# Singularities of generic mean curvature flow

## Joint work with Bill Minicozzi

# MCF of closed embedded hyper-surfaces in $\mathbf{R}^{n+1}$.

A one-parameter family of smooth embedded hypersurfaces $M_t$ in $\mathbf{R}^{n+1}$ flow by mean curvature if

$$\partial_t x^\perp = \bar{H}\,.$$

Here $x$ is the position vector, $\mathbf{n}$ is the unit normal to the hyper-surface,

$$\bar{H} = -H\,\mathbf{n}\,.$$

and

$$H = \text{div}(\mathbf{n})\,.$$

The unit sphere moves through round concentric spheres and the flow stays smooth until the sphere disappears in a point.

Precisely: round concentric spheres with radii $\sqrt{-2nt}$ (for $t < 0$) flow by mean curvature and disappear at $t = 0$.

Planes (and, more generally, minimal surfaces) are static under the MCF.

Cylinders $\mathbf{S}^k \times \mathbf{R}^{n-k}$ with radii $\sqrt{-2kt}$ flow by MCF until they become extinct in an $(n-k)$-dimensional plane.

# Huisken's monotonicity.

Suppose now that the hyper-surfaces $M_t$ are closed and define a non-negative function on $\mathbf{R}^{n+1} \times (-\infty, 0)$ by

$$\Phi(x,t) = (-4\pi t)^{-\frac{n}{2}} \, \mathrm{e}^{\frac{|x|^2}{4t}} \, .$$

The Huisken volume is

$$\int_{M_t} \Phi \, .$$

A computation shows

$$\frac{d}{dt} \int_{M_t} \Phi = - \int_{M_t} \left| H\mathbf{n} + \frac{x^\perp}{2t} \right|^2 \Phi = - \int_{M_t} \left| H + \frac{\langle x, \mathbf{n} \rangle}{2t} \right|^2 \Phi \, .$$

Hence, the Huisken volume is monotone non-increasing under the MCF.

Similarly, for $t_0 \in \mathbf{R}$ and $x_0 \in \mathbf{R}^{n+1}$ if we define a non-negative function $\Phi_{x_0,t_0}$ on $\mathbf{R}^{n+1} \times (-\infty, t_0)$ by

$$\Phi_{x_0,t_0}(x,t) = \Phi(x - x_0, t - t_0)\,,$$

then

$$\int_{M_t} \Phi_{x_0,t_0}$$

defines a volume 'centered' at $(x_0, t_0)$ instead of at $(0, 0)$.

We have the monotonicity

$$\frac{d}{dt}\int_{M_t} \Phi_{x_0,t_0} = -\int_{M_t} \left| H - \frac{\langle x - x_0, \mathbf{n}\rangle}{2(t_0 - t)} \right|^2 \Phi_{x_0,t_0}\,.$$

## Self-similar shrinkers.

A MCF is said to be a self-similar shrinker if $M_t = \sqrt{-t}M_{-1}$ for $t < 0$. This can easily be seen to be equivalent to that $\Sigma = M_{-1}$ satisfies the equation

$$H = \frac{\langle x, \mathbf{n} \rangle}{2} \, . \tag{0.1}$$

That is, if $M_t = \sqrt{-t}M_{-1}$, then $M_{-1}$ satisfies (0.1) and conversely if $\Sigma$ is a hyper-surface satisfying (0.1), then $M_t = \sqrt{-t}\,\Sigma$ flow by mean curvature.

Self-shrinkers are exactly the hyper-surfaces where there is equality in Huisken's monotonicity formula.

## Examples of self-similar shrinkers:

Hyperplanes.

Round spheres.

General cylinders $\mathbf{R}^k \times \mathbf{S}^{n-k}$.

Angenent's shrinking donut.

Numerical examples of Angenent, Ilmanen, Chopp, Nguyen, Sethian, and others.

## The $F$ functional - critical points are self-shrinkers.

Suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is a hyper-surface with Euclidean volume growth (or more generally polynomial volume growth). That is,

$$\mathrm{Vol}(B_r \cap \Sigma) \leq C r^n \, .$$

For $t_0 > 0$ and $x_0 \in \mathbf{R}^{n+1}$ we define an $F_{x_0,t_0}$ functional by

$$\begin{aligned} F_{x_0,t_0} = F_{x_0,t_0}(\Sigma) &= (4\pi t_0)^{-n/2} \int_\Sigma \mathrm{e}^{-\frac{|x-x_0|^2}{4t_0}} \, d\mu \\ &= \int_\Sigma \Phi_{x_0,t_0}(\cdot, 0) \, . \end{aligned}$$

$F = F_{0,1}$

The following gives a useful way of thinking of a self-shrinker:

**Lemma 0.2.** $\Sigma$ is a self-shrinker
$\Longleftrightarrow$
$\Sigma$ is a critical point of the $F$ functional
$\Longleftrightarrow$
$\Sigma$ is a minimal surface in the metric $g_{ij} = \mathrm{e}^{-\frac{|x|^2}{2n}} \delta_{ij}$.

*Proof.* Suppose that $\Sigma_s$ is a variation of $\Sigma$ with normal variational field $f\,\mathbf{n}$ where $f$ is compactly supported. Then an easy computation gives

$$\frac{d}{ds} F(\Sigma_s) = (4\pi)^{-\frac{n}{2}} \int_\Sigma f \left( \frac{\langle x, \mathbf{n} \rangle}{2} - H \right) \mathrm{e}^{-\frac{|x|^2}{4}} \,.$$

□

## Smooth compactness theorem for self-shrinkers.

The following smooth compactness theorem for self-shrinkers in $\mathbf{R}^3$ will be useful later on.

**Theorem 0.3.** (C-Minicozzi).

Given an integer $g \geq 0$ and a constant $V > 0$, the space of smooth complete embedded self-shrinkers $\Sigma \subset \mathbf{R}^3$ with genus at most $g$, $\partial \Sigma = \emptyset$, and the scale-invariant area bound

$$\text{Area } (B_R(x_0) \cap \Sigma) \leq V \, R^2$$

for all $x_0 \in \mathbf{R}^3$ and all $R > 0$ is compact.

Namely, any sequence has a subsequence that converges in the topology of $C^k$ convergence on compact subsets for any $k \geq 2$.

Self-shrinkers model singularities in mean curvature flow ("MCF") so this compactness theorem gives some control over the possible singularities.

The genus bound and scale-invariant local area bound will automatically come from corresponding bounds on the initial surface in a MCF.

This compactness theorem will play an important role in understanding generic mean curvature flow.

# Tangent flows.

Self-shrinkers play an important role in the study of mean curvature flow, not least because they describe all possible blow ups at a given singularity of a mean curvature flow.

To explain this, we will need the notion of a tangent flow, which generalizes the tangent cone construction from minimal surfaces.

# Tangent flows continued.

The basic idea is that we can rescale a MCF in space and time to obtain a new MCF thereby expanding a small neighborhood of the point that we want to focus on.

Huisken's monotonicity gives uniform control over these rescalings and a standard compactness theorem then gives a subsequence converging to a limiting solution of the MCF. This limit is called a tangent flow.

A tangent flow will achieve equality in Huisken's monotonicity formula and, thus, must be a self-shrinker.

# Tangent flows; precise definition.

The precise definition of a *tangent flow* at a point $(x_0, t_0)$ in space-time of a MCF $M_t$ is as follows:

First translate $M_t$ in space-time to move $(x_0, t_0)$ to $(0, 0)$ and then take a sequence of parabolic dilations $(x, t) \to (c_j\, x, c_j^2\, t)$ with $c_j \to \infty$ to get MCF's

$$M_t^j = c_j \left( M_{c_j^2(t-t_0)} - x_0 \right) .$$

Using Huisken's monotonicity formula, and Ilmanens compactness theorem for Brakke flows, White and Ilmanen show that a subsequence of the $M_t^j$'s converges weakly to a limiting flow $\mathcal{T}_t$ that we will call a *tangent flow* at $(x_0, t_0)$.

Another application of Huisken's monotonicity shows that $\mathcal{T}_t$ is a self-shrinker.

Uniqueness of $\mathcal{T}_t$ is unknown. That is, whether different sequences of dilations might lead to different tangent flows.

## Conjecture about regularity of tangent flows.

Ilmanen: in $\mathbf{R}^3$ tangent flows at the first singular time must be smooth.

**Conjecture 0.4.** (Ilmanen). For a smooth one-parameter family of closed embedded surfaces in $\mathbf{R}^3$ flowing by mean curvature, every tangent flow at the first singular time has multiplicity one.

If this conjecture holds, then it would follow from Brakke's regularity theorem that near a singularity the flow can be written as a graph of a function with small gradient over the tangent flow.

# Smooth up to the first singular time

We will say that a MCF is smooth up to and including the first singular time if every tangent flow at the first singular time is smooth and has multiplicity one.

Conjecturally, all MCF in $\mathbf{R}^3$ are smooth up to the first singular time.

## Regularity conjecture in higher dimension.

It is a well-known conjecture that the space-time singular set of any (non-fattening) MCF starting at a closed smooth embedded $n$-dimensional hypersurface has codimension at least three (recall that time is counted as if it had dimension two, so that the space-time track that the hyper-surface fan-out has space-time dimension $n+2$).

Similarly, it is conjectured that any time slice of a tangent flow is an $n$-dimensional self-shrinker with singular set of dimension at most $n-3$; this is equivalent to that the singular set of the tangent flow has dimension at most $n-1$.

**Conjecture 0.5.** For a smooth closed embedded hypersurface $M_0$ in $\mathbf{R}^{n+1}$. A time slice of any tangent flow of the MCF starting at $M_0$ has multiplicity one and the singular set is of dimension at most $n-3$.

# Back to MCF starting at a smooth closed hyper-surface.

Results of Grayson, Huisken, Hamilton, Gage, White, Huisken-Sinestrari.

# Examples

The snake, the dumbbell.

# Virtually no hope of general classification.

Virtually no hope of general classification of tangent flows and self-shrinkers.

Recall Angenent's shrinking donut, numerical examples of Angenent, Ilmanen, Chopp, Nguyen, Sethian, and others.

# Recall the $F$ functional. The entropy.

Given $x_0 \in \mathbf{R}^{n+1}$ and $t_0 > 0$, define the functional $F_{x_0,t_0}$ by

$$F_{x_0,t_0}(\Sigma) = (4\pi t_0)^{-n/2} \int_\Sigma \mathrm{e}^{\frac{-|x-x_0|^2}{4t_0}} \, d\mu \, .$$

The *entropy* $\lambda = \lambda(\Sigma)$ of $\Sigma$ will be the supremum of the $F_{x_0,t_0}$ functionals

$$\lambda = \sup_{x_0,t_0} \; F_{x_0,t_0}(\Sigma) \, .$$

The key properties of the entropy $\lambda$ are:

- $\lambda$ is non-negative and invariant under dilations, rotations, or translations of $\Sigma$.
- $\lambda(M_t)$ is non-increasing in $t$ if the hypersurfaces $M_t$ flow by mean curvature or by the rescaled mean curvature flow.
- The critical points of $\lambda$ are self-shrinkers for the mean curvature flow.

# Entropies

In $\mathbf{R}^3$ we have the following entropies:

The plane has entropy 1.

The sphere has entropy $\frac{4}{\mathrm{e}} = 1.4715\ldots$.

The cylinder has entropy $\sqrt{\frac{2\pi}{\mathrm{e}}} = 1.5203\ldots$.

## $F$-stability.

The main point of these functionals is that $\Sigma$ is a critical point of $F_{x_0,t_0}$ precisely when it is the time $t = -t_0$ slice of a self-shrinking solution of the mean curvature flow that becomes extinct at $x = x_0$ and $t = 0$. It turns out that every critical point $\Sigma$ is unstable in the usual sense; i.e., there are always nearby hypersurfaces where the $F_{x_0,t_0}$ functional is strictly less. This is because translating a self-shrinker in space (or time) lowers the functional.

The stability that we are interested in, which we will call $F$-stability, mods out for these translations:

A critical point $\Sigma$ for $F_{x_0,t_0}$ is *$F$-stable* if for every bounded variation $\Sigma_s$ with $\Sigma_0 = \Sigma$, there exist variations $x_s$ of $x_0$ and $t_s$ of $t_0$ that make $F'' = (F_{x_s,t_s}(\Sigma_s))'' \geq 0$ at $s = 0$.

A bounded variation $\Sigma_s$ over $\Sigma$ is a one-parameter family of graphs over $\Sigma$ given by $\{x + s\, f(x)\, \mathbf{n}(x) \,|\, x \in \Sigma\}$ where $\mathbf{n}$ is the unit normal and $f$ is a bounded function on $\Sigma$.

# Entropy stability.

Recall that the key properties of the entropy $\lambda$ are:

- $\lambda$ is non-negative and invariant under dilations, rotations, or translations of $\Sigma$.
- $\lambda(M_t)$ is non-increasing in $t$ if the hypersurfaces $M_t$ flow by mean curvature or by the rescaled mean curvature flow.
- The critical points of $\lambda$ are self-shrinkers for the mean curvature flow.

These properties are the main advantages of the entropy functional over the $F$ functionals. The main disadvantage of the entropy is that it need not depend smoothly on $\Sigma$.

To deal with this, we will say that a self-shrinker is *entropy stable* if it is a strict local minimum for the entropy functional.

# Classification of entropy stable self-shrinkers.

To illustrate our results, we will first specialize to the case where $n = 2$, that is to mean curvature flow of surfaces in $\mathbf{R}^3$.

**Theorem 0.6.** (C-Minicozzi).

Suppose $\Sigma \subset \mathbf{R}^3$ is a self-shrinker with polynomial volume growth, but is not a sphere, a plane, or a cylinder.
Then $\Sigma$ can be perturbed to an arbitrarily close hypersurface $\tilde{\Sigma}$ where the entropy is strictly less.

In particular, $\Sigma$ cannot arise as a tangent flow to the MCF starting from $\tilde{\Sigma}$.

# $F$-stable in $\mathbf{R}^3$.

The next theorem will play a key role in the proof of the previous theorem (Theorem 0.6).

**Theorem 0.7.** (C-Minicozzi).

The sphere and the plane are the only smooth self-shrinkers in $\mathbf{R}^3$ with polynomial volume growth that are $F$-stable with respect to variations with compact support.

# Piecewise MCF.

A piece-wise MCF is a finite collection of MCF's $M^i_t$ on time intervals $[t_i, t_{i+1}]$ so that each $M^{i+1}_{t_{i+1}}$ is the graph over $M^i_{t_{i+1}}$ of a function $u_{i+1}$ and

$$\text{Area}\left(M^{i+1}_{t_{i+1}}\right) = \text{Area}\left(M^{i}_{t_{i+1}}\right) \, ,$$
$$\lambda\left(M^{i+1}_{t_{i+1}}\right) \leq \lambda\left(M^{i}_{t_{i+1}}\right) \, .$$

With this definition, area and entropy are non-increasing in $t$ even across the jumps.

We will define a piece-wise rescaled MCF similarly, except that each $M^i_t$ moves by a rescaled mean curvature flow.

# Generalization of Huisken and Grayson's theorems.

For simplicity, we assume in below that the MCF is smooth up to and including the first singular time. As mentioned, this would be the case for any MCF in $\mathbf{R}^3$ if the multiplicity one conjecture holds.

# Generalization of the results of Gage-Hamilton, Grayson, and Huisken.

**Theorem 0.8.** (C-Minicozzi).

For any closed embedded surface $\Sigma \subset \mathbf{R}^3$, there exists a piece-wise MCF $M_t$ starting at $\Sigma$ and defined up to time $t_0$ where the surfaces become singular. Moreover, $M_t$ can be chosen so that if

$$\liminf_{t \to t_0} \frac{\mathrm{diam} M_t}{\sqrt{t_0 - t}} < \infty \,,$$

then $M_t$ becomes extinct in a round point.

One consequence of this theorem (Theorem 0.8) is that if the initial surface is topologically not a sphere, then the piece-wise flow must develop a non-compact (after rescaling) singularity.

Remember the example of a dumbell.

# Higher dimensions

The next two theorems are the first steps for the higher dimensional version.

**Theorem 0.9.** (C-Minicozzi).

Suppose $\Sigma$ is a smooth self-shrinker with polynomial volume growth, but is not a sphere and does not split off a line isometrically.

Then $\Sigma$ can be perturbed to a graph $\tilde{\Sigma}$ over $\Sigma$ of a function with arbitrarily small $C^k$ norm (for any fixed $k$) so that the entropy of $\tilde{\Sigma}$ is strictly less than the entropy of $\Sigma$.

In particular, $\Sigma$ cannot arise as a tangent flow to the MCF starting from $\tilde{\Sigma}$.

## Higher dimensions continued.

The next theorem will be a key step in the proof of the previous theorem; Theorem 0.9.

**Theorem 0.10.** (C-Minicozzi).

If $\Sigma$ is a smooth self-shrinker in $\mathbf{R}^{n+1}$ with polynomial volume growth that is $F$-stable with respect to compactly supported variations, then it is either the round sphere or a hyperplane.

## Regularity of tangent flows.

As mentioned above, it is conjectured that the space-time singular set of any MCF starting at a closed smooth embedded $n$-dimensional hypersurface has codimension at least three (recall that time is counted as if it had dimension two, so that the space-time track that the hypersurface fan-out has space-time dimension $n + 2$).

Similarly, it is conjectured that the singular set of a tangent flow has dimension at most $n - 1$. Thus, in particular, since any tangent flow is self-similar it follows that any time slice of a tangent flow is an $n$-dimensional self-shrinker with singular set of dimension at most $n - 3$.

# Regularity of tangent flows continued.

In both theorems from two slides ago if $n \leq 6$, then we only need to assume that $\Sigma$ is an oriented integral varifold that is smooth off of a singular set with locally finite $(n-2)$-dimensional Hausdorff measure.