A NOTE ON CANONICAL RICCI FORMS ON 2-STEP NILMANIFOLDS

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Abstract. In this note we prove that any left-invariant almost Hermitian structure on a 2-step nilmanifold is Ricci-flat with respect to the Chern connection and that it is Ricci -flat with respect to another canonical connection if and only if it is cosymplectic (i.e. $d^*\omega = 0$).

1. introduction

Let (M, g, J, ω) be an almost Hermitian manifold. Gauduchon introduced in [13] a 1-parameter family ∇^t of canonical Hermitian connections which can be distinguished by the properties of the torsion tensor T. In this family ∇^1 corresponds to so-called Chern connection which can be defined as the unique Hermitian connection whose $(1,1)$-part of the torsion vanishes. In the *quasi-Kähler* case (i.e. when $\overline{\partial}\omega = 0$), the line $\{\nabla^t\}$ degenerates to a single point and the Chern connection is the unique canonical connection.

Any canonical connection ∇^t induces the so-called *Ricci form* $\rho^t(X,Y) = 2\,\mathrm{i}\,\mathrm{tr}_\omega R^t(X,Y)$, where R^t denotes the curvature of ∇^t. It turns out that ρ^t is always a closed form which can be locally written as the derivative of the 1-form $\theta^t(X) = \sum_{r=1}^n g(\nabla^t_X Z_r, Z_{\overline{r}})$, where $\{Z_r\}$ is a (local) unitary frame. Moreover, in the cosymplectic case (i.e. when $d\omega^{n-1} = 0$) the line $\{\theta^t\}$ degenerates to a single point (see Corollary 3.3) and all the canonical connections have the same Ricci form.

The aim of this paper is to study the Ricci forms ρ^t on 2-step nilmanifolds equipped with a left-invariant almost Hermitian structure. We recall that by definition a *k-step nilmanifold* is a compact quotient of a k-step nilpotent Lie group G by lattice. Since we are considering *left-invariant* almost Hermitian structures, we can work on Lie algrebras in an algebraic fashion. Our main result is the following

Theorem 1. *Let $(\mathfrak{g}, g, J, \omega)$ be a 2-step nilpotent Lie algebra with an almost Hermitian structure. Then (g, J) is Ricci-flat with respect to the Chern connection and it is Ricci-flat with respect to another canonical connection if and only if it is cosymplectic (i.e. $d^*\omega = 0$).*

This theorem has the following immediate consequence:

Corollary 1.1. *Every left-invariant almost Hermitian structure on a nilmanifold associated to a 2-step Lie group is Ricci-flat with respect to the Chern connection.*

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2. Preliminaries on canonical connections

Let (M, g, J, ω) be an almost Hermitian manifold, where ω is the fundamental form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. The almost complex structure J extends to r-forms as

$$J\alpha(X_1, \dots, X_n) = (-1)^r \alpha(JX_1, \dots, JX_n)$$

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inducing the splittings

$$TM\otimes\mathbb{C}=T^{1,0}M\oplus T^{0,1}M\,,\quad \Lambda^r(M,\mathbb{C})=\bigoplus_{p+q=r}\Lambda^{p,q}M\,,$$

where $\Lambda^r(M,\mathbb{C})$ is the vector bundle of complex r-forms on M. In particular Λ^3M splits as

$$\Lambda^3M=\Lambda^+M\oplus\Lambda^-M\,,$$

where $\Lambda^+M=(\Lambda^{2,1}M\oplus\Lambda^{1,2}M)\cap\Lambda^3M$ and $\Lambda^-M=(\Lambda^{3,0}M\oplus\Lambda^{0,3}M)\cap\Lambda^3M$. Given a 3-form γ we denote by γ^+ and γ^- the projection onto Λ^+M and Λ^-M, respectively. Moreover, denoting by $\Omega^2(TM)$ the vector space of smooth sections of $\Lambda^2M\otimes TM$, we have the splitting

$$\Omega^2(TM)=\Omega^{2,0}(TM)\oplus\Omega^{1,1}(TM)\oplus\Omega^{0,2}(TM)$$

where

$$\begin{aligned}\Omega^{2,0}(TM)&=\{B\in\Omega^2(TM)\ :\ B(JX,Y)=JB(X,Y)\}\,;\\ \Omega^{1,1}(TM)&=\{B\in\Omega^2(TM)\ :\ B(JX,JY)=B(X,Y)\}\,;\\ \Omega^{0,2}(TM)&=\{B\in\Omega^2(TM)\ :\ B(JX,Y)=-JB(X,Y)\}\,.\end{aligned}$$

Hence any $B\in\Omega^2(TM)$ can be written as $B=B^{2,0}+B^{1,1}+B^{0,2}$. Notice that in terms of complex vector fields of type $(1,0)$ we have

$$B^{2,0}(Z_i,Z_j)=B(Z_i,Z_j)+B(Z_i,Z_j)-\mathrm{i}\,JB(Z_i,Z_j)-\mathrm{i}\,JB(Z_i,Z_j)=2B(Z_i,Z_j)-2\,\mathrm{i}\,JB(Z_i,Z_j)\,.$$

In particular the condition $B^{2,0}=0$ can be written in terms of $(1,0)$ vector fields as $B(Z_i,Z_j)\in T^{0,1}M$. Furthermore $\Omega^2(TM)$ splits as

$$\Omega^2(TM)=\Omega^2_b(TM)\oplus\Omega^2_c(TM)$$

where

$$\begin{aligned}g(B_b(X,Y),Z)&=\frac{1}{2}(g(B(X,Y),Z)-g(B(Z,X),Y)-g(B(Y,Z),X))\,,\\ g(B_c(X,Y),Z)&=\frac{1}{2}(g(B(X,Y),Z)+g(B(Z,X),Y)+g(B(Y,Z),X))\,.\end{aligned}$$

Now we consider connections on M. A connection ∇ on M is called *Hermitian* if $\nabla J=0$, $\nabla g=0$. It is well-known that every almost Hermitian manifold admits Hermitian connections. We denote by $\mathcal{C}$ the space of Hermitian connection on M. Gauduchon introduced in [13] the following special class of Hermitian connections:

Definition 2.1. *A connection $\nabla\in\mathcal{C}$ is called* canonical *if its torsion T satisfies* $T_b^{1,1}=0$.

From [13] it follows that any canonical connection ∇ can be written as

$$\begin{aligned}g(\nabla_XY,Z)=&\,g(D_XY,Z)+\frac{t-1}{4}(d^c\omega)^+(X,Y,Z)+\frac{t+1}{4}(d^c\omega)^+(X,JY,JZ)-g(X,N(Y,Z))+\\ &\frac{1}{2}(d^c\omega)^-(X,Y,Z)\,.\end{aligned}\tag{2.1}$$

for some $t\in\mathbb{R}$, where d^c is the operator acting on r-forms as $d^c=(-1)^rJdJ$ and N denotes the Nijenhuis tensor $N(X,Y)=[JX,JY]-[X,Y]-J([JX,Y]+[X,JY])$.

For $t\in\mathbb{R}$ we denote by ∇^t the corresponding canonical connection. In the special case of a quasi-Kähler structure (i.e. $\overline{\partial}\omega=0$) the space of canonical connections reduces to a single point, while if J is integrable (i.e. $N=0$) equation (2.1) reduces to

$$g(\nabla^t_XY,Z)=g(D_XY,Z)+\frac{t-1}{4}(d^c\omega)(X,Y,Z)+\frac{t+1}{4}(d^c\omega)(X,JY,JZ)\,.$$

For the parameters $t=1,0,-1$, the family (2.1) gives the following remarkable cases

- $t=1$. In this case ∇^1 is called the *Chern connetion*. This connection can be defined as the unique Hermitian connection satisfying $T^{1,1}=0$.
- $t=0$. In this case ∇^0 is called the *first canonical connection*. This connection can be defined as the unique Hermitian connection whose torsion satisfies $T^{2,0}=0$.

- $t=-1$. In this case the connection ∇^{-1} is important in the complex case where it is known as the *Bismut connection*. Indeed, if J is integrable, then ∇^{-1} can be defined as the unique Hermitian connection having totally skew-symmetric torsion (see [6]).

3. Canonical Ricci forms

Let (M^{2n},g,J) be an almost Hermitian manifold and let $\mathcal{C}$ the space of the associated Hermitian connections. For any $\nabla\in\mathcal{C}$ it is defined the Ricci form $\rho(X,Y)=2\,\mathrm{i}\,\mathrm{tr}_\omega R(X,Y)$, where R is the curvature tensor $R(X,Y):=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$. Such a form is always closed and it locally satisfies $\rho=d\theta$, where $\theta(X)=\sum_{r=1}^n g(\nabla_X Z_r,Z_{\overline{r}})$ and $\{Z_r\}$ is a local unitary frame. In the case of a canonical connection $\nabla^t\in\mathcal{C}$ we use notation ρ^t and θ^t. We denote by $\natural$ the natural isomorphism between vector fields and 1-forms induced by g. Namely, if X a vector field, then we denote by $X^\natural$ the 1-form $X^\natural(Y)=g(X,Y)$. We have the following

Proposition 3.1. *θ^t is locally defined by*

$$\theta^t(X)=\sum_{r=1}^n \mathrm{i}\,\Im\mathfrak{m}\,\{g([X+t\,\mathrm{i}\,JX,Z_r],Z_{\overline{r}})\}+\frac{1}{2}\,\mathrm{i}(t-1)g(d^*\omega,X^\natural)\,. \tag{3.1}$$

for any vector field X.

Proof. First of all we note that if Z_r is a vector field of type $(1,0)$, then

$$N(Z_r,Z_{\overline{r}})=(d^c\omega)^-(X,Z_r,Z_{\overline{r}})=0\,,\quad (d^c\omega)^+(X,Z_r,Z_{\overline{r}})=d^c\omega(X,Z_r,Z_{\overline{r}})=-d\omega(JX,Z_r,Z_{\overline{r}})\,.$$

Hence if $\{Z_r\}$ is a local unitary frame using equation (2.1) we get

$$\begin{aligned}\theta^t(X)=&\sum_{r=1}^n\Big\{g(D_XZ_r,Z_{\overline{r}})+\frac{t-1}{4}(d^c\omega)(X,Z_r,Z_{\overline{r}})+\frac{t+1}{4}(d^c\omega)(X,JZ_r,JZ_{\overline{r}})\Big\}\\=&\sum_{r=1}^n\Big\{g(D_XZ_r,Z_{\overline{r}})-\frac{t}{2}d\omega(JX,Z_r,Z_{\overline{r}})\Big\}\,.\end{aligned}$$

Now

$$\begin{aligned}2g(D_XZ_r,Z_{\overline{r}})=&Xg(Z_r,Z_{\overline{r}})-Z_{\overline{r}}g(X,Z_r)+Z_rg(X,Z_{\overline{r}})+g([X,Z_r],Z_{\overline{r}})+g([Z_{\overline{r}},X],Z_r)-g([Z_r,Z_{\overline{r}}],X)\\=&-Z_{\overline{r}}g(X,Z_r)+Z_rg(X,Z_{\overline{r}})+g([X,Z_r],Z_{\overline{r}})+g([Z_{\overline{r}},X],Z_r)-g([Z_r,Z_{\overline{r}}],X)\end{aligned}$$

and

$$\begin{aligned}d\omega(JX,Z_r,Z_{\overline{r}})=&(JX)\omega(Z_r,Z_{\overline{r}})-Z_r\omega(JX,Z_{\overline{r}})+Z_{\overline{r}}\omega(JX,Z_r)-\omega([JX,Z_r],Z_{\overline{r}})\\&-\omega([Z_{\overline{r}},JX],Z_r)-\omega([Z_r,Z_{\overline{r}}],JX)\\=&Z_rg(X,Z_{\overline{r}})-Z_{\overline{r}}g(X,Z_r)+\mathrm{i}\,g([JX,Z_r],Z_{\overline{r}})+\mathrm{i}\,g([Z_{\overline{r}},JX],Z_r)-g([Z_r,Z_{\overline{r}}],X)\,.\end{aligned}$$

Then we have

$$\begin{aligned}\theta^t(X)=&\frac{1}{2}\sum_{r=1}^n\Big\{g([X+t\,\mathrm{i}\,JX,Z_r],Z_{\overline{r}})-g([X-t\,\mathrm{i}\,JX,Z_{\overline{r}}],Z_r)+g([Z_r,Z_{\overline{r}}],tX-X)\\&+(1-t)Z_rg(X,Z_{\overline{r}})-(1-t)Z_{\overline{r}}g(X,Z_r)\Big\}\\=&\sum_{r=1}^n\Big\{\,\mathrm{i}\,\Im\mathfrak{m}\,\{g([X+t\,\mathrm{i}\,JX,Z_r],Z_{\overline{r}})+(1-t)Z_rg(X,Z_{\overline{r}})\}-\frac{1}{2}(1-t)g([Z_r,Z_{\overline{r}}],X)\Big\}\,.\end{aligned}$$

So in order to prove the statement we have to show that

$$\sum_{r=1}^n\Big\{\Im\mathfrak{m}\,\{Z_rg(X,Z_{\overline{r}})\}+\mathrm{i}\,\frac{1}{2}g([Z_r,Z_{\overline{r}}],X)\Big\}=-\frac{1}{2}\,g(\omega,dX^\natural)\,. \tag{3.2}$$

We can write $X=\sum_{r=1}^n(X_rZ_r+X_{\overline{r}}Z_{\overline{r}})$ and $X^\natural=\sum_{r=1}^n(X_r\zeta^r+X_{\overline{r}}\zeta^{\overline{r}})$, where $\{\zeta^r\}$ is the coframe dual to of $\{Z_r\}$. Then we get

$$\begin{aligned} g(\omega,dX^\natural)&=\mathrm{i}\sum_{k=1}^n g(\zeta^k\wedge\zeta^{\overline{k}},dX^\natural)\\ &=\mathrm{i}\sum_{k,r=1}^n(Z_r(X_{\overline{r}})-Z_{\overline{r}}(X_r)-X^\natural([Z_r,Z_{\overline{r}}]))g(\zeta^k\wedge\zeta^{\overline{k}},\zeta^r\wedge\zeta^{\overline{r}})\\ &=\mathrm{i}\sum_{k=1}^n Z_k(X_{\overline{k}})-Z_{\overline{k}}(X_k)-X^\natural([Z_k,Z_{\overline{k}}]))\\ &=-2\sum_{k=1}^n\Im\mathfrak{m}\{Z_k(X_{\overline{k}})\}-\mathrm{i}\sum_{k,s=1}^n(B^s_{k\overline{k}}X_s+B^{\overline{s}}_{k\overline{k}}X_{\overline{s}})\\ &=-\sum_{k=1}^n\left(2\,\Im\mathfrak{m}\{Z_k(X_{\overline{k}})\}+\mathrm{i}\,g([Z_k,Z_{\overline{k}}],X)\right)\,, \end{aligned}$$

where with B we denote the components of the brackets. □

Corollary 3.2. *The following formulae hold*

- $\theta^1(X)=2\,\mathrm{i}\sum_{r=1}^n\Im\mathfrak{m}\,g([X^{0,1},Z_r],Z_{\overline{r}})\,;$
- $\theta^0(X)=\mathrm{i}\sum_{r=1}^n\Im\mathfrak{m}\,\{g([X,Z_r],Z_{\overline{r}})\}-\mathrm{i}\,\frac{1}{2}g(d^*\omega,X^\natural)\,;$
- $\theta^{-1}(X)=2\,\mathrm{i}\sum_{r=1}^n\Im\mathfrak{m}\,\{g([X^{1,0},Z_r],Z_{\overline{r}})\}-\mathrm{i}\,g(d^*\omega,X^\natural)\,.$

It is useful to write down formula (3.1) in real coordinates. In order to do this we write $Z_r=\frac{1}{\sqrt{2}}(e_r-\mathrm{i}\,Je_r)$ for a suitable orthonormal frame $\{e_1,\dots,e_n,Je_1,\dots,Je_n\}$. Then a direct computation gives

$$\begin{aligned} 2\Im\mathfrak{m}\,\{g([X+t\,\mathrm{i}\,JX,Z_r],Z_{\overline{r}})\}&=\Im\mathfrak{m}\,\{g([X+t\,\mathrm{i}\,JX,e_r-\mathrm{i}\,Je_r],e_r+\mathrm{i}\,Je_r)\}\\ &=g([X,e_r],Je_r)-g([X,Je_r],e_r)+tg([JX,e_r],e_r)+tg([JX,Je_r],Je_r) \end{aligned}$$

and

$$\begin{aligned} \theta^t(X)=&\frac{1}{2}\,\mathrm{i}\sum_{r=1}^n\{g([X,e_r],Je_r)-g([X,Je_r],e_r)+tg([JX,e_r],e_r)+tg([JX,Je_r],Je_r)\}\\ &+\frac{1}{2}\,\mathrm{i}(t-1)g(d^*\omega,X^\natural)\,. \end{aligned} \tag{3.3}$$

A remarkable consequence of formula (3.1) is the following

Corollary 3.3. *All canonical connections of a cosymplectic structure have the same Ricci form.*

Proof. It is enough to show that $\theta^1=\theta^{-1}$. Since the cosymplectic condition $d^*\omega=0$ implies

$$\theta^{-1}(X)=\sum_{r=1}^n 2\,\mathrm{i}\,\Im\mathfrak{m}\,\{g([X^{1,0},Z_r],Z_{\overline{r}})\}$$

we have

$$\begin{aligned} \theta^{-1}(X)=&-\sum_{r=1}^n 2\,\mathrm{i}\,\Im\mathfrak{m}\,\{g([X^{0,1},Z_{\overline{r}}],Z_r)\}=-\sum_{r=1}^n 2\,\mathrm{i}\,\Im\mathfrak{m}\,\{g(D_{X^{0,1}}Z_{\overline{r}},Z_r)-g(D_{Z_{\overline{r}}}X^{0,1},Z_r)\}\\ =&\sum_{r=1}^n 2\,\mathrm{i}\,\Im\mathfrak{m}\,\{g(D_{X^{0,1}}Z_r,Z_{\overline{r}})-g(D_{Z_{\overline{r}}}X^{0,1},Z_r)\}\\ =&\,\theta^1(X)+\sum_{r=1}^n 2\,\mathrm{i}\,\Im\mathfrak{m}\,\{g(D_{Z_r}X^{0,1},Z_{\overline{r}})-g(D_{Z_{\overline{r}}}X^{0,1},Z_r)\}\,. \end{aligned}$$

Now we observe that $\sum g(D_{Z_r}X^{0,1}, Z_{\overline{r}}) = -\sum g(X^{0,1}, D_{Z_r}Z_{\overline{r}}) = 0$, since the cosymplectic condition forces $\sum D_{Z_{\overline{r}}}Z_r$ to be of type $(1,0)$ (see e.g. [16]). The last step consists to show that $\sum \mathfrak{Im}\left\{g(D_{Z_{\overline{r}}}X^{0,1}, Z_r)\right\} = 0$. Here it is enough to consider the identity

$$\sum_{r=1}^{n}\Big\{\mathfrak{Im}\left\{Z_r g(X, Z_{\overline{r}})\right\} + \mathrm{i}\,\frac{1}{2}g([Z_r, Z_{\overline{r}}], X)\Big\} = \sum_{r=1}^{n}\mathfrak{Im}\left\{g(D_{Z_{\overline{r}}}X^{0,1}, Z_r)\right\}$$

which can be checked performing a direct computation. Then equation (3.2) implies the statement. □

Remark 3.4. In the Hermitian case this last result was already known. In fact, it can be deduced from formula (8) of [14]. Another proof of this fact can be found in [17].

4. Canonical Ricci forms on Lie algebras

Now we restrict our attention to left-invariant almost Hermitian structures on Lie groups (or more generally on left-invariant almost Hermitian structures on quotient of Lie groups by lattices). Since here all the computations are purely algebraic, we may assume to work on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ equipped with an almost Hermitian structure (g, J). An almost Hermitian structure on a Lie algebra is a pair (g, J), where J is an endomorphism of $\mathfrak{g}$ satisfying $J^2 = -\mathrm{Id}$ and g is a J-Hermitian inner product. The bracket of $\mathfrak{g}$ has not a priori any relation with J. The pair (g, J) induces as usual the fundamental form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$.

Proposition 3.1 implies the following

Proposition 4.1. *Let $(\mathfrak{g}, [\cdot,\cdot], g, J)$ be a Lie algebra with an almost Hermitian structure. For any $t \in \mathbb{R}$ the following formula holds*

$$\theta^t(X) = \frac{1}{2}\,\mathrm{i}\left\{-\mathrm{tr}(\mathrm{ad}_X \circ J) + t\,\mathrm{tr}\,\mathrm{ad}_{JX} + (t-1)\,g(d^*\omega, X^\natural)\right\}\,. \tag{4.1}$$

Moreover if $(\mathfrak{g}, [\cdot,\cdot])$ is unimodular (i.e. $\mathrm{tr}\,\mathrm{ad}_X = 0$ for any $X \in \mathfrak{g}$); then

$$\rho^t(X, Y) = \frac{1}{2}\,\mathrm{i}\,\mathrm{tr}(\mathrm{ad}_{[X,Y]} \circ J) - \frac{1}{2}\,\mathrm{i}(t-1)\,g(d^*\omega, [X,Y]^\natural) \tag{4.2}$$

and ρ^t is the same for any t if and only if (g, J) cosymplectic.

Proof. The only non-trivial part of the statement is the last assertion. So we have just to show that condition $g(\omega, d[X,Y]^\natural) = 0$ is equivalent to $d^*\omega = 0$. We can write $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus [\mathfrak{g},\mathfrak{g}]^\perp$. Let $X \in [\mathfrak{g},\mathfrak{g}]^\perp$, then

$$dX^\natural(Z, W) = -g(X, [Z, W]) = 0\,.$$

Hence for every $X \in [\mathfrak{g},\mathfrak{g}]^\perp$, $dX^\natural = 0$. This implies that $d\mathfrak{g}^* = d([\mathfrak{g},\mathfrak{g}]^\natural)$ and the claim follows. □

Now we can prove Theorem 1:

Proof of Theorem 1. Let $(\mathfrak{g}, [\cdot,\cdot], g, J)$ is a 2-step nilpotent Lie algebra with an almost Hermitian structure. Then, taking into account that $\mathfrak{g}$ is unimodular, the 2-step condition implies that $[\mathfrak{g},\mathfrak{g}]$ is contained in the center of $\mathfrak{g}$ and $\mathrm{tr}(\mathrm{ad}_{[X,Y]} \circ J) = 0$ for every $X, Y \in \mathfrak{g}$. Then formula (4.2) reduces to

$$\rho^t(X, Y) = \frac{1}{2}\,\mathrm{i}(1-t)\,g(d^*\omega, [X,Y]^\natural) \tag{4.3}$$

and first claim follows. □

Proposition 4.1 allows us to describe the behavior of $\{\rho^t\}$ for some special almost Hermitian structures:

Proposition 4.2. *Let $(\mathfrak{g}[\cdot,\cdot], g, J)$ be an almost Hermitian Lie algebra.*

- *If J is bi-invariant (i.e. $[J\cdot,\cdot] = J[\cdot,\cdot]$), then*

$$\theta^t(X) = (t-1)\,\mathrm{i}\,\mathrm{tr}(\mathrm{ad}_{JX})\,,\quad \rho^t(X, Y) = \mathrm{i}(1-t)\,\mathrm{tr}(\mathrm{ad}_{[JX,Y]})\,.$$

- *If J is anti-bi-invariant (i.e. $[J\cdot,\cdot] = -J[\cdot,\cdot]$), then*

$$\theta^t = 0\,,\quad \rho^t = 0\,.$$

- *If J is abelian (i.e. $[J\cdot, J\cdot] = [\cdot,\cdot]$), then*

$$\theta^t(X) = \frac{1}{2}\,\mathrm{i}\left\{(1+t)\,\mathrm{tr}(\mathrm{ad}_{JX}) + (t-1)\,g(d^*\omega, X^\natural)\right\}\,,$$
$$\rho^t(X) = \frac{1}{2}\,\mathrm{i}\left\{-(1+t)\,\mathrm{tr}(\mathrm{ad}_{J[X,Y]}) + (1-t)\,g(d^*\omega, [X,Y]^\natural)\right\}\,.$$

- *If J is anti-abelian (i.e. $[J\cdot, J\cdot] = -[\cdot,\cdot]$), then*

$$\theta^t(X) = \frac{1}{2}\,\mathrm{i}(1+t)\,\mathrm{tr}(\mathrm{ad}_{JX})\,, \quad \rho^t(X,Y) = -\frac{1}{2}\,\mathrm{i}(1+t)\,\mathrm{tr}(\mathrm{ad}_{J[X,Y]})\,.$$

In particular in the unimodular case bi-invariant, anti-bi-invariant and anti-abelian almost Hermitian structures are Ricci-flat with respect to any canonical connection, while in the abelian case ρ^t is given by the following formula

$$\rho^t(X,Y) = \frac{1}{2}\,\mathrm{i}(1-t)\,g(d^*\omega, [X,Y]^\natural).$$

and $\rho^t = 0$ for $t \neq 0$ if and only if (g, J) is a cosymplectic structure.

Remark 4.3. We remark the following facts:

- The bi-invariant condition $[J\cdot,\cdot] = J[\cdot,\cdot]$ is equivalent to require that the simply-connected Lie group associated to $(\mathfrak{g}, J)$ is a complex Lie group. The fact that a bi-invariant almost Hermitian structure on an unimodular Lie algebra is Ricci-flat with respect any canonical connection has been already proved by Grantcharov in [15].
- The anti-bi-invariant condition $[J\cdot,\cdot] = -J[\cdot,\cdot]$ is equivalent to require that any J-compatible inner product on $\mathfrak{g}$ is quasi-Kähler and flat with respect to the Chern connection ∇^1 (see [9]).
- The abelian condition $[J\cdot, J\cdot] = [\cdot,\cdot]$ was introduced in [4] and was intensely studied in [3, 5, 12, 8, 18]. This condition is equivalent to require that $\mathfrak{g}^{1,0}$ is an abelian Lie algebra.
- Finally, the anti-abelian condition $[J\cdot, J\cdot] = -[\cdot,\cdot]$ was studied in [11].

Remark 4.4. Theorem 1 can be applied to the Heisenberg Lie algebras $\mathfrak{h}_n(\mathbb{R})$ and $\mathfrak{h}_n(\mathbb{C})$. That accords to Theorem 4.1 of [20] and Proposition 4.10 and 4.11 of [10]. Moreover things work differently either in the 3-step nilpotent case or in the 2-step solvable case (see [10]).

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