Mathematics > Differential Geometry

An index formula for simple graphs

Oliver Knill

(Submitted on 2 May 2012)

Gauss-Bonnet for simple graphs G assures that the sum of curvatures K(x) over the vertex set V of G is the Euler characteristic X(G). Poincare-Hopf tells that for any injective function f on V the sum of i(f,x) is X(G). We also know that averaging the indices E[i(f,x)] over all functions gives curvature K(x).
We explore here the situation when G is geometric of dimension d: that is if each unit sphere S(x) is geometric of dimension d-1 and that X(S(x))=0 for even d and X(S(x))=2 for odd d. The dimension of G is inductively defined as the average of 1+dim(S(x)) over all S(x) assuming the empty graph has dimension -1.
We prove that any odd dimensional geometric graph G has zero curvature. This is done with the help of an index formula j(f,x) = 1-X(S(x))/2-X(B(f,x))/2, where j(x)=[i(f,x)+i(-f,x)]/2. The graph B(f,x) is the discrete level surface {y | f(y) = f(x)} intersected with S(x). It is a subgraph of the line graph of G and geometric if G is geometric.
The index formula simplifies for geometric graphs: for even d it is j(f,x) = 1-X(B(f,x))/2, where B(f,x) is a (d-2)-dimensional graph. For odd d it becomes j(f,x) =-X(B(f,x))/2, where B(f,x) is an odd dimensional graph. Because by induction with respect to d, the X(B(f,x))=0 we know now that that j(f,x) is zero for all x and so, by taking expectation over f that curvature K(x) is zero for all x.
We also point out that all these results hold almost verbatim for compact Riemannian manifolds and actually are much simpler there. The same integral geometric index formula is valid if f is a Morse function, i(f,x) is the index of the gradient vector field and if S(x) is a sufficiently small geodesic sphere around x and B(f,x) which is S(x) intersected with the level surface {y | f(y)=f(x)}. Also in the continuum, the symmetric index j(f,x) is constant zero everywhere if d is odd.

Comments:	24 pages, 6 figures
Subjects:	**Differential Geometry (math.DG)**; Computational Geometry (cs.CG); General Topology (math.GN)
MSC classes:	05C10, 57M15, 68R10, 53A55 60B99, 94C99, 97K30
Cite as:	**arXiv:1205.0306 [math.DG]**
	(or **arXiv:1205.0306v1 [math.DG]** for this version)

Submission history

Download:

- PDF
- Other formats

Current browse context:

math.DG

< prev | next >

new | recent | 1205

Change to browse by:

cs
cs.CG
math
math.GN

References & Citations

- NASA ADS

Bookmark(what is this?)