

# ISOMETRIC DEFORMATIONS OF MINIMAL SURFACES IN $S^4$

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ABSTRACT. We consider the isometric deformation problem for oriented non simply connected immersed minimal surfaces  $f : M \rightarrow S^4$ . We prove that the space of all isometric minimal immersions of  $M$  into  $S^4$  with the same normal curvature function is, within congruences, either finite or a circle. Furthermore, we show that for any compact immersed minimal surface in  $S^4$  with nontrivial normal bundle there are at most finitely many noncongruent immersed minimal surfaces in  $S^4$  isometric to it with the same normal curvature function.

## 1. INTRODUCTION

A classical question about isometric immersions is to decide if given an isometric immersion  $f : M \rightarrow N$ , this is, up to ambient isometries, the unique way of immerse isometrically the Riemannian manifold  $M$  into the Riemannian manifold  $N$ . When  $f$  is a minimal immersion, one can ask if this is the unique isometric minimal immersion of  $M$  into  $N$ , up to ambient isometries. If this the case,  $f$  is called *minimally rigid*. The rigidity aspects of minimal hypersurfaces in a Euclidean space or in a sphere have drawn several author's attention. A conclusive result due to Dajczer and Gromoll [8] states that a complete minimal hypersurfaces in  $\mathbb{R}^{n+1}$  or in  $S^{n+1}$ ,  $n \geq 4$ , is minimally rigid (in the Euclidean case the assumption that the hypersurface does not contain  $\mathbb{R}^{n-3}$  as a factor is needed).

This result fails to hold in general for surfaces. However, Choi, Meeks and White [7] proved that a properly embedded minimal surface in  $\mathbb{R}^3$  with more than one end is minimally rigid. The case where the Euclidean space is replaced by a sphere is more difficult. A result due to Barbosa [3] says that a minimally immersed 2-sphere in a sphere is minimally rigid, while Ramanathan [15] has shown that for each compact surface minimally immersed in  $S^3$ , there are only finitely many noncongruent immersed minimal surfaces isometric to it.

It is interesting in minimal surface theory to determine whether a given minimal surface can be deformed in a nontrivial way. The oldest known example is the deformation of the catenoid into the helicoid. We are interested in isometric deformations of immersed minimal surfaces  $f : M \rightarrow S^4$  which

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2010 *Mathematics Subject Classification*. Primary 53A10; Secondary 53C42.

*Key words and phrases*. Minimal surfaces in  $S^4$ , isometric deformation, normal curvature function.

preserve the normal curvature function. If  $M$  is simply connected, then there exists a smooth  $2\pi$ -periodic parameter deformation  $f_\theta$  of  $f$ , the *associated family*, through isometric minimal immersions with the same normal curvature function. The procedure is to rotate the second fundamental form of  $f$  by  $\theta$  and then integrate the system of Gauss, Codazzi and Ricci equations (cf. [19]). The members of this family are noncongruent, unless  $f$  is superminimal. We recall that superminimal surfaces are the minimal surfaces whose Hopf differential vanishes identically, or equivalently, the curvature ellipse is a circle at each point.

Thus the rigidity for simply connected minimal surfaces fails in a natural way, and consequently the rigidity problem for minimal surfaces has a global nature. The above procedure cannot be carried out in general in the presence of nontrivial fundamental group  $\pi_1(M)$ . In this case, the obstruction is described as a homeomorphism from  $\pi_1(M)$  to the isometry group  $\text{Isom}(S^4)$  of  $S^4$ .

Our aim is to study the space of isometric minimal immersions of  $M$  into  $S^4$  with the same normal curvature function. For any connected minimal surface, not necessarily compact, we have the following result that was inspired by a recent paper due to Smyth and Tinaglia [18].

**Theorem 1.** *Let  $f : M \rightarrow S^4$  be an isometric minimal immersion of an oriented connected 2-dimensional Riemannian manifold  $M$  into  $S^4$  with normal curvature function  $K_N$ . Then, within congruences, the space of all isometric minimal immersions of  $M$  into  $S^4$  with the same normal curvature function  $K_N$  is either finite or a circle.*

The main result deals with the number of isometric minimal immersions of compact surfaces with the same normal curvature function. We prove that, within this kind of deformations, compact immersed minimal surfaces in  $S^4$  with nontrivial normal bundle are minimally rigid up to finiteness.

**Theorem 2.** *Let  $f : M \rightarrow S^4$  be an isometric minimal immersion of a compact oriented 2-dimensional Riemannian manifold  $M$  into  $S^4$  with nontrivial normal bundle and normal curvature function  $K_N$ . Then there exist at most finitely many noncongruent minimal immersions of  $M$  into  $S^4$  with the same normal curvature function  $K_N$ .*

As an application, we show that if a compact minimal surface allows a one parameter group of intrinsic isometries that preserve the normal curvature function, then these isometries extend to extrinsic ones.

**Corollary 1.** *Let  $f : M \rightarrow S^4$  be an isometric minimal immersion of a compact oriented 2-dimensional Riemannian manifold  $M$  into  $S^4$  with nontrivial normal bundle. Suppose that  $M$  admits a one parameter group of isometries  $\varphi_t : M \rightarrow M, t \in \mathbb{R}$ , that preserve the orientation and the normal curvature function. Then there exists a one parameter group of isometries  $\tau_t : S^4 \rightarrow S^4$  such that  $f \circ \varphi_t = \tau_t \circ f$  for all  $t \in \mathbb{R}$ .*

The paper is organized as follows: In section 2, we fix the notation and give the local theory of minimal surfaces in  $S^4$ . In section 3, we discuss the associated family of simply connected minimal surfaces. In section 4, we deal with the deformability of not necessarily simply connected minimal surfaces and prove Theorem 1. Finally, in the last section we give the proof of the main result, namely Theorem 2.

## 2. LOCAL THEORY OF MINIMAL SURFACES IN $S^4$

Let  $f : M \rightarrow S^4$  be an immersed minimal surface, i.e., an isometric minimal immersion of a connected oriented 2-dimensional Riemannian manifold  $M$ , with normal bundle  $Nf$  and second fundamental form  $B^f = B$ .

Let  $\{e_j\}$  be a local orthonormal frame field on  $S^4$ , which agrees with the orientation of  $TS^4$ , and let  $\{\omega_j\}$  be the coframe dual to  $\{e_j\}$ . The structure equations of  $S^4$  are

$$(2.1) \quad d\omega_j = \sum_k \omega_{jk} \wedge \omega_k,$$

$$(2.2) \quad d\omega_{jk} = \sum_l \omega_{jl} \wedge \omega_{lk} - \omega_j \wedge \omega_k,$$

where the connection form  $\omega_{jk}$  is given by  $\omega_{jk}(X) = \langle \tilde{\nabla}_X e_j, e_k \rangle$ ,  $\tilde{\nabla}$  is the Levi-Civita connection with respect to the standard metric  $\langle \cdot, \cdot \rangle$  of  $S^4$ . We choose the frame such that, restricted to  $M$ ,  $e_1$  and  $e_2$  are tangent and agree with the given orientation of  $M$  and consequently  $e_3$  and  $e_4$  are normal to the surface. Then we have  $\omega_\alpha = 0$ . By (2.1) and Cartan's Lemma, we get

$$\omega_{j\alpha} = \sum_k h_{jk}^\alpha \omega_k, \quad h_{jk}^\alpha = h_{kj}^\alpha, \quad j, k \in \{1, 2\}, \quad \alpha \in \{3, 4\}.$$

The assumption that  $f$  is minimal is equivalent to  $h_{11}^\alpha + h_{22}^\alpha = 0$ . Restricting equations (2.1) and (2.2) to  $M$ , we get the Cartan structure equations of  $f$ .

We may view  $M$  as a Riemann surface with complex structure determined as usual by the metric and the orientation. The complexified tangent bundle  $TM \otimes \mathbb{C}$  is decomposed into the eigenspaces of the complex structure  $J$ , called  $T'M$  and  $T''M$ , corresponding to the eigenvalues  $i$  and  $-i$ . The second fundamental form  $B$ , which takes values in  $Nf$ , can be complex linearly extended to  $TM \otimes \mathbb{C}$  with values in the complexified vector bundle  $Nf \otimes \mathbb{C}$  and then decomposed into its  $(p, q)$ -components,  $p+q=2$ , which are tensor products of  $p$  many 1-forms vanishing on  $T''M$  and  $q$  many 1-forms vanishing on  $T'M$ . The minimality of  $f$  implies that the  $(1, 1)$ -component of  $B$  vanishes. Consequently, for a local complex coordinate  $z = x + iy$ , we have the following decomposition

$$B = B^{(2,0)} + B^{(0,2)},$$

where

$$B^{(2,0)} = B\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)dz^2, \quad B^{(0,2)} = \overline{B^{(2,0)}} \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

The Hopf differential is by definition the differential form of type  $(4, 0)$

$$\Phi := \langle B^{(2,0)}, B^{(2,0)} \rangle.$$

The components of the second fundamental form are given by

$$h_1^\alpha := h_{11}^\alpha = \langle B(e_1, e_1), e_\alpha \rangle, \quad h_2^\alpha := h_{12}^\alpha = \langle B(e_1, e_2), e_\alpha \rangle,$$

where  $\alpha = 3$  or  $4$ . We use complex vectors, and we put

$$H_\alpha = h_1^\alpha + ih_2^\alpha, \quad E = e_1 - ie_2 \quad \text{and} \quad \varphi = \omega_1 + i\omega_2.$$

If we choose a local complex coordinate  $z = x + iy$  such that  $\varphi = \rho dz$ , for some smooth real function  $\rho$ , then we easily get

$$\Phi = \frac{1}{4}(\overline{H}_3^2 + \overline{H}_4^2)\varphi^4.$$

The *curvature ellipse*, for each point  $p$  in  $M$ , is the subset of the normal space  $N_p f$  given by

$$\mathcal{E}(p) = \{B(X, X) : X \in T_p M, |X| = 1\}.$$

It is known that  $\mathcal{E}(p)$  is indeed an ellipse (possibly degenerated). The zeros of  $\Phi$  are precisely the points where the curvature ellipse is a circle. The minimal surface is called *superminimal* if  $\Phi$  is identically zero, or equivalently, if the curvature ellipse is a circle at each point.

It is a consequence of the structure equations that the Hopf differential is holomorphic (cf. [5, 6]). Hence either a minimal surface is superminimal, or the points where the curvature ellipse is a circle are isolated.

The *normal curvature function*  $K_N$  (cf. [1]), which depends on the orientation of the normal bundle  $Nf$ , is defined by the equation

$$d\omega_{34} = -K_N \omega_1 \wedge \omega_2,$$

or equivalently,

$$(2.3) \quad K_N = i(H_3 \overline{H}_4 - \overline{H}_3 H_4).$$

It is not hard to verify that

$$|K_N| = \frac{2}{\pi} \text{Area}(\mathcal{E}).$$

Let  $\kappa \geq \mu \geq 0$  be the length of the semi-axes of the curvature ellipse. Then

$$|K_N| = 2\kappa\mu.$$

The length of the second fundamental form is given by

$$(2.4) \quad \|B\|^2 = 2(|H_3|^2 + |H_4|^2).$$

**Lemma 1.** *Assume that  $f : M \rightarrow S^4$  is not superminimal and let  $M_1$  be the set of isolated points where the curvature ellipse is a circle. Around each point in  $M \setminus M_1$ , there exist a local complex coordinate  $(U, z)$  with  $U \subset M \setminus M_1$  and orthonormal frames  $\{e_1, e_2\}$  in  $TM|_U$ ,  $\{e_3, e_4\}$  in  $Nf|_U$  which agree with the given orientations such that*

(i) the Riemannian metric of  $M$  is given by

$$ds^2 = \frac{|dz|^2}{(\kappa_1^2 - \mu_1^2)^{1/2}} \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{E}{2(\kappa_1^2 - \mu_1^2)^{1/4}},$$

(ii)  $e_3$  and  $e_4$  give respectively the directions of the major and the minor axes of the curvature ellipse, and

(iii)  $H_3 = \kappa_1, H_4 = i\mu_1$ , where  $\kappa_1$  and  $\mu_1$  are smooth real valued functions with

$$\kappa = |\kappa_1|, \quad \mu = |\mu_1|.$$

Moreover, the connection and the normal connection forms, with respect to this frame, are given by

$$(2.5) \quad \omega_{12} = -\frac{1}{4} * d \log (\kappa_1^2 - \mu_1^2), \quad \omega_{34} = * \frac{\kappa_1 d\mu_1 - \mu_1 d\kappa_1}{\kappa_1^2 - \mu_1^2},$$

where  $*$  stands for the Hodge operator.

*Proof.* Assume that the curvature ellipse is not a circle at a point  $x \in M$ . By continuity the same property holds on an open set  $U_1$  around  $x$ . Let  $\{e_3, e_4\}$  be an arbitrary oriented orthonormal frame in  $Nf|U_1$ . We introduce Cartesian coordinates  $(x_1, x_2)$  in each fiber of  $Nf|U_1$  adapted to this frame. In view of

$$B(e_1, e_1) = h_1^3 e_3 + h_1^4 e_4 \quad \text{and} \quad B(e_2, e_2) = h_2^3 e_3 + h_2^4 e_4,$$

we deduce that the quadratic equation of the curvature ellipse is given by

$$\begin{aligned} & |H_4|^2 x_1^2 - 2\operatorname{Re}(H_3 \overline{H_4}) x_1 x_2 + |H_3|^2 x_2^2 \\ & = |H_3|^2 |H_4|^2 - (\operatorname{Re}(H_3 \overline{H_4}))^2. \end{aligned}$$

On  $U_1$  we now choose the frame so that  $e_3$  and  $e_4$  give the directions of the major and minor axes respectively (cf. [2]). This means that the coefficient of  $x_1 x_2$  above must vanish, i.e.,  $H_3 \overline{H_4}$  is imaginary. It is clear that the length of the semi-axes of the curvature ellipse are given by  $\kappa = |H_3|$  and  $\mu = |H_4|$ . Since  $\kappa = |H_3| > 0$  on  $U_1$ , we may choose  $e_1$  and  $e_2$  so that  $h_2^3 = 0$ . Then the fact that  $H_3 \overline{H_4}$  is imaginary implies that  $h_1^4 = 0$ . We set  $\kappa_1 := h_1^3$  and  $\mu_1 := h_2^4$ .

We consider local coordinates  $(u, v)$  on a neighborhood  $U \subset U_1$  of  $x$  such that

$$e_1 = \frac{1}{r_1} \frac{\partial}{\partial u} \quad \text{and} \quad e_2 = \frac{1}{r_2} \frac{\partial}{\partial v}.$$

Then  $\omega_1 = r_1 du$  and  $\omega_2 = r_2 dv$ . From (2.1) we find that

$$\omega_{12} = -\frac{(r_1)_v}{r_2} du - \frac{(r_2)_u}{r_1} dv.$$

Taking the exterior derivative of

$$\omega_{13} = \kappa_1 r_1 du, \quad \omega_{23} = -\kappa_1 r_2 dv, \quad \omega_{14} = \mu_1 r_2 dv, \quad \omega_{24} = \mu_1 r_1 du$$

and using the structure equations (2.2), we obtain

$$\begin{aligned} 2\kappa_1(r_1)_v + (\kappa_1)_v r_1 + \mu_1 r_1 r_2 \omega_{34}(e_1) &= 0, \\ 2\kappa_1(r_2)_u + (\kappa_1)_u r_2 - \mu_1 r_1 r_2 \omega_{34}(e_2) &= 0, \\ 2\mu_1(r_2)_u + (\mu_1)_u r_2 - \kappa_1 r_1 r_2 \omega_{34}(e_2) &= 0, \\ 2\mu_1(r_1)_v + (\mu_1)_v r_1 + \kappa_1 r_1 r_2 \omega_{34}(e_1) &= 0. \end{aligned}$$

Eliminating  $\omega_{34}(e_1)$  and  $\omega_{34}(e_2)$ , we get

$$\begin{aligned} 2(\kappa_1^2 - \mu_1^2)(r_2)_u + (\kappa_1(\kappa_1)_u - \mu_1(\mu_1)_u)r_2 &= 0, \\ 2(\kappa_1^2 - \mu_1^2)(r_1)_v + (\kappa_1(\kappa_1)_v - \mu_1(\mu_1)_v)r_1 &= 0, \end{aligned}$$

and so  $r_1^2(\kappa_1^2 - \mu_1^2)^{1/2}$  depends only on  $u$  and  $r_2^2(\kappa_1^2 - \mu_1^2)^{1/2}$  depends only on  $v$ . Then we introduce the complex coordinate  $z = x + iy$  given by

$$x = \int r_1(\kappa_1^2 - \mu_1^2)^{1/4} du \quad \text{and} \quad y = \int r_2(\kappa_1^2 - \mu_1^2)^{1/4} dv.$$

Now it is easy to verify that

$$ds^2 = \frac{|dz|^2}{(\kappa_1^2 - \mu_1^2)^{1/4}} \quad \text{and} \quad E = 2(\kappa_1^2 - \mu_1^2)^{1/4} \frac{\partial}{\partial z}.$$

This completes the proof of parts (i)-(iii).

Taking the exterior derivatives of

$$\omega_{13} = \kappa_1 \omega_1, \quad \omega_{23} = -\kappa_1 \omega_1, \quad \omega_{14} = \mu_1 \omega_2, \quad \omega_{24} = \mu_1 \omega_1$$

and using the structure equations, we obtain (2.5).  $\square$

**2.1. The splitting of the Hopf differential.** Using the null frame field

$$\eta = e_3 + ie_4, \quad \bar{\eta} = e_3 - ie_4$$

of the complexified normal bundle  $Nf \otimes \mathbb{C}$ , we have

$$\langle B^{(2,0)}, B^{(2,0)} \rangle = \langle B^{(2,0)}, \eta \rangle \langle B^{(2,0)}, \bar{\eta} \rangle.$$

Therefore, from the definition of the Hopf differential, we get

$$\Phi = \frac{1}{4}(\overline{H}_3^2 + \overline{H}_4^2)\varphi^4 = \frac{1}{4}k_+ k_- \varphi^4,$$

where

$$k_{\pm} := \overline{H}_3 \pm i\overline{H}_4.$$

The functions

$$a_{\pm} := |k_{\pm}|$$

are globally well-defined. Their geometric meaning is that they both determine the geometry of the curvature ellipse. Indeed, since the Gaussian curvature  $K$  of  $M$  is given by

$$K = 1 - \frac{1}{2}\|B\|^2,$$

it follows from (2.3) and (2.4) that

$$a_{\pm} = (1 - K \pm K_N)^{1/2} = \kappa \pm \varepsilon \mu,$$

where  $\varepsilon = \pm 1$ , according to whether  $K_N \geq 0$  or  $K_N \leq 0$ .

We use the above mentioned notation throughout the paper.

### 3. ASSOCIATED FAMILY OF MINIMAL SURFACES IN $S^4$

Let  $f : M \rightarrow S^4$  be an isometric minimal immersion of a simply connected oriented 2-dimensional Riemannian manifold  $M$  with second fundamental form  $B$  and normal bundle  $Nf$ . For each  $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ , we consider the orthogonal and parallel tensor field

$$J_\theta = \cos \theta I + \sin \theta J,$$

where  $I$  is the identity map and  $J$  is the complex structure determined by the orientation and the metric of  $M$ . It is easy to see that the symmetric section  $\Gamma_\theta$  of the homomorphism bundle  $\text{Hom}(TM \times TM, Nf)$  given by

$$\Gamma_\theta(X, Y) := B(J_\theta X, Y),$$

where  $X$  and  $Y$  are tangent to  $M$ , satisfies the Gauss, Codazzi and Ricci equations with respect to the normal connection  $\nabla^\perp$  of  $Nf$  (cf. [19]). Hence there exist an isometric immersion  $f_\theta : M \rightarrow S^4$  and a vector bundle isomorphism

$$T_\theta : Nf \rightarrow Nf_\theta,$$

which is parallel and orthogonal, such that

$$B^{f_\theta}(X, Y) = T_\theta(B(J_\theta X, Y))$$

for all  $X$  and  $Y$ , where  $B^{f_\theta}$  is the second fundamental form of  $f_\theta$ . Obviously,  $f_\theta$  is also minimal. The  $2\pi$ -periodic family  $f_\theta$  is the *associated family* of  $f$ . The members of the associated family are noncongruent, unless  $f$  is superminimal and so each  $f_\theta$  is congruent to  $f$  (cf. [11]).

The normal curvature function of  $f_\theta$  coincides with the normal curvature function of  $f$ , where the orientation of  $Nf_\theta$  is naturally induced from the orientation on  $Nf$  and the bundle isomorphism  $T_\theta$ . Indeed, for a local orthonormal frame  $\{e_3, e_4\}$  of  $Nf$ , we consider the frame of  $Nf_\theta$  given by

$$e_3^\theta := T_\theta(e_3), \quad e_4^\theta := T_\theta(e_4).$$

Then it is easy to see that  $H_3, H_4$  and the corresponding functions  $H_3^\theta, H_4^\theta$  for  $f_\theta$  are related by

$$H_3^\theta = \exp(-2i\theta)H_3 \quad \text{and} \quad H_4^\theta = \exp(-2i\theta)H_4.$$

From these, by virtue of (2.3), it follows that  $f$  and  $f_\theta$  have the same normal curvature function.

Actually the associated family is the only way to isometrically deform any simply connected immersed minimal surface in  $S^4$  preserving the normal curvature function. This has already been proved by Eschenburg and Tribuzy in [10].

In order to state their result, we recall the notion of absolute value type functions introduced in [9, 10]. A smooth complex valued function  $u$  defined on a connected oriented surface  $M$  is called of *holomorphic type* if locally

$u = u_0 u_1$ , where  $u_0$  is holomorphic and  $u_1$  is smooth without zeros. A function  $a : M \rightarrow [0, +\infty)$  on  $M$  is called of *absolute value type* if there is a function  $u$  of holomorphic type on  $M$  such that  $a = |u|$ . The zero set of such a function is either isolated or the whole of  $M$ , and outside its zeros the function is smooth.

**Theorem 3.** [10] *Let  $f : M \rightarrow S^4$  be an immersed minimal surface which is not nonsuperminimal with Gaussian curvature  $K$  and normal curvature function  $K_N$ . Then the functions  $a_{\pm} = (1 - K \pm K_N)^{1/2}$  are of absolute value type and satisfy*

$$(3.1) \quad \Delta \log a_{\pm} = 2K \mp K_N,$$

where  $\Delta$  stands for the Laplace operator of  $M$ . Furthermore, if  $M$  is simply connected, then any other minimal immersion of  $M$  into  $S^4$  having the same normal curvature function  $K_N$  is congruent to some  $f_{\theta}$ .

#### 4. ISOMETRIC DEFORMATIONS OF MINIMAL SURFACES IN $S^4$ PRESERVING THE NORMAL CURVATURE FUNCTION

We consider the following

**Problem 1.** *Given an immersed minimal surface  $f : M \rightarrow S^4$  with normal curvature function  $K_N^f = K_N$ , describe the space of all isometric minimal immersions of  $M$  into  $S^4$  with the same normal curvature function  $K_N$ .*

From the holomorphicity of the Hopf differential we know that either  $f$  is superminimal or the curvature ellipse is a circle at isolated points only. The answer to the above problem is already known in the case where  $f$  is superminimal, since superminimal surfaces are rigid among superminimal surfaces (cf. [14, 20]).

Hereafter we assume that  $f$  is not superminimal. Let  $g : M \rightarrow S^4$  be another immersed minimal surface with the same normal curvature  $K_N$ . If  $M$  is not simply connected, then we consider the Riemannian covering map  $p : \tilde{M} \rightarrow M$ ,  $\tilde{M}$  being the universal cover of  $M$  equipped with the metric and the orientation that makes  $p$  an orientation preserving local isometry. Then the immersed minimal surfaces  $\tilde{f} := f \circ p$  and  $\tilde{g} := g \circ p$  have the same normal curvature  $\tilde{K}_N = K_N \circ p$ . The orientation on each bundle  $N\tilde{f}$  and  $N\tilde{g}$  is naturally induced from that of  $Nf$  and  $Ng$ , respectively. According to Theorem 3,  $\tilde{g}$  is congruent to some  $\tilde{f}_{\theta}$  in the associated family of  $\tilde{f}$ . Now the question is whether  $\tilde{f}_{\theta}$  projects to an isometric minimal immersion  $f_{\theta} : M \rightarrow S^4$ .

Hence the study of the space of all isometric minimal immersions of  $M$  into  $S^4$  with the same normal curvature  $K_N$  is reduced to the study of the set

$$\mathcal{S}(f) := \left\{ \theta \in [0, 2\pi] : \text{there exists } f_{\theta} : M \rightarrow S^4 \text{ so that } \tilde{f}_{\theta} = f_{\theta} \circ p \right\}.$$



Obviously  $0 \in \mathcal{S}(f)$ . Moreover, for each  $\theta \in \mathcal{S}(f)$ ,  $f$  and  $f_\theta$  have the same normal curvature, where the orientation of the normal bundle of  $f_\theta$  is inherited in a natural way from the orientation of  $N\tilde{f}_\theta$ . Indeed, for any  $x \in M$ , the normal curvature  $K_N^{f_\theta}$  of  $f_\theta$  is given by

$$\begin{aligned} K_N^{f_\theta}(x) &= K_N^{f_\theta} \circ p(\tilde{x}) = K_N^{f_\theta \circ p}(\tilde{x}) = K_N^{\tilde{f}_\theta}(\tilde{x}) \\ &= K_N^{\tilde{f}}(\tilde{x}) = K_N^f \circ p(\tilde{x}) = K_N(x), \end{aligned}$$

where  $\tilde{x} \in p^{-1}(x)$ .

**Lemma 2.** *For any  $\sigma$  in the group  $\mathcal{D}$  of deck transformations of the covering map  $p : \tilde{M} \rightarrow M$ , the minimal immersions  $\tilde{f}_\theta$  and  $\tilde{f}_\theta \circ \sigma$  are congruent.*

*Proof.* It is enough to prove the existence of an orthogonal and parallel isomorphism between the normal bundles of  $\tilde{f}_\theta$  and  $\tilde{f}_\theta \circ \sigma$  that preserves the second fundamental forms. If  $T_\theta$  is the isomorphism between the normal bundles of  $\tilde{f}$  and  $\tilde{f}_\theta$ , then we define the bundle isomorphism

$$\Sigma_\theta : N\tilde{f}_\theta \rightarrow N(\tilde{f}_\theta \circ \sigma)$$

so that at any point  $\tilde{x} \in \tilde{M}$

$$\Sigma_\theta|_{\tilde{x}} : N_{\tilde{x}}\tilde{f}_\theta \rightarrow N_{\tilde{x}}(\tilde{f}_\theta \circ \sigma)$$

is given by

$$\Sigma_\theta|_{\tilde{x}}(\xi) := T_\theta|_{\sigma(\tilde{x})}(T_\theta^{-1}|_{\tilde{x}}(\xi)), \quad \xi \in N_{\tilde{x}}\tilde{f}_\theta.$$

For any  $\tilde{v}, \tilde{w} \in T_{\tilde{x}}\tilde{M}$  the second fundamental form of  $\tilde{f}_\theta \circ \sigma$  is given by

$$\begin{aligned} B^{\tilde{f}_\theta \circ \sigma}|_{\tilde{x}}(\tilde{v}, \tilde{w}) &= B^{\tilde{f}_\theta}|_{\sigma(\tilde{x})}(d\sigma_{\tilde{x}}(\tilde{v}), d\sigma_{\tilde{x}}(\tilde{w})) \\ &= T_\theta|_{\sigma(\tilde{x})}(B^{\tilde{f}}|_{\sigma(\tilde{x})}(\tilde{J}_\theta \circ d\sigma_{\tilde{x}}(\tilde{v}), d\sigma_{\tilde{x}}(\tilde{w}))), \end{aligned}$$

where  $B^{\tilde{f}_\theta}$  is the second fundamental form of  $\tilde{f}_\theta$ ,

$$\tilde{J}_\theta = \cos \theta \tilde{I} + \sin \theta \tilde{J}$$

and  $\tilde{J}$  is the complex structure of  $\tilde{M}$ . Since  $\sigma$  is a deck transformation, we have

$$\tilde{f} \circ \sigma = \tilde{f} \quad \text{and} \quad \tilde{J}_\theta \circ d\sigma = d\sigma \circ \tilde{J}_\theta.$$

Thus it follows that

$$\begin{aligned} B^{\tilde{f}_\theta \circ \sigma}|_{\tilde{x}}(\tilde{v}, \tilde{w}) &= T_\theta|_{\sigma(\tilde{x})}(B^{\tilde{f}}|_{\sigma(\tilde{x})}(d\sigma_{\tilde{x}} \circ \tilde{J}_\theta(\tilde{v}), d\sigma_{\tilde{x}}(\tilde{w}))) \\ &= T_\theta|_{\sigma(\tilde{x})}(B^{\tilde{f} \circ \sigma}|_{\tilde{x}}(\tilde{J}_\theta(\tilde{v}), \tilde{w})), \end{aligned}$$

or equivalently,

$$B^{\tilde{f}_\theta \circ \sigma}|_{\tilde{x}}(\tilde{v}, \tilde{w}) = T_\theta|_{\sigma(\tilde{x})}(B^{\tilde{f}}|_{\tilde{x}}(\tilde{J}_\theta(\tilde{v}), \tilde{w})).$$

Then bearing in mind the definition of  $\Sigma_\theta$  and the relation

$$B^{\tilde{f}_\theta}|_{\tilde{x}}(\tilde{v}, \tilde{w}) = T_\theta|_{\tilde{x}}(B^{\tilde{f}}|_{\tilde{x}}(\tilde{J}_\theta(\tilde{v}), \tilde{w})),$$

we find

$$B^{\tilde{f}_\theta \circ \sigma}|_{\tilde{x}}(\tilde{v}, \tilde{w}) = \Sigma_\theta(B^{\tilde{f}_\theta}|_{\tilde{x}}(\tilde{v}, \tilde{w})),$$

which shows that  $\Sigma_\theta$  preserves the second fundamental forms of  $\tilde{f}_\theta$  and  $\tilde{f}_\theta \circ \sigma$ .

Now let  $\xi = T_\theta(\eta)$  be an arbitrary section of  $N\tilde{f}_\theta$ , where  $\eta$  is a section of  $N\tilde{f}$ . Then

$$\Sigma_\theta(\xi) = T_\theta(\eta \circ \sigma^{-1}) \circ \sigma$$

and consequently for any  $\tilde{X}$  tangent to  $\tilde{M}$  we have

$$\begin{aligned} (\nabla_{\tilde{X}}^\perp \Sigma_\theta)\xi &= \nabla_{\tilde{X}}^\perp \Sigma_\theta(\xi) - \Sigma_\theta(\nabla_{\tilde{X}}^\perp \xi) \\ &= \nabla_{\tilde{X}}^\perp (T_\theta(\eta \circ \sigma^{-1}) \circ \sigma) - T_\theta(\nabla_{\tilde{X}}^\perp (\eta \circ \sigma^{-1})) \circ \sigma \\ &= (\nabla_{d\sigma(\tilde{X})}^\perp T_\theta(\eta \circ \sigma^{-1})) \circ \sigma - T_\theta(\nabla_{\tilde{X}}^\perp (\eta \circ \sigma^{-1})) \circ \sigma \\ &= T_\theta(\nabla_{d\sigma(\tilde{X})}^\perp (\eta \circ \sigma^{-1})) \circ \sigma - T_\theta(\nabla_{\tilde{X}}^\perp (\eta \circ \sigma^{-1})) \circ \sigma \\ &= T_\theta(\nabla_{d\sigma(\tilde{X})}^\perp (\eta \circ \sigma^{-1}) - \nabla_{\tilde{X}}^\perp (\eta \circ \sigma^{-1})) \circ \sigma, \end{aligned}$$

where, by abuse of notation,  $\nabla^\perp$  stands for the normal connection of every involved immersion. Since  $\eta \circ \sigma^{-1}$  is a section of the normal bundle of  $\tilde{f} = f \circ p$ , we may write  $\eta \circ \sigma^{-1} = \delta \circ p$  for some local section  $\delta$  of the normal bundle of  $f$ . We observe that

$$\begin{aligned} \nabla_{d\sigma(\tilde{X})}^\perp (\eta \circ \sigma^{-1}) - \nabla_{\tilde{X}}^\perp (\eta \circ \sigma^{-1}) &= \nabla_{d\sigma(\tilde{X})}^\perp (\delta \circ p) - \nabla_{\tilde{X}}^\perp (\delta \circ p) \\ &= \nabla_{dp \circ d\sigma(\tilde{X})}^\perp \delta - \nabla_{d\tilde{X}}^\perp \delta = 0. \end{aligned}$$

Therefore  $\Sigma_\theta$  is parallel, orthogonal and preserves the second fundamental forms of  $\tilde{f}_\theta$  and  $\tilde{f}_\theta \circ \sigma$ , and this completes the proof.  $\square$

Lemma 2 allows us to define a map

$$\Phi_\theta : \mathcal{D} \rightarrow \text{Isom}(S^4)$$

for each  $\theta \in [0, 2\pi]$ , such that

$$\tilde{f}_\theta \circ \sigma = \Phi_\theta(\sigma) \circ \tilde{f}_\theta$$

for any  $\sigma \in \mathcal{D}$ . It is easy to see that  $\Phi_\theta$  is a homomorphism for each  $\theta \in [0, 2\pi]$ . Furthermore,  $\theta \in \mathcal{S}(f)$  if and only if  $\Phi_\theta(\mathcal{D}) = \{I\}$ . In case where the image of  $f$  is contained in a totally geodesic  $S^3$ , then  $\Phi_\theta$  maps  $\mathcal{D}$  into  $\text{Isom}(S^3)$ .

Now we are ready to prove Theorem 1 which describes the structure of the set  $\mathcal{S}(f)$ .

*Proof of Theorem 1.* Assume that  $\mathcal{S}(f)$  is infinite. Then there exists a sequence  $\{\theta_m\}$  in  $\mathcal{S}(f)$  which we may assume converges to some  $\theta_0 \in [0, 2\pi]$ , by passing to a subsequence if necessary. From  $\Phi_{\theta_m}(\mathcal{D}) = \{I\}$  for all  $m \in \mathbb{N}$ ,

we immediately obtain  $\Phi_{\theta_0}(\mathcal{D}) = \{I\}$ . Let  $\sigma \in \mathcal{D}$ . By applying the Mean Value Theorem to each entry  $(\Phi_\theta(\sigma))_{jk}$  of the corresponding matrix, we get

$$\frac{d}{d\theta}(\Phi_\theta(\sigma))_{jk}(\dot{\theta}_m) = 0$$

for some  $\dot{\theta}_m$  which lies between  $\theta_0$  and  $\theta_m$ . By continuity we get

$$\frac{d}{d\theta}(\Phi_\theta(\sigma))_{jk}(\theta_0) = 0$$

for each  $\sigma \in \mathcal{D}$ . Applying repeatedly the Mean Value Theorem, we conclude that

$$\frac{d^n}{d\theta^n}(\Phi_\theta(\sigma))_{jk}(\theta_0) = 0$$

for each  $\sigma \in \mathcal{D}$  and any integer  $n \geq 1$ . Since  $\Phi_\theta(\sigma)$  is an analytic curve (cf. [12]) in  $\text{Isom}(S^4)$ , we infer that  $\Phi_\theta(\sigma) = I$ , and so  $\mathcal{S}(f) = [0, 2\pi]$ .  $\square$

**Lemma 3.** *Let  $f : M \rightarrow S^4$  be an immersed minimal surface whose image is not contained in any totally geodesic  $S^3$ . Then for each  $\theta \in \mathcal{S}(f)$  there exists a parallel and orthogonal bundle isomorphism  $T_\theta : Nf \rightarrow Nf_\theta$  such that the second fundamental forms of  $f$  and  $f_\theta$  are related by*

$$B^{f_\theta}(X, Y) = T_\theta(B^f(J_\theta X, Y))$$

for all  $X, Y$  tangent to  $M$ .

*Proof.* Let  $\theta \in \mathcal{S}(f)$ . We claim that for each point there exist an open neighborhood  $U$  and a parallel and orthogonal bundle isomorphism

$$T_\theta^U : Nf|_U \rightarrow Nf_\theta|_U$$

such that the second fundamental forms of  $f|_U$  and  $f_\theta|_U$  are related by

$$(4.1) \quad B^{f_\theta|_U}(X, Y) = T_\theta^U(B^{f|_U}(J_\theta X, Y))$$

for all  $X, Y$  tangent to  $U$ .

Indeed, let  $\tilde{U} \subset \tilde{M}$  and  $U \subset M$  are chosen so that the Riemannian covering map  $p : \tilde{M} \rightarrow M$  maps  $\tilde{U}$  isometrically onto  $U$ ,  $\tilde{M}$  being the universal cover of  $M$ . We define the orthogonal isomorphism  $T_\theta^U$  between the bundles  $Nf|_U$  and  $Nf_\theta|_U$  by

$$T_\theta^U(\xi) := \tilde{T}_\theta(\xi \circ p) \circ (p|_{\tilde{U}})^{-1},$$

where  $\xi$  is section of  $Nf|_U$  and

$$\tilde{T}_\theta : N\tilde{f} \rightarrow N\tilde{f}_\theta$$

is the orthogonal and parallel isomorphism between the normal bundles of the minimal surfaces  $\tilde{f} = f \circ p$  and  $\tilde{f}_\theta = f_\theta \circ p$ , so that

$$(4.2) \quad B^{\tilde{f}_\theta}(\tilde{X}, \tilde{Y}) = \tilde{T}_\theta(\tilde{B}(\tilde{J}_\theta \tilde{X}, \tilde{Y}))$$

for all  $\tilde{X}, \tilde{Y}$  tangent to  $\tilde{M}$ .

The second fundamental forms of the minimal surfaces  $\tilde{f} = f \circ p$  and  $\tilde{f}_\theta = f_\theta \circ p$  are given by

$$B^{\tilde{f}}(\tilde{X}, \tilde{Y}) = B^f(dp(\tilde{X}), dp(\tilde{Y}))$$

and

$$B^{\tilde{f}_\theta}(\tilde{X}, \tilde{Y}) = B^{f_\theta}(dp(\tilde{X}), dp(\tilde{Y}))$$

for all  $\tilde{X}$  and  $\tilde{Y}$ . Then it is easy to see that (4.2) implies (4.1).

Let  $\xi$  is section of  $Nf|_U$ . For any  $X = dp(\tilde{X})$  tangent to  $U$ , we have

$$\begin{aligned} (\nabla_{\tilde{X}}^\perp T_\theta^U)\xi &= \nabla_{dp(\tilde{X})}^\perp T_\theta^U(\xi) - T_\theta^U(\nabla_{dp(\tilde{X})}^\perp \xi) \\ &= \nabla_{dp(\tilde{X})}^\perp (\tilde{T}_\theta(\xi \circ p|_{\tilde{U}}) \circ (p|_{\tilde{U}})^{-1}) - T_\theta^U(\nabla_{\tilde{X}}^\perp (\xi \circ p|_{\tilde{U}}) \circ (p|_{\tilde{U}})^{-1}) \\ &= (\nabla_{\tilde{X}}^\perp \tilde{T}_\theta(\xi \circ p|_{\tilde{U}})) \circ (p|_{\tilde{U}})^{-1} - \tilde{T}_\theta(\nabla_{\tilde{X}}^\perp (\xi \circ p|_{\tilde{U}})) \circ (p|_{\tilde{U}})^{-1}, \end{aligned}$$

where, by abuse of notation,  $\nabla^\perp$  stands for the normal connection of every involved immersion. The above shows that  $T_\theta^U$  is parallel, since  $\tilde{T}_\theta$  is parallel.

Let  $V$  be another open subset of  $M$  with  $U \cap V \neq \emptyset$  and corresponding bundle isomorphism

$$T_\theta^V : Nf|_V \rightarrow Nf_\theta|_V$$

such that the second fundamental forms of  $f|_V$  and  $f_\theta|_V$  are related by

$$(4.3) \quad B^{f_\theta|_V}(X, Y) = T_\theta^V(B^{f|_V}(J_\theta X, Y))$$

for all  $X$  and  $Y$  tangent to  $V$ . We consider the set  $M_0$  of points where the normal curvature vanishes, or equivalently, the set of points where the first normal space is a proper subset of the normal space. Our assumption implies that the set  $M \setminus M_0$  is dense in  $M$ . From (4.1) and (4.3) we see that  $T_\theta^U = T_\theta^V$  on  $U \cap V \setminus M_0$ . By continuity, we infer that  $T_\theta^U = T_\theta^V$  on  $U \cap V$ . Thus  $T_\theta^U$  is globally well-defined.  $\square$

## 5. PROOF OF THE MAIN RESULT

Before we proceed to the proof of Theorem 2, we recall some useful facts. At first, we need the topological restrictions for minimal surfaces in  $S^4$  that were obtained by Eschenburg and Tribuzy [10]. To this purpose, we review some properties of absolute value type functions.

The zero set of an absolute value type function  $a$  on a connected compact oriented surface  $M$  is either isolated or the whole of  $M$ , and outside its zeros, the function is smooth. If  $a$  is a nonzero absolute value type function, i.e., locally  $a = |t_0|a_1$ , with  $t_0$  holomorphic, the order  $k \geq 1$  of any  $x \in M$  with  $a(x) = 0$  is the order of  $t_0$  at  $x$ . Let  $N(a)$  be the sum of all orders for all zeros of  $a$ . Then  $\Delta \log a$  is bounded on  $M \setminus \{a = 0\}$  and its integral is given by

$$\int_M \Delta \log a dA = -2\pi N(a).$$

The following lemma, due to Eschenburg and Tribuzy [10], follows immediately from Theorem 3 just by integrating (3.1) and using the Gauss-Bonnet

Theorem and the fact that the Euler number  $\chi(Nf)$  of the normal bundle is given by

$$\int_M K_N dA = 2\pi\chi(Nf).$$

**Lemma 4.** *Let  $f : M \rightarrow S^4$  be a compact oriented immersed minimal surface. If  $f$  is not superminimal, then the Euler number  $\chi(Nf)$  of the normal bundle and the Euler-Poincaré characteristic  $\chi(M)$  of  $M$  satisfy*

$$2\chi(M) \pm \chi(Nf) = -N(a_{\mp}).$$

We also need some facts about holomorphic bundle-valued forms (cf. [4]). Let  $M$  be a 2-dimensional oriented Riemannian manifold with the canonically defined complex structure and  $\mathbf{E}$  be a complex vector bundle over  $M$  equipped with a connection  $\nabla$ . For any  $\mathbf{E}$ -valued  $r$ -covariant tensor field  $F$  on  $M$  the covariant derivative is defined in the usual way, where  $M$  is equipped with the Levi-Civita connection. If  $F$  is of holomorphic type  $(r, 0)$ , we say that  $F$  is holomorphic if its covariant derivative is of holomorphic type  $(r+1, 0)$ . In terms of a local complex coordinate  $(U, z)$  on  $M$ , a tensor field  $F$  of holomorphic type  $(r, 0)$  is written on  $U$  in the form

$$F = udz^r,$$

where  $u : U \rightarrow \mathbf{E}$  is given by

$$u = F\left(\frac{\partial}{\partial z}, \dots, \frac{\partial}{\partial z}\right).$$

Then  $F$  is holomorphic if and only if

$$\nabla_{\frac{\partial}{\partial \bar{z}}} u = 0.$$

The following, which we quote from [4], is crucial for the proof of the main result.

**Theorem 4.** *Assume that the  $\mathbf{E}$ -valued tensor field  $F$  on  $M$  is holomorphic, and let  $x \in M$  be such that  $F(x) = 0$ . Let  $(U, z)$  be a local complex coordinate on  $M$  with  $z(x) = 0$ . Then either  $F \equiv 0$  on  $U$ , or  $F = z^m F^*$ , where  $m$  is a positive integer and  $F^*(x) \neq 0$ .*

Now let  $f : M \rightarrow S^4$  be an immersed minimal surface which is not contained in any totally geodesic  $S^3$ . Assume hereafter that  $f$  is not superminimal. The set  $M_1$  of points where the curvature ellipse is a circle consists of isolated points only. For each point  $x \in M \setminus M_1$ , we consider an orthonormal frame  $\{e_1, e_2, e_3, e_4\}$  on a neighborhood  $U_x \subset M \setminus M_1$  of  $x$  as in Lemma 1 with normal connection form  $\omega_{34}$ .

For arbitrary  $\theta_j \in \mathcal{S}(f)$ ,  $j = 1, \dots, n$ , we consider the local orthonormal frame  $\{e_3^{\theta_j}, e_4^{\theta_j}\}$  of the normal bundle of  $f_{\theta_j}$  defined by

$$e_3^{\theta_j} := T_{\theta_j}(e_3), \quad e_4^{\theta_j} := T_{\theta_j}(e_4),$$

where

$$T_{\theta_j} : Nf \rightarrow Nf_{\theta_j}$$

is the bundle isomorphism of Lemma 3. Obviously,  $\omega_{34}$  is also the normal connection form of  $f_{\theta_j}$  with respect to this frame. By virtue of Lemma 3, we easily find that  $H_3, H_4$  and the corresponding functions  $H_3^{\theta_j}, H_4^{\theta_j}$  for  $f_{\theta_j}$  are related by

$$(5.1) \quad H_3^{\theta_j} = \exp(-2i\theta_j)H_3 \quad \text{and} \quad H_4^{\theta_j} = \exp(-2i\theta_j)H_4.$$

Using (5.1) and the Weingarten formula for  $f_{\theta_j}$ , we obtain

$$(5.2) \quad D_E e_3^{\theta_j} = -\kappa_1 \exp(i\theta_j) df_{\theta_j}(\bar{E}) + \omega_{34}(E) e_4^{\theta_j}$$

and

$$(5.3) \quad D_E e_4^{\theta_j} = i\mu_1 \exp(i\theta_j) df_{\theta_j}(\bar{E}) - \omega_{34}(E) e_3^{\theta_j},$$

where

$$E = e_1 - ie_2$$

and  $D$  stands for the usual connection in the induced bundle  $(i_1 \circ f)^*(T\mathbb{R}^5)$ ,  $i_1 : S^4 \rightarrow \mathbb{R}^5$  being the inclusion map.

The following auxiliary lemma is needed for the proof of the main result.

**Lemma 5.** *Assume that there exist vectors  $v_j \in \mathbb{R}^5, j = 1, \dots, n$ , such that*

$$(5.4) \quad \sum_{j=1}^n \langle f_{\theta_j}, v_j \rangle = 0 \quad \text{on} \quad U_x.$$

(i) *Then*

$$(5.5) \quad \sum_{j=1}^n \exp(i\theta_j) \left( \kappa_1 \langle e_3^{\theta_j}, v_j \rangle - i\mu_1 \langle e_4^{\theta_j}, v_j \rangle \right) = 0.$$

(ii) *Away from points where  $\omega_{34}$  vanishes, we have*

$$(5.6) \quad \sum_{j=1}^n \exp(i\theta_j) \langle e_4^{\theta_j}, v_j \rangle = \frac{\kappa_1}{2\omega_{34}(E)} \sum_{j=1}^n \exp(2i\theta_j) \langle df_{\theta_j}(\bar{E}), v_j \rangle$$

and

$$(5.7) \quad 2\omega_{34}(E) \sum_{j=1}^n \exp(2i\theta_j) \langle f_{\theta_j}, v_j \rangle = L \sum_{j=1}^n \exp(2i\theta_j) \langle df_{\theta_j}(\bar{E}), v_j \rangle,$$

where  $L$  is the complex valued function given by

$$L = -E(\omega_{34}(E)) - 3i\omega_{12}(E)\omega_{34}(E).$$

(iii) *Furthermore, we have*

$$(5.8) \quad \bar{E} \left( \sum_{j=1}^n \exp(i\theta_j) \langle e_4^{\theta_j}, v_j \rangle \right) = -\omega_{34}(\bar{E}) \sum_{j=1}^n \exp(i\theta_j) \langle e_3^{\theta_j}, v_j \rangle.$$

*Proof.* Our assumption implies that

$$(5.9) \quad \sum_{j=1}^n \langle df_{\theta_j}, v_j \rangle = 0.$$

Differentiating, using the Gauss formula and (5.4), we immediately see that the second fundamental forms of  $f_{\theta_j}$  satisfy

$$\sum_{j=1}^n \langle B^{f_{\theta_j}}, v_j \rangle = 0.$$

This on account of (5.1) yields

$$\sum_{j=1}^n \exp(i\theta_j) \left( \overline{H}_3 \langle e_3^{\theta_j}, v_j \rangle + \overline{H}_4 \langle e_4^{\theta_j}, v_j \rangle \right) = 0.$$

Since the frame  $\{e_3, e_4\}$  is chosen as in Lemma 1, we have  $H_3 = \kappa_1$ ,  $H_4 = i\mu_1$ , and the above immediately implies (5.5).

Differentiating (5.5) with respect to  $E$  and using (5.2) and (5.3) we obtain

$$\begin{aligned} \sum_{j=1}^n \exp(i\theta_j) \left( (E(\kappa_1) + i\mu_1\omega_{34}(E)) \langle e_3^{\theta_j}, v_j \rangle - (iE(\mu_1) - \kappa_1\omega_{34}(E)) \langle e_4^{\theta_j}, v_j \rangle \right) \\ = (\kappa_1^2 - \mu_1^2) \sum_{j=1}^n \exp(2i\theta_j) \langle df_{\theta_j}(\overline{E}), v_j \rangle. \end{aligned}$$

We view (5.5) and the above equation as a linear system with unknowns

$$\sum_{j=1}^n \exp(i\theta_j) \langle e_3^{\theta_j}, v_j \rangle, \quad \sum_{j=1}^n \exp(i\theta_j) \langle e_4^{\theta_j}, v_j \rangle.$$

The determinant of this system is given by

$$a = (\kappa_1^2 - \mu_1^2)\omega_{34}(E) - i(\kappa_1 E(\mu_1) - \mu_1 E(\kappa_1)).$$

Equations (2.5) yield

$$(5.10) \quad E(\kappa_1) = -2i\kappa_1\omega_{12}(E) + i\mu_1\omega_{34}(E)$$

and

$$(5.11) \quad E(\mu_1) = -2i\mu_1\omega_{12}(E) + i\kappa_1\omega_{34}(E).$$

Then the determinant is written as

$$a = 2(\kappa_1^2 - \mu_1^2)\omega_{34}(E).$$

Using (5.10), (5.11) and solving the linear system, we easily obtain (5.6).

Now differentiating (5.6) with respect to  $E$ , using (5.2), (5.3), (5.5), (5.6) and the Gauss formula, we find

$$\begin{aligned} \left( E \left( \frac{\kappa_1}{\omega_{34}(E)} \right) - \frac{i\kappa_1\omega_{12}(E)}{\omega_{34}(E)} - i\mu_1 \right) \sum_{j=1}^n \exp(2i\theta_j) \langle df_{\theta_j}(\bar{E}), v_j \rangle \\ = -\frac{2\kappa_1}{\omega_{34}(E)} \sum_{j=1}^n \exp(2i\theta_j) \langle f_{\theta_j}, v_j \rangle, \end{aligned}$$

or equivalently, by virtue of (5.10),

$$\begin{aligned} \kappa_1 \left( E(\omega_{34}(E)) + 3i\omega_{12}(E)\omega_{34}(E) \right) \sum_{j=1}^n \exp(2i\theta_j) \langle df_{\theta_j}(\bar{E}), v_j \rangle \\ = -2\kappa_1\omega_{34}(E) \sum_{j=1}^n \exp(2i\theta_j) \langle f_{\theta_j}, v_j \rangle. \end{aligned}$$

This last equation is equivalent to (5.7), since  $\kappa_1^2 > \mu_1^2$  on  $U_x$ .

Appealing to (5.3), we observe that

$$\begin{aligned} \bar{E} \left( \sum_{j=1}^n \exp(i\theta_j) \langle e_4^{\theta_j}, v_j \rangle \right) &= -i\mu_1 \sum_{j=1}^n \langle df_{\theta_j}(E), v_j \rangle \\ &\quad - \omega_{34}(\bar{E}) \sum_{j=1}^n \exp(i\theta_j) \langle e_3^{\theta_j}, v_j \rangle, \end{aligned}$$

which in view of (5.9) immediately yields (5.8).  $\square$

Now we are ready to give the proof of the main result.

*Proof of Theorem 2.* Let  $f : M \rightarrow S^4$  be an isometric minimal immersion of a compact oriented 2-dimensional Riemannian manifold  $M$  into  $S^4$  with nontrivial normal bundle. We may assume that  $f$  is not superminimal, otherwise there is nothing to prove. According to Theorem 1, either there are only finitely many noncongruent immersed minimal surfaces isometric to  $f$  with the same normal curvature, or the space of all minimal surfaces in  $S^4$  with these properties is a circle.

Arguing indirectly, we suppose that  $\mathcal{S}(f) = [0, 2\pi]$ . The strategy is to prove that the coordinate functions of the minimal surfaces  $f_\theta, \theta \in [0, 2\pi]$ , are linearly independent. On the other hand, these functions are eigenfunctions of the Laplace operator of  $M$  with corresponding eigenvalue 2. This leads to a contradiction since the eigenspaces of the Laplace operator are finite dimensional.

**Claim 1.** *Let  $0 < \theta_1 < \dots < \theta_n \leq 2\pi$ . If for vectors  $v_j \in \mathbb{R}^5, j = 1, \dots, n$ , the following holds*

$$(5.12) \quad \sum_{j=1}^n \langle f_{\theta_j}, v_j \rangle = 0,$$



then  $v_j = 0$  for all  $j = 1, \dots, n$ .

Assume to the contrary that each vector  $v_j, j = 1, \dots, n$ , is nonzero. Let  $M_1$  be the set of isolated points where the curvature ellipse is a circle. Obviously  $M_1$  is finite. We set  $M_1 = \{x_1, \dots, x_k\}$ . Around each point  $x \in M \setminus M_1$ , we consider a local complex coordinate  $(U_x, z)$  and an orthonormal frame  $\{e_1, e_2, e_3, e_4\}$  along  $f$  on  $U_x \subset M \setminus M_1$  as in Lemma 1 with normal connection form  $\omega_{34}$ .

We also consider the local orthonormal frame  $\{e_3^{\theta_j}, e_4^{\theta_j}\}$  of the normal bundle of  $f_{\theta_j}$  defined by

$$e_3^{\theta_j} := T_{\theta_j}(e_3), \quad e_4^{\theta_j} := T_{\theta_j}(e_4),$$

where

$$T_{\theta_j} : Nf \rightarrow Nf_{\theta_j}$$

is the bundle isomorphism of Lemma 3.

We now define the smooth complex valued functions  $\varphi, \psi : U_x \rightarrow \mathbb{C}$  by

$$\varphi := \sum_{j=1}^n \exp(i\theta_j) \langle e_3^{\theta_j}, v_j \rangle \quad \text{and} \quad \psi := \sum_{j=1}^n \exp(i\theta_j) \langle e_4^{\theta_j}, v_j \rangle.$$

These functions are locally defined and obviously depend on the choice frame  $\{e_3, e_4\}$ . If for another point  $\hat{x} \in M \setminus M_1$ , with corresponding neighborhood  $U_{\hat{x}} \subset M \setminus M_1$  and frame  $\{\hat{e}_3, \hat{e}_4\}$  chosen as in Lemma 1, we have  $U_x \cap U_{\hat{x}} \neq \emptyset$ , then either  $\hat{e}_3 = e_3$  and  $\hat{e}_4 = e_4$  or  $\hat{e}_3 = -e_3$  and  $\hat{e}_4 = -e_4$  on  $U_x \cap U_{\hat{x}}$ . The corresponding functions  $\hat{\varphi}, \hat{\psi} : U_{\hat{x}} \rightarrow \mathbb{C}$  are given by

$$\hat{\varphi} := \sum_{j=1}^n \exp(i\theta_j) \langle \hat{e}_3^{\theta_j}, v_j \rangle \quad \text{and} \quad \hat{\psi} := \sum_{j=1}^n \exp(i\theta_j) \langle \hat{e}_4^{\theta_j}, v_j \rangle,$$

where

$$\hat{e}_3^{\theta_j} := T_{\theta_j}(\hat{e}_3), \quad \hat{e}_4^{\theta_j} := T_{\theta_j}(\hat{e}_4).$$

It is obvious that  $\varphi^2 = \hat{\varphi}^2$  and  $\psi^2 = \hat{\psi}^2$  on  $U_x \cap U_{\hat{x}}$ . This means that the functions  $\varphi^2, \psi^2$  are globally well-defined on  $M \setminus M_1$ .

From (5.5) we have

$$\varphi = \frac{i\mu_1}{\kappa_1} \psi,$$

while (2.5) yields

$$\omega_{34}(\overline{E}) = \frac{i}{\kappa_1^2 - \mu_1^2} (\kappa_1 \overline{E}(\mu_1) - \mu_1 \overline{E}(\kappa_1)).$$

Then using (5.8) and the last two equations, we easily verify that

$$\overline{E} \left( \psi^2 \left( 1 - \frac{\mu_1^2}{\kappa_1^2} \right) \right) = 0,$$

or equivalently,

$$\frac{\partial}{\partial \bar{z}} \left( \psi^2 \left( 1 - \frac{\mu^2}{\kappa^2} \right) \right) = 0,$$

where  $z$  is the local complex coordinate of Lemma 1. This shows that the function

$$\psi^2\left(1 - \frac{\mu^2}{\kappa^2}\right) : M \setminus M_1 \rightarrow \mathbb{C}$$

is holomorphic with isolated singularities. From the inequality

$$\left|\psi^2\left(1 - \frac{\mu^2}{\kappa^2}\right)\right| \leq \left(\sum_{j=1}^n |v_j|\right)^2,$$

we see that it is also bounded. Thus its singularities are removable and we end up with a  $\mathbb{C}$ -valued holomorphic function on the compact Riemann surface  $M$ . Hence there exists a constant  $c \in \mathbb{C}$  such that

$$(5.13) \quad \psi^2(\kappa^2 - \mu^2) = c\kappa^2 \quad \text{on } M \setminus M_1.$$

We will show that  $c = 0$ . Indeed if there exists a point  $x_l \in M_1$  with  $\kappa(x_l) = \mu(x_l) > 0$ , then taking the limit in (5.13) along a sequence of points in  $M \setminus M_1$  which converges to  $x_l$  and using the boundedness of  $\psi^2$ , we deduce that  $c = 0$ .

Suppose now that for all points in  $M_1$  the curvature ellipse degenerates into a point, i.e.,  $\kappa(x_l) = \mu(x_l) = 0$  for all  $l = 1, \dots, k$ . In other words, all points in  $M_1$  are totally geodesic points. For each  $l = 1, \dots, k$ , let  $(V, z)$  be a local complex coordinate around  $x_l$  with  $z(x_l) = 0$ . It is a well known consequence of the Codazzi equation that the  $(2, 0)$ -part

$$B^{(2,0)} = B\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) dz^2$$

of the second fundamental form of  $f$  is holomorphic as a  $Nf \otimes \mathbb{C}$ -valued tensor field (cf. [13] or [16]). Since  $B^{(2,0)}$  is not identically zero and  $x_l$  is a zero of it, according to Theorem 4, we may write

$$(5.14) \quad B^{(2,0)} = z^{m_l} B^{*(2,0)} \quad \text{on } V$$

for a positive integer  $m_l$ , where  $B^{*(2,0)}$  is a tensor field on  $V$  of type  $(2, 0)$  with  $B^{*(2,0)}|_{x_l} \neq 0$ . We now define the  $Nf$ -valued tensor field on  $V$

$$B^* := B^{*(2,0)} + \overline{B^{*(2,0)}}.$$

Since the  $(1, 1)$ -part of  $B^*$  vanishes, it follows easily that  $B^*$  maps the unit tangent circle at each tangent plane on  $V$  into an ellipse on the corresponding normal space with length of the semi-axes  $\kappa^* \geq \mu^* \geq 0$ .

We also consider the differential form of type  $(4, 0)$

$$\Phi^* := \langle B^{*(2,0)}, B^{*(2,0)} \rangle$$

which, in view of (5.14), is related to the Hopf differential of  $f$  by

$$\Phi = z^{2m_l} \Phi^*.$$

We now consider arbitrary orthonormal frames  $\{\xi_1, \xi_2\}$  and  $\{\xi_3, \xi_4\}$  of  $TM|V$  and  $Nf|V$  respectively, that agree with the given orientations. Then we split

the form  $\Phi^*$ , with respect to this frame, in the same manner we spitted the Hopf differential in section 2, i.e.,

$$\Phi = \frac{1}{4}(\overline{H}_3^2 + \overline{H}_4^2)\varphi^4 = \frac{1}{4}k_+k_-\varphi^4$$

and

$$\Phi^* = \frac{1}{4}(\overline{H}_3^{*2} + \overline{H}_4^{*2})\varphi^4 = \frac{1}{4}k_+^*k_-^*\varphi^4,$$

where

$$k_{\pm} = \overline{H}_3 \pm i\overline{H}_4, \quad k_{\pm}^* = \overline{H}_3^* \pm i\overline{H}_4^*$$

and

$$H_{\alpha} = h_1^{\alpha} + ih_2^{\alpha}, \quad H_{\alpha}^* = h_1^{*\alpha} + ih_2^{*\alpha}.$$

The components of  $B$  and  $B^*$  are given respectively by

$$h_1^{\alpha} = \langle B(\xi_1, \xi_1), \xi_{\alpha} \rangle, \quad h_2^{\alpha} = \langle B(\xi_1, \xi_2), \xi_{\alpha} \rangle$$

and

$$h_1^{*\alpha} = \langle B^*(\xi_1, \xi_1), \xi_{\alpha} \rangle, \quad h_2^{*\alpha} = \langle B^*(\xi_1, \xi_2), \xi_{\alpha} \rangle$$

for  $\alpha = 3, 4$ . Then, in view of (5.14), we obtain  $\overline{H}_{\alpha} = z^{m_l}\overline{H}_{\alpha}^*$ , or  $k_{\pm} = z^{m_l}k_{\pm}^*$ . Hence

$$a_{\pm} = |z|^{m_l}a_{\pm}^*,$$

or equivalently,

$$(5.15) \quad \kappa \pm \mu = |z|^{m_l}(\kappa^* \pm \mu^*).$$

From this we deduce that

$$\kappa = |z|^{m_l}\kappa^* \quad \text{and} \quad \mu = |z|^{m_l}\mu^*$$

and (5.13) now yields

$$(5.16) \quad \psi^2(\kappa^{*2} - \mu^{*2}) = c\kappa^{*2} \quad \text{on} \quad V \setminus \{x_l\}.$$

To prove that  $c = 0$ , we now argue in the following way. If  $\kappa^*(x_l) > \mu^*(x_l)$  for all  $1 \leq l \leq k$ , then (5.15) implies that

$$N(a_+) = \sum_{l=1}^k m_l = N(a_-).$$

Hence Lemma 4 yields  $\chi(Nf) = 0$ , which contradicts our assumption. Thus  $\kappa^*(x_l) = \mu^*(x_l)$  for some  $1 \leq l \leq k$ . Taking the limit in (5.16), along a sequence of points in  $V \setminus \{x_l\}$  which converges to  $x_l$ , we obtain  $c\kappa^{*2}(x_l) = 0$ . Obviously  $\kappa^*(x_l) > 0$ , since  $B^*|_{x_l} \neq 0$ , and consequently we infer that  $c = 0$ .

From (5.13), we conclude that  $\psi^2 = 0$  everywhere on  $M \setminus M_1$ . We note that  $\omega_{34}(E)$  cannot vanish on an open subset of  $M \setminus M_1$ . Indeed, if  $\omega_{34}(E) = 0$  on an open subset  $U \subset M \setminus M_1$ , then (2.5) would imply that the ratio  $\mu_1/\kappa_1$  is constant on  $U$ . Using Theorem 3, it is easy to see that the Gaussian curvature  $K$  satisfies the Ricci condition

$$\Delta \log(1 - K) = 4K$$

on  $U$ . According to a result due to Sakaki [17], and bearing in mind the fact that minimal surfaces are real analytic, we infer that our minimal surface lies in a totally geodesic  $S^3$ , which contradicts the assumption on the normal bundle.

Hence we may appeal to (5.6) and (5.7) to obtain

$$\sum_{j=1}^n \exp(2i\theta_j) \langle f_{\theta_j}, v_j \rangle = 0.$$

Combining this with (5.12), we get

$$\sum_{j=2}^n \langle f_{\theta_j}, w_j \rangle = 0,$$

where  $w_j := \lambda_j v_j \neq 0, j = 2, \dots, n$ , and  $\lambda_j$  is either  $\cos 2\theta_n - \cos 2\theta_1$  or  $\sin 2\theta_n - \sin 2\theta_1$ . Then repeating the same argument, we inductively conclude at the end that  $\langle f_{\theta_n}, w \rangle = 0$ , for some nonzero vector  $w$ . So  $f_{\theta_n}$  lies in a totally geodesic  $S^3$ , contradiction.

Therefore we have proved Claim 1. This means that the coordinate functions of  $f_\theta, \theta \in [0, 2\pi]$ , are linearly independent and the proof of the theorem is complete.  $\square$

*Proof of Corollary 1.* From Theorem 2, we know that

$$\mathcal{S}(f) = \{\theta_0, \theta_1, \dots, \theta_n\}$$

for some positive integer  $n$  with  $0 = \theta_0 < \theta_1 < \dots < \theta_n \leq 2\pi$ . So  $f_{\theta_j}, j = 1, \dots, n$ , is the maximal family of noncongruent minimal surfaces in  $S^4$  which are isometric to  $f$  and have the same normal curvature. We consider the immersed minimal surfaces  $f_t := f \circ \varphi_t$ . From the assumptions it follows that each  $f_t$  is isometric to  $f$  and has the same normal curvature with  $f$ . According to Theorem 2 and since the second fundamental form of  $f_t$  depends continuously on the parameter, we deduce that  $f_t$  is congruent to exactly one  $f_{\theta_j}$  for all  $t$ . Since  $f \circ \varphi_0 = f$ , we conclude that  $f_t$  is congruent to  $f$  for all  $t$ .  $\square$

## REFERENCES

- [1] A.C. Asperti, *Immersiones of surfaces into 4-dimensional spaces with nonzero normal curvature*, Ann. Mat. Pura Appl. **125** (1980), 313-328.
- [2] A.C. Asperti, D. Ferus and L. Rodriguez, *Surfaces with non-zero normal curvature tensor*, Atti Accad. Naz. Lincei **73** (1982), 109-115.
- [3] J.L. Barbosa, *On minimal immersions of  $S^2$  into  $S^{2m}$* , Trans. Amer. Math. Soc. **210** (1975), 75-106.
- [4] J. Bolton, T.J. Willmore and L.M. Woodward, *Immersiones of surfaces into space forms*, Global differential geometry and global analysis 1984 (Berlin, 1984), 46-58, Lecture Notes in Math., 1156, Springer, Berlin, 1985.
- [5] S.S. Chern, *On the minimal immersions of the two-sphere in a space of constant curvature*. Problems in Analysis, 27-40. Princeton: University Press 1970.
- [6] S.S. Chern and J.D. Wolfson, *Minimal surfaces by moving frames*, Amer. J. Math. **105** (1983), 59-83.

- [7] H. Choi, W. Meeks and B. White, *A rigidity theorem for properly embedded minimal surfaces in  $\mathbb{R}^3$* , J. Differential Geom. **32** (1990), 65-76.
- [8] M. Dajczer and D. Gromoll, *Gauss parametrizations and rigidity aspects of submanifolds*, J. Differential Geom. **22** (1985), 1-12.
- [9] J.H. Eschenburg, I.V. Guadalupe and R. Tribuzy, *The fundamental equations of minimal surfaces in  $\mathbb{C}P^2$* , Math. Ann. **270** (1985), 571-598.
- [10] J.H. Eschenburg and R. Tribuzy, *Constant mean curvature surfaces in 4-space forms*, Rend. Semin. Mat. Univ. Padova **79** (1988), 185-202.
- [11] J.H. Eschenburg and R. Tribuzy, *Associated families of pluriharmonic maps and isotropy*, Manuscripta Math. **95** (1998), 295-310.
- [12] J.H. Eschenburg and P. Quast, *The spectral parameter of pluriharmonic maps*, Bull. Lond. Math. Soc. **42** (2010), 229-236.
- [13] I.V. Guadalupe and L. Rodriguez, *Normal curvature of surfaces in space forms*, Pacific J. Math. **106** (1983), 95-103.
- [14] G.D. Johnson, *An intrinsic characterization of a class of minimal surfaces in constant curvature manifolds*, Pacific J. Math. **149** (1991), 113-125.
- [15] J. Ramanathan, *Rigidity of minimal surfaces in  $S^3$* , Manuscripta Math. **60** (1988), 417-422.
- [16] E.A. Ruh, *Minimal immersions of 2-spheres in  $S^4$* , Proc. Amer. Math. Soc. **28** (1971), 219-222.
- [17] M. Sakaki, *Minimal surfaces with the Ricci condition in 4-dimensional space forms*, Proc. Amer. Math. Soc. **121** (1994), 573-577.
- [18] B. Smyth and G. Tinaglia, *The number of constant mean curvature isometric immersions of a surface*, arXiv:0811.1231v2 [math. DG].
- [19] R. Tribuzy and I.V. Guadalupe, *Minimal immersions of surfaces into 4-dimensional space forms*, Rend. Semin. Mat. Univ. Padova **73** (1985), 1-13.
- [20] Th. Vlachos, *Congruence of minimal surfaces and higher fundamental forms*, Manuscripta Math. **110** (2003), 77-91.

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