol_g(x),$$

where $\mathcal{L}$ is the Lie derivative on M_r with respect to the given vector field.

Proof of Lemma 6. Since $r_t = e^w r_0 = e^w r$, we have

$$dr_t = r_t dw + e^w dr.$$

Apply D to both sides of above equation, and use Lemma 4 and (18),

$$
\begin{aligned}
D(dr_t) &= rd\phi + r\phi\, dw|_{t=0} + \phi\, dr \\
&= rd\phi + \phi dr.
\end{aligned}
$$

We then wedge above expression with $dvol_g(x)$, and observe that the term $rd\phi(x) \wedge dvol_g(x) = 0$ at the point $p = (r, x)$, we have thus established statement (a).

Statement (b) follows directly by Leibniz rule and (19).

To prove statement (c), we apply Leibniz rule again and get

$$D[dvol_{g_t}(x_t)] = D[dvol_g(x_t)] + D[dvol_{g_t}(x)].$$

Notice that when restricted to M, $g_t = e^{2t\phi}g$, the result follows easily.

We now give the proof of Theorem 2.

Proof of Theorem 2. Starting with the following basic equation

$$
\begin{aligned}
0 &= D(dvol_{g^+}(p)) \\
&= D(r_t^{-n-1}(\sum_k v^{(k)}(x_t, g_t)r_t^k)dr_t dvol_{g_t}(x_t)),
\end{aligned}
$$

we use Leibniz Rule and apply (17) and Lemma 5 above to get

$$
\begin{aligned}
0 &= \sum_k r^{k-n-1}\{D[v^{(k)}(x_t, g_t)] + k\phi(x)v^{(k)}(x, g)\}dr \wedge dvol_g(x) \\
&\quad + \sum_k r^{k-n-1}dr\mathcal{L}_{F_r}[v^{(k)}dvol_g(x)]. \qquad\qquad (21)
\end{aligned}
$$

We integrate (21) over M_r which is identified to M via the canonical diffeomorphism. Since the form involving the Lie derivative is exact, it will vanish after the integration. Thus, we get

$$
dr \sum r^{k-n-1} \int_M \{D[v^{(k)}(x_t, g_t)] + k\phi(x)v^{(k)}(x, g)\}dvol_g(x) = 0.
$$
$$(22)$$

Now notice that the above equation holds for all small r, we prove the following identity:

Claim 7

$$
\int_M \{D[v^{(k)}(x, g_t)] + k\phi(x)v^{(k)}(x, g)\}dvol_g = 0. \qquad (23)
$$

We now finish the proof of Theorem 2.

Given $g_t = e^{2t\phi}g$ a variation of metrics on M in the conformal class of g, denote $V = \int_M dvol_g$. Then,

$$
\begin{aligned}
D[\mathcal{F}_k(g_t)] &= \frac{1}{V^{1-\frac{2k}{n}}} \int_M D[v^{(2k)}(x, g_t)] \, dvol_g(x) + n \int_M v^{(2k)}(x, g)\phi(x) \, dvol_g(x) \\
&\quad - \frac{n-2k}{nV^{2-\frac{n-2k}{n}}} \int_M n\phi \, dvol_g \int_M v^{(2k)}(x, g) \, dvol_g(x) \\
&= \frac{n-2k}{V^{1-\frac{2k}{n}}} \int_M [v^{(2k)} - \frac{\int_M v^{(2k)}(g, x)dvol_g(x)}{V}]\phi \, dvol_g.
\end{aligned}
$$

9

It implies that, when $n > 2k$, the critical metric g of the functional $\mathcal{F}_k$ satisfies

$$v^{(2k)}(g) = constant.$$

When $n = 2k$, the computation shows that the functional is invariant under the conformal deformation.

3 Computation of $v^{(4)}$ for $n > 4$

In this Section, we verify the formula for $v^{(4)}$. In particular, we prove that for any dimension $n > 4$, $v^{(4)}$ equals to $\sigma_2(A)$ up to a constant. We remark that this formula is stated without proof in [GJ06], and the method of derivation is known to experts in this field, thus we will be brief in our derivation.

We start with the basic equations

$$Ric(g_+) = -ng_+, \tag{24}$$

$$g_+ = \frac{1}{r^2}(dr^2 + h(r, \cdot)) \tag{25}$$

$$h = g + r^2 g^{(2)} + r^4 g^{(4)} + \cdots + \tilde{h} r^n log r + \cdots. \tag{26}$$

For future convenience, we denote

$$C_k = g^{(k)} g^{-1}.$$

First, we have the following Lemma, which follows from a simple computation.

Lemma 8

$$v^{(2)} = \frac{1}{2} \text{Tr } C_2,$$

$$v^{(4)} = \frac{1}{2}[\text{Tr}C_4 + \sigma_2(C_2) - \frac{1}{4}(\text{Tr } C_2)^2]. \tag{27}$$

We set up a local coordinate $\{x_1, \cdots, x_n\}$ on M_r; thus, $\{x_1, \cdots, x_n, x_{n+1} = r\}$ is a coordinate on X. By (24) and (25), we get,

$$rh''_{ij} + (1-n)h'_{ij} - h^{kl}h'_{kl}h_{ij} - rh^{kl}h'_{ik}h'_{lj} + \frac{r}{2}h^{kl}h'_{kl}h'_{ij} - 2r \; Ric(h)_{ij} = 0, \tag{28}$$

10

where we use $'$ to denote the derivative with respect to r and $Ric(h)$ is the Ricci curvature of the submanifold M_r with respect to the restricted metric $h(r, \cdot)$.

We analyze (28) by (26). Since $n > 4$ we get

$$h' = 2rg^{(2)} + 4r^3g^{(4)} + O(r^4),$$
$$h'' = 2g^{(2)} + 12r^2g^{(4)} + O(r^3). \tag{29}$$

Studying the coefficient of r in (28), we get a tensor equation over M,

$$2g^{(2)} + (1-n)(2g^{(2)}) - 2g^{kl}g^{(2)}_{kl}g - 2Ric(g) = 0. \tag{30}$$

Taking trace with respect to g, which we will denote as Tr_g, we get

$$R_g = (2 - 2n)\text{Tr}_g g^{(2)},$$

or,

$$J_g = \frac{R_g}{(2n-2)} = -\text{Tr}_g g^{(2)}. \tag{31}$$

Combine (30) and (31), we get

$$g^{(2)} = \frac{1}{(2-n)}(Ric_g - J_g) = -P_g. \tag{32}$$

and

$$v^{(2)} = \frac{-1}{2}J_g.$$

We now apply the same method to compute $v^{(4)}$. Studying the coefficient of r^3 in (28), we have

$$12g^{(4)} + (1-n)(4g^{(4)}) - \alpha - \beta + \gamma - 2\delta = 0, \tag{33}$$

where $\alpha, \beta, \gamma, \delta$ are the coefficients of r^3 in $\text{Tr}[h^{-1}h']h$, $rh'h^{-1}h'$, $\frac{r}{2}h^{-1}h'h'$ and $r\,Ric(h)$, respectively.

We now compute these coefficients. First notice that

$$\begin{aligned}h^{-1} &= [(\text{Id} + \text{B})\text{g}]^{-1} = \text{g}^{-1}(\text{Id} - \text{B} + \text{B}^2 - \text{B}^3 + \cdots), \\ &= g^{-1}(1 - r^2C_2 - r^4C_4 + r^4C_2^2 + o(r^4)). \end{aligned} \tag{34}$$

Combing with (28), we get

$$\alpha = -2(\text{Tr } C_2^2)g + \text{Tr}_g[(4g^{(4)})]g + \text{Tr}_g[2g^{(2)}]g^{(2)}. \tag{35}$$

11

Similarly, we can get,

$$\beta = 2g^{(2)} \, g^{-1} \, (2g^{(2)}), \tag{36}$$

$$\gamma = \frac{1}{2}\mathrm{Tr}_g[2g^{(2)}](2g^{(2)}), \tag{37}$$

Thus, by (33), we get

$$0 = 12g^{(4)} + (1-n)(4g^{(4)}) + 2(\mathrm{Tr}\ C_2^2)g - \mathrm{Tr}_g[(4g^{(4)})]g$$
$$-\mathrm{Tr}_g[2g^{(2)}]g^{(2)} - 4g^{(2)} \, g^{-1} \, g^{(2)} + 2\mathrm{Tr}_g[g^{(2)}](g^{(2)}) - 2\delta \tag{38}$$

Taking trace with respect to g, we get

$$(16 - 8n)\mathrm{Tr}_g g^{(4)} + (2n - 4)(\mathrm{Tr}\ C_2^2) - 2\mathrm{Tr}_g\delta = 0;$$

or, equivalently we get the fomula

$$[(8 - 4n)\mathrm{Tr}\ C_4 + (n - 2)(\mathrm{Tr}\ C_2^2)] - \mathrm{Tr}_g\delta = 0. \tag{39}$$

Regarding to the term involving δ, we will prove the following

Lemma 9

$$\mathrm{Tr}_g\delta = (4 - 4n)\mathrm{Tr}\ C_4 + (n - 1)\mathrm{Tr}\ C_2^2.$$

This lemma can be verified by relating the scalar curvature of $\bar{g}$ to that of the scalar curvature of g_+ and $h(r, .)$. The computation is tedious but relatively routine. We will skip the detail here.

Combine the formula in 39 and 9, we have

$$\mathrm{Tr}\ C_4 = \mathrm{Tr}\ C_2^2. \tag{40}$$

By Lemma 8 and (40), we get

$$v^{(4)} = \frac{1}{2}[\mathrm{Tr}\ C_4 + \sigma_2(C_2) - (\frac{1}{4})(\mathrm{Tr}C_2^2)] = \frac{1}{2}(\sigma_2(C_2) - \frac{1}{2}\sigma_2(C_2)) = \frac{1}{4}\sigma_2(C_2).$$

Noticing that $C_2 = -P_g$, where P_g is the Schouten tensor, we have established the following

Theorem 10 *For $n > 4$, we have*

$$v^{(4)}(g) = \frac{1}{4}\sigma_2(g).$$

12

4 Variational property of $v^{(6)}$

In this section, we study the properties of

$$v^{(6)}(g) = -\frac{1}{8}\left[\sigma_3(g) + \frac{1}{3(n-4)}(P_g)^{ij}(B_g)_{ij}\right].$$

We give a direct proof of the following special case of Theorem 2.

Theorem 11 *For any metric h on M and $6 \leq n = \dim M$, define a functional*

$$\mathcal{F}_3(h) = \int_M v^{(6)}(h)dvol_h / (\int_M dvol_h)^{\frac{n-6}{n}}, \tag{41}$$

then $\mathcal{F}_3$ is variational within the conformal class; i.e., the critical metric in $[h]$ satisfies the equation

$$v^{(6)} = constant. \tag{42}$$

For $n = 6$, we have that $F_3(h)$ is constant in the conformal class $[h]$.

To prove the theorem, we first recall some basic conformal transformation law for the curvature invariants involved.

Lemma 12 *For a fixed smooth function ϕ defined on M, we consider two conformal equivalent metrics, g and $g_\phi = e^{2\phi}g$ on M. Then we have, under a local coordinate system,*

$$P(g_\phi)_{ij} = P_{ij} - \phi_{ij} + \phi_i\phi_j - \frac{1}{2}|\nabla\phi|^2 g_{ij}, \tag{43}$$

$$B(g_\phi)_{ij} = e^{-2\phi}[B_{ij} - (n-4)(C_{ikj} + C_{jki})\phi^k - (n-4)W_{kijl}\phi^k\phi^l], \tag{44}$$

where W and C are the Weyl tensor and Cotten tensor of g, respectively.

We now consider a family of conformal metrics on M, $g_t = e^{2t\phi}g$ and denote $D = \frac{d}{dt}|_{t=0}$ as before.

Now we can compute $D[\mathcal{F}_3(g_t)]$. We separate the computation in two steps.

First,

$$D \int B_{ij}(g_t) P^{ij}(g_t) dvol(g_t)$$

$$= \int \{D[B_{ij}(g_t)]P^{ij} + B_{ij}D[P^{ij}(g)t]\}dvol + B_{ij}P^{ij}D[dvol(g_t)]\}$$

$$= \int [-2\phi B_{ij}P^{ij} - 2(n-4)C_{ijk}P^{ij}\phi^k - B_{ij}\phi^{ij} + (n-4)\phi B_{ij}P^{ij}]dvol$$

$$= \int [-2\phi B_{ij}P^{ij} - 2(n-4)C_{ijk}P^{ij}\phi^k + \nabla^i B_{ij}\phi^j + (n-4)\phi B_{ij}P^{ij}]dvol$$

$$= \int [-2(n-4)C_{ijk}P^{ij}\phi^k - (n-4)C_{ijk}P^{ij}\phi^k + (n-6)\phi B_{ij}P^{ij}]dvol$$

$$= \int [-3(n-4)C_{ijk}P^{ij}\phi^k + (n-6)\phi B_{ij}P^{ij}]dvol,$$

Second, define the Newton tensor as

$$T^{ij} = \sigma_2(g)g^{ij} - \sigma_1(g)P^{ij} + (P^2)^{ij}.$$

Then we have

$$D\left[\int \sigma_3(g_t)dvol(g_t)\right]$$

$$= \int [\sigma_3(g)(n-6)\phi - T^{ij}\phi_{ij}]dvol$$

$$= \int [\sigma_3(g)(n-6)\phi + T^{ij}_{,j}\phi_i]dvol. \tag{45}$$

By the Bianchi identity, we have

$$P^{ij}_{,j} = \nabla^i J;$$
$$(P^2)^{ij}_{,j} = (P^k_i P^{kj})_{,j}$$
$$= P^i_k J^{,k} + P^i_{k,j}P^{kj}$$
$$= P^i_k J^{,k} + g^{is}C_{ksj}P^{kj} + P_{kj}P^{,i}_{kj}$$
$$= P^i_k J^{,k} + C^{kij}P_{kj} + \frac{1}{2}\nabla^i[\mathrm{Tr}_g(P^2)]$$
$$= P^{ik}J_{,k} + C^{kij}P_{kj} + \frac{1}{2}\nabla^i[\mathrm{Tr}_g(P^2)]. \tag{46}$$

Thus, we have

$$\int T^{ij}_{,j}\phi_i dvol$$

$$= \int \{\sigma_2^{,i} - J_{,j}P^{ij} - JJ^{,i} + P^{ik}J_{,k} + C^{kij}P_{kj} + \frac{1}{2}\nabla^i[\text{Tr}_g(P^2)]\}\phi_i dvol$$

$$= \int [C^{kij}P_{kj}\phi_i + \nabla^i[\sigma_2 + \frac{1}{2}(\text{Tr}_g P^2 - J^2)]\phi_i dvol$$

$$= \int C^{kij}P_{kj}\phi_i dvol. \tag{47}$$

Finally, we can combine these to get

$$DF_3(g_t)$$
$$= -\frac{1}{8}[D \left(\int \sigma_3(g_t)dvol(g_t)\right) + \frac{1}{3(n-4)}D \int B_{ij}(g_t)P^{ij}(g_t)dvol(g_t)]$$
$$= (n-6)\int v^{(6)}\phi dvol. \tag{48}$$

Theorem 11 then follows easily.

Remark 13 *From (43) and (44), we see that the equation*

$$v^{(6)}(g_w) = \text{constant}. \tag{49}$$

is a second order PDE in terms of the conformal factor w. This is in analogue of $\sigma_k(g_w) = \text{constant}$ equation which has been intensively studied in recent years. It remains to see under what conditions can the PDE (49) be solved for metrics in a fixed conformal class and if the sign of the integral $\int v^{(6)}(g)dv_g$ plays some role and carries geometric information as in the case for $\int v^{(4)}(g)dv_g$ on manifolds of dimension 3 and 4 (cf. [GV01, CGY02]). The authors wish to report some further study of this problem in the future.

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