Persistent Homology of Filtered Covers

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Abstract

We prove an extension to the simplicial Nerve Lemma which establishes isomorphism of persistent homology groups, in the case where the covering spaces are filtered. While persistent homology is now widely used in topological data analysis, the usual Nerve Lemma does not provide isomorphism of persistent homology groups. Our argument involves some homological algebra: the key point being that although the maps produced in the standard proof of the Nerve Lemma do not commute as maps of chain complexes, the maps they induce on homology do.

Persistent homology has become a central tool in topological data analysis (TDA). The purpose of the present paper is to update the Nerve Lemma accordingly[1]. Specifically we prove an extension to the (finite) simplicial version of the Nerve Lemma, which is sufficient for the usual TDA applications. Our proof is self-contained and elementary. After the writing of this paper, it was brought to our attention that Chazal and Oudot [6] have proved an analogous result in the topological category, however their proof is more involved and relies on earlier work. In this light we feel there is merit in publishing our own argument for two reasons: 1. it provides a proof in the simplicial category of this basic result (updating the semi-classical Nerve Lemma) and 2. its simplicity allows one to see

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[1] This result was announced in the tech report [12], which also surveys relevant recent work in TDA; however, the sketch of proof suggested there did not correctly anticipate the issue discussed in Remark 4.

explicitly why the relevant chain maps must commute on homology level (see Remark 4 for a discussion of this issue).

Our main result is the following.

Proposition 1. *Let Δ be a simplicial complex and $\{\Delta^\ell\}_{\ell\in I}$ a filtration of Δ, as a topological space, such that each Δ^ℓ is also a simplicial complex. For each $\ell \in I$, suppose $(\Delta_i^\ell)_{i\in I}$ is a locally finite family of subcomplexes of Δ^ℓ. such that $\Delta^\ell = \bigcup_{i\in I} \Delta_i^\ell$ and every nonempty finite intersection $\Delta_{i_1}^\ell \cap \cdots, \cap \Delta_{i_t}^\ell$ is contractible. Suppose now for $\ell, \ell+p \in I, p > 0$ we have*

$$\Delta_i^\ell \text{ is a subcomplex of } \Delta_i^{\ell+p}. \tag{1}$$

Then for each $k \in \mathbb{N}$,

$$H_k^p(\Delta^\ell) \cong H_k^p(\mathcal{N}(\Delta_i^\ell)),$$

where $\mathcal{N}(\Delta_i^\ell)$ is the nerve of the collection $(\Delta_i^\ell)_{i\in I}$. In other words the p-persistent homology groups coincide at level ℓ in the filtrations $\Delta^\ell \subset \Delta^{\ell+p}$ and $\mathcal{N}(\Delta_i^\ell) \subset \mathcal{N}(\Delta_i^{\ell+p})$.

We define persistent homology below. For more detail the reader is referred to Weinberger's short expository article [17], or Zomorodian's thesis [18].

APPLICATION TO TOPOLOGICAL DATA ANALYSIS   Much recent TDA work [2, 9, 10, 3, 4, 5, 7, 8, 16, 18, 19] uses α-complexes to recover topological invariants of a submanifold $\mathcal{M} \subset \mathbb{R}^N$ from a point cloud Z associated to $\mathcal{M}$ while [14, 15] (in the case of smooth $\mathcal{M}$) uses Čech complexes and addresses more general sampling.

Given a (finite) point cloud $Z \subset \mathbb{R}^d$, denote $K(Z,\alpha)$ the α-complex, first defined by Edelsbrunner in 1995, [11]; denote by Čech(Z,α) the Čech complex. In each case, vertices are points of Z. The **Čech complex** is defined by:

$$\sigma = [z_0 z_1 \ldots z_p] \text{ is a } p\text{-simplex iff } \bigcap_{j=0}^{p} B(z_j,\alpha) \neq \emptyset$$

.

The α-**complex** $A(Z,\alpha)$ is defined[2] by:

[2]Warning: usually the notation $A(Z,\alpha)$ is used to denote the analogous complex obtained using balls of radius $\alpha/2$ intersected with Voronoi cells.

$$\sigma = [z_0 z_1 \dots z_p] \text{ is a } p\text{-simplex iff } \bigcap_{j=0}^{p} [B(z_j, \alpha) \cap V_j] \neq \emptyset,$$

where $V_j = \{x \in \mathbb{R}^N : \forall z \in Z,\ d(x, z_j) \leq d(x, z)\}$ is the Voronoi cell of $z_j \in Z$.

Let U be the union of balls of radius α around points of Z. Both of the collections of sets – $\{B(z_j, \alpha) : z_j \in Z\}$ and $\{B(z_j, \alpha) \cap V_j : z_j \in Z\}$ – are finite covers of U and the complexes just defined are their **nerves** (see Hatcher [13] or Bjoerner [1]). Moreover the sets in these covers are convex. The Nerve Lemma [1] therefore implies the nerves are homotopy equivalent to U, and hence to each other.

Proposition 1, as an extension of the simplicial Nerve Lemma, shows that the persistent homology groups coincide. Indeed for Čech complexes we may triangulate the collection of larger balls[3] so that the smaller balls and intersections are subcomplexes, and similarly for α-complexes; their union in either case is U.

PERSISTENT HOMOLOGY Using the standard notation $C_k(X), Z_k(X), B_k(X)$ for k-chains, k-cycles and k-boundaries respectively of a simplicial complex X, we recall:

Definition 2 (Persistent Homology)**.** *Given integers $p, k > 0$, and a filtered topological space $X = \bigcup_{\ell=0}^{\infty} X^\ell$ with $\ell < \ell' \Rightarrow X^\ell \subset X^{\ell'}$. The p-persistent k-th homology of X^ℓ is the image of $H_k(X^\ell)$ in $H_k(X^{\ell+p})$ induced by inclusion. Equivalently, it may be defined as*

$$H_k^p(X^\ell) := \frac{Z_k(X^\ell)}{B_k(X^{\ell+p}) \cap Z_k(X^\ell)}.$$

DETAILS OF THE SIMPLICIAL NERVE LEMMA: POSETS AND ORDER COMPLEXES This section is a summary of the relevant exposition in Bjoerner [1]. We use *poset* as a shorthand for *partially ordered set*.

The **face poset** $P(\Delta)$ of a simplicial complex Δ is the set of faces (simplices) of Δ ordered by inclusion. The **order complex** $\Delta(P)$ of a poset P with partial order $\leq$ is the simplicial complex with vertex set P such that $[x_0 \dots x_n]$ a k-simplex if and only if $x_0 < \dots < x_k$. Given a simplicial complex Δ, the simplicial complex

[3] The radius α plays the role of ℓ in the Proposition.

$\Delta(P(\Delta))$ is called the **barycentric subdivision** of Δ; it is homeomorphic to Δ (using geometric realizations). For readability we write $\Delta P(\Delta)$.

From now on, we **assume the hypotheses of Proposition 1.** The simplicial version of the Nerve Lemma is proved in [1] by showing that a certain continuous map

$$\Theta^\ell : \Delta P(\Delta^\ell) \to \Delta P(\mathcal{N}(\Delta_i^\ell))$$

is a homotopy equivalence, so Δ^ℓ and $\mathcal{N}(\Delta_i^\ell)$ are homotopy equivalent. In particular, Θ^ℓ induces an isomorphism between homology groups. The map Θ^ℓ is defined starting with the poset map $f^\ell : P(\Delta^\ell)) \to P(\mathcal{N}(\Delta_i^\ell))$ given by

$$\pi \mapsto \{i \in I \colon \pi \in \Delta_i\}.$$

This is an order-reversing poset map and so induces a simplicial map,

$$\Theta^\ell \colon \Delta P(\Delta^\ell) \to \Delta P(\mathcal{N}(\Delta_i^\ell)),$$

whose effect on vertices is given by f^ℓ. In fact Θ^ℓ can also be defined in this way on all of $\Delta P(\Delta^{\ell+p})$, and we will assume this.

Remark 3. We remark that $\Delta P(\Delta^\ell)$ is a subcomplex of $\Delta P(\Delta^{\ell+p})$ because Δ^ℓ is a subcomplex of $\Delta^{\ell+p}$ (hence any face of Δ^ℓ is a face of $\Delta^{\ell+p}$ and moreover nested faces $x_0 < \ldots < x_k$ of Δ^ℓ are nested faces of $\Delta^{\ell+p}$). Also, $\Delta P(\mathcal{N}(\Delta_i^\ell))$ is a subcomplex of $\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))$ by the same reasoning, since $\mathcal{N}(\Delta_i^\ell)$ is a subcomplex of $\mathcal{N}(\Delta_i^{\ell+p})$ by Equation (1). Indeed, by (1), a nonempty intersection $\Delta_{i_1}^\ell \cap \ldots \cap \Delta_{i_k}^\ell$ implies a nonempty intersection $\Delta_{i_1}^{\ell+p} \cap \ldots \cap \Delta_{i_k}^{\ell+p}$. We will not write these subcomplex inclusions explicitly; as commented earlier, we assume Θ^ℓ is defined on all of $\Delta P(\Delta^{\ell+p})$.

Remark 4. Given $\sigma \in \Delta P(\Delta^\ell)$, it is *not true* in general that $\Theta^{\ell+p}(\sigma) = \Theta^\ell(\sigma)$. In other words the following diagram does not commute:

$$\begin{tikzcd}
\Delta P(\Delta^{\ell+p}) \arrow[r, "\Theta^{\ell+p}"] & \Delta P(\mathcal{N}(\Delta_i^{\ell+p})) \\
\Delta P(\Delta^\ell) \arrow[u, hook, "\subseteq"] \arrow[r, "\Theta^\ell"] & \Delta P(\mathcal{N}(\Delta_i^\ell)). \arrow[u, hook, "\subseteq"']
\end{tikzcd}$$

Indeed, this may be seen already at vertex level: the poset map f^ℓ takes a simplex π of Δ^ℓ to the set of all indices i such that π is a subsimplex of Δ_i^ℓ, while $f^{\ell+p}$ takes π to the set of all i such that π is a subsimplex of $\Delta_i^{\ell+p}$. The second set contains the first by Equation (1), but may be strictly larger. In that case Θ^ℓ and $\Theta^{\ell+p}$ map the vertex π of $\Delta P(\Delta^\ell) \subset \Delta P(\Delta^{\ell+p})$ to distinct vertices of $\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))$.

We will, however, show that the induced chain maps Θ_*^ℓ and $\Theta_*^{\ell+p}$ differ on k-cycles of $\Delta P(\Delta^{\ell+p})$ by boundaries of $\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))$. In other words, the homology-level diagram induced by the above diagram *does* commute, giving an isomorphism of the respective p-persistent homology groups, $H_k^p(\Delta P(\Delta^\ell)) \cong H_k^p(\Delta P(\mathcal{N}(\Delta_i^\ell)))$, and therefore $H_k^p(\Delta^\ell) \cong H_k^p(\mathcal{N}(\Delta_i^\ell))$.

TECHNICAL LEMMA Given two k-simplices σ and τ with a fixed ordering of the vertices of each, we define a preferred simplicial decomposition of the mapping cylinder of the simplicial map that sends one simplex to the other preserving vertex order. Each of the original simplices belongs to this abstract simplicial complex.

Remark 5. This is a simpler version of the usual simplicial mapping cylinder, as we do not take a barycentric subdivision of one of the simplices.

We write $[v_0 v_1 \dots v_k]^o$ to denote the k-simplex $[v_0 v_1 \dots v_k]$ with this explicit vertex ordering and refer to it as an **ordered simplex**.

Definition 6 (Simplicial Mapping Cylinder)**.** *Given two ordered k-simplices $\sigma = [v_0 v_1 \dots v_k]^o$ and $\tau = [w_0 w_1 \dots w_k]^o$, define*

$$\mathrm{Cyl}(\sigma, \tau) := \sum_{t=0}^{k} (-1)^{t+1} [v_0 \dots v_t w_t \dots w_k],$$

a formal linear combination of abstract $(k+1)$-simplices on the vertex set $\{v_0, \dots, v_k\} \sqcup \{w_0, \dots, w_k\}$. Let μ_1, μ_2 be k-chains of a simplicial complex X with vertex set V. If we have

$$\mu_1 = \sum_{i=0}^{m} a_i \sigma_i \text{ and } \mu_2 = \sum_{i=0}^{m} a_i \tau_i$$

then we say μ_1 and μ_2 are compatible *and define (for a fixed ordering of the vertices of each σ_i and τ_i)*

$$\mathrm{Cyl}\Big(\sum_{i=0}^{m} a_i \sigma_i, \sum_{i=0}^{m} a_i \tau_i\Big) := \sum_{i=0}^{m} a_i \,\mathrm{Cyl}(\sigma_i, \tau_i)$$

as a formal linear combination of abstract $(k+1)$*-simplices on the vertex set* $V \sqcup V$.

In fact, $\mathrm{Cyl}(\sigma,\tau)$ in the definition, provides a simplicial decomposition of the topological mapping cylinder of the map given by $v_i \mapsto w_i$. We will only need the following (which we prove in the Appendix):

Lemma 7. *Given two compatible* k*-chains* μ_1 *and* μ_2*,*

$$\partial\,\mathrm{Cyl}(\mu_1,\mu_2) = \mu_1 - \mu_2 - \mathrm{Cyl}(\partial\mu_1,\partial\mu_2).$$

Therefore,

Corollary 8. *If* μ_1 *and* μ_2 *are compatible* k*-cycles,*

$$\partial\,\mathrm{Cyl}(\mu_1,\mu_2) = \mu_1 - \mu_2.$$

The reason for defining Cyl() in this manner is its well-behaved interaction with Θ^ℓ and $\Theta^{\ell+p}$ which we now describe. Let V be the vertex set of $\Delta P(\Delta^{\ell+p})$. Suppose we use apostrophes to indicate the elements of $V \sqcup V$ which come from the second factor; so $V \sqcup V = V \cup \{v' \colon v \in V\}$. Let $\mu \in Z_k(\Delta P(\Delta^{\ell+p}))$ be a k-cycle. Denote by μ' the corresponding k-cycle using the vertices v'. These are compatible k-chains and so $\mathrm{Cyl}(\mu,\mu')$ is well-defined (for any fixed ordering of the vertices of simplices of μ). It is a linear combination of abstract $(k+1)$-simplices on the vertex set $V \sqcup V$ and so we may apply to it the chain map φ induced by

$$v \mapsto f^\ell(v),\; v' \mapsto f^{\ell+p}(v).$$

Both of these images are vertices of $\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))$. By Corollary 8, we have

$$\begin{aligned}\partial\varphi\,\mathrm{Cyl}(\mu,\mu') &= \varphi\partial\,\mathrm{Cyl}(\mu,\mu')\\ &= \varphi(\mu-\mu')\\ &= \Theta^\ell_*(\mu) - \Theta^{\ell+p}_*(\mu),\end{aligned}$$

where Θ^ℓ_* and $\Theta^{\ell+p}_*$ are the chain maps induced by Θ^ℓ and $\Theta^{\ell+p}$ respectively. The latter, we assume, are both defined on all of $\Delta P(\Delta^{\ell+p})$, mapping into $\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))$ (see Remark 3). Here $\varphi\,\mathrm{Cyl}(\mu,\mu')$ is a formal linear combination of *abstract* $(k+1)$-simplices on the vertex set of $\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))$; none of these $(k+1)$-simplices need a priori be actual simplices of $\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))$. The following techical lemma shows, however, that they are, assuming a natural ordering of the vertices in each simplex of μ.

Lemma 9. *Let σ be a k-simplex of $\Delta P(\Delta^\ell)$), with the canonical vertex order inherited from the underlying poset. Then $\varphi\,\mathrm{Cyl}(\sigma,\sigma) \in C_{k+1}(\Delta P(\mathcal{N}(\Delta_i^{\ell+p})))$. Hence, for any $\mu \in Z_k(\Delta P(\Delta^{\ell+p}))$, $\Theta_*^\ell(\mu) - \Theta_*^{\ell+p}(\mu) \in B_k(\Delta P(\mathcal{N}(\Delta_i^{\ell+p})))$.*

Proof. Note that if x is a vertex of $\Delta P(\Delta^\ell)$ then x is a simplex of Δ^ℓ and by Remark 4, $f^\ell(x) \leq f^{\ell+p}(x)$. Suppose the k-simplex σ of $\Delta P(\Delta^\ell)$) is defined by nested simplices[4] $x_k < \ldots < x_0$ of Δ^ℓ and take any t, $1 \leq t \leq k$. We have

$$f^\ell(x_0) \leq \ldots \leq f^\ell(x_t) \leq f^{\ell+p}(x_t) \leq \ldots f^{\ell+p}(x_k).$$

Therefore, for all t, $1 \leq t \leq k$,

$$\{f^\ell(x_0), \ldots, f^\ell(x_t), f^{\ell+p}(x_t), \ldots, f^{\ell+p}(x_k)\}$$

is a simplex of $\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))$ (possibly of dimension less than k). And so, in the sum for $\mathrm{Cyl}(\sigma,\sigma)$, the abstract k-simplices $[x_0 \ldots x_t x_t' \ldots x_k']$ which are not killed off by the chain map φ will be mapped to actual k-simplices

$$[f^\ell(x_0) \ldots f^\ell(x_t) f^{\ell+p}(x_t) \ldots f^{\ell+p}(x_k)]$$

of $\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))$ (apostrophes denoting vertices in the second factor of $V \sqcup V$, as before). The final statement of the Lemma follows immediately; it suffices to assume the above-mentioned canonical vertex order in each simplex of μ. $\square$

PROOF OF THE PROPOSITION

Proof of Proposition 1. We now consider the homology level diagram induced by the diagram of Remark 4. By the proof of the Nerve Lemma, the chain maps

$$\Theta_*^\ell : C_k(\Delta P(\Delta^\ell)) \to C_k(\Delta P(\mathcal{N}(\Delta_i^\ell)))$$

and

$$\Theta_*^{\ell+p} : C_k(\Delta P(\Delta^{\ell+p})) \to C_k(\Delta P(\mathcal{N}(\Delta_i^{\ell+p})))$$

descend to isomorphisms on homology (we retain the same names for the new

[4]The indexing is done this way to make order-reversed images via f^ℓ and $f^{\ell+p}$ easier to read.

maps). So we have,

$$\begin{tikzcd}
H_k(\Delta P(\Delta^{\ell+p})) \arrow[r, "\Theta_*^{\ell+p}", "\cong"'] & H_k(\Delta P(\mathcal{N}(\Delta_i^{\ell+p}))) \\
H_k(\Delta P(\Delta^{\ell})) \arrow[u] \arrow[r, "\Theta_*^{\ell}", "\cong"'] & \Delta H_k(P(\mathcal{N}(\Delta_i^{\ell}))), \arrow[u]
\end{tikzcd}$$

where the vertical maps are those induced by inclusion. By Lemma 9, *this diagram commutes*. Indeed, given a homology class $[\mu]$ in the bottom left corner, with μ a cycle representing it, the Lemma implies that $\Theta_*^{\ell}(\mu)$ and $\Theta_*^{\ell+p}(\mu)$ differ by a boundary in the top right corner.

Therefore, the image of $H_k(\Delta P(\Delta^{\ell}))$ in $H_k(\Delta P(\Delta^{\ell+p}))$ is isomorphic to the image of $H_k(P(\mathcal{N}(\Delta_i^{\ell})))$ in $H_k(\Delta P(\mathcal{N}(\Delta_i^{\ell+p})))$; i.e.,

$$H_k^p(\Delta P(\Delta^{\ell})) \cong H_k^p(\Delta P(\mathcal{N}(\Delta_i^{\ell}))).$$

$\square$

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APPENDIX

Proof of Lemma 7. Recall that

$$\partial[v_0 v_1 \dots v_k] = \sum_{j=0}^{k} (-1)^j [v_0 \dots \hat{v}_j \dots v_k].$$

We prove the Lemma for k-simplices; it follows for compatible k-chains.

$$
\begin{aligned}
&\partial \operatorname{Cyl}([v_0 \dots v_k],[w_0 \dots w_k]) \\
&\quad = \partial \sum_{t=0}^{k} (-1)^{t+1} [v_0 \dots v_t w_t \dots w_k] \\
&\quad = -\partial [v_0 w_0 \dots w_k] + (-1)^{k+1} \partial [v_0 \dots v_k w_k] + \sum_{t=1}^{k-1} (-1)^{t+1} \partial [v_0 \dots v_t w_t \dots w_k] \\
&\quad = -[w_0 \dots w_k] + (-1)^{2(k+1)} [v_0 \dots v_k] \\
&\qquad\quad + \sum_{j=0}^{k} (-1)^j [v_0 w_0 \dots \hat{w}_j \dots w_k] + (-1)^{k+1} \sum_{j=0}^{k} (-1)^j [v_0 \dots \hat{v}_j \dots v_k w_k] \\
&\qquad\quad + \sum_{t=1}^{k-1} (-1)^{t+1} \sum_{j=0}^{t} (-1)^j [v_0 \dots \hat{v}_j \dots v_t w_t \dots w_k] \\
&\qquad\quad + \sum_{t=1}^{k-1} (-1)^{t+1} \sum_{j=t}^{k} (-1)^{j+1} [v_0 \dots v_t w_t \dots \hat{w}_j \dots w_k] \\
&\quad = [v_0 \dots v_k] - [w_0 \dots w_k] - \operatorname{Cyl}(\partial [v_0 \dots v_k], \partial [w_0 \dots w_k])
\end{aligned}
$$

because

$$
\begin{aligned}
&\operatorname{Cyl}(\partial [v_0 \dots v_k], \partial [w_0 \dots w_k]) \\
&= \sum_{j=0}^{k} (-1)^j \operatorname{Cyl}([v_0 \dots \hat{v}_j \dots v_k],[w_0 \dots \hat{w}_j \dots w_k]) \\
&= \sum_{j=0}^{k} (-1)^j \left\{ \sum_{t=0}^{j-1} (-1)^{t+1} [v_0 \dots v_t w_t \dots \hat{w}_j \dots w_k] + \sum_{t=j+1}^{k} (-1)^t [v_0 \dots \hat{v}_j \dots v_t w_t \dots w_k] \right\} \\
&= \sum_{t=0}^{k-1} (-1)^j \sum_{j=t+1}^{k} (-1)^{t+1} [v_0 \dots v_t w_t \dots \hat{w}_j \dots w_k] \\
&\qquad + \sum_{t=1}^{k} (-1)^j \sum_{j=0}^{t-1} (-1)^t [v_0 \dots \hat{v}_j \dots v_t w_t \dots w_k]
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{t=0}^{k-1}\sum_{j=t+1}^{k}(-1)^{t+j}[v_0\dots v_t w_t\dots \hat{w}_j\dots w_k]\\
&\quad -\sum_{t=1}^{k}\sum_{j=0}^{t-1}(-1)^{t+j+1}[v_0\dots \hat{v}_j\dots v_t w_t\dots w_k]\\
&= -\left\{\sum_{t=0}^{k-1}\sum_{j=t}^{k}(-1)^{t+j}[v_0\dots v_t w_t\dots \hat{w}_j\dots w_k]-\sum_{t=0}^{k-1}(-1)^{2t}[v_0\dots v_t w_{t+1}\dots w_k]\right\}\\
&\quad -\left\{\sum_{t=1}^{k}\sum_{j=0}^{t}(-1)^{t+j+1}[v_0\dots \hat{v}_j\dots v_t w_t\dots w_k]-\sum_{t=1}^{k}(-1)^{2t+1}[v_0\dots v_{t-1} w_t\dots w_k]\right\}\\
&= -\left\{\sum_{t=0}^{k-1}\sum_{j=t}^{k}(-1)^{t+j}[v_0\dots v_t w_t\dots \hat{w}_j\dots w_k]+\sum_{t=1}^{k}\sum_{j=0}^{t}(-1)^{t+j+1}[v_0\dots \hat{v}_j\dots v_t w_t\dots w_k]\right\}\\
&= -\left\{\sum_{t=1}^{k-1}\sum_{j=t}^{k}(-1)^{t+j}[v_0\dots v_t w_t\dots \hat{w}_j\dots w_k]+\sum_{t=1}^{k-1}\sum_{j=0}^{t}(-1)^{t+j+1}[v_0\dots \hat{v}_j\dots v_t w_t\dots w_k]\right.\\
&\quad \left.+\sum_{j=0}^{k}(-1)^{j}[v_0 w_0\dots \hat{w}_j\dots w_k]++\sum_{j=0}^{k}(-1)^{k+j+1}[v_0\dots \hat{v}_j\dots v_k w_k]\right\}.
\end{aligned}
$$

□

References

[1] A. Bjoerner, *Topological Methods*. In Handbook of Combinatorics (eds. R. Graham, M. Groetschel and L. Lovasz), North-Holland, Amsterdam, (1995), 1819-1872.

[2] E. Carlsson, G. Carlsson, V. de Silva, *An Algebraic Topological Method for Feature Identification*. Int. J. Comput. Geometry Appl. **16** (2006), 291-314.

[3] F. Chazal, A. Lieutier, *Smooth Manifold Reconstruction from Noisy and Non Uniform Approximation with Guarantees*. Computational Geometry: Theory and Applications, **40** (2008), 156-170.

[4] F. Chazal, A. Lieutier, *Weak feature size and persistent homology: computing homology of solids in $\mathbb{R}^N$ from noisy data samples*. In Proc. 21st Annual Symposium on Computational Geometry (2005), 255-262.

[5] F. Chazal, D. Cohen-Steiner, M. Glisse, L.J. Guibas, and S. Y. Oudot, *Proximity of Persistence Modules and their Diagrams*. In Proc. 25th Annual Symposium on Computational Geometry (2009), 237-246.

[6] F. Chazal, S. Oudot, *Towards Persistence-Based Reconstruction in Euclidean Spaces*. In Proc. 24th Annual Symposium on Computational Geometry (2008), 232-241.

[7] D. Cohen-Steiner, H. Edelsbrunner and J. Harer, *Stability of persistence diagrams*. Discrete Comput. Geom. **37** (2007), 103-120.

[8] V. de Silva, G. Carlsson, *Topological estimation using witness complexes*. Eurographics Symposium on Point-Based Graphics (2004).

[9] H. Edelsbrunner, D. Letscher and A. Zomorodian, *Topological persistence and simplification*. Discrete Comput. Geom. **28** (2002), 511-533.

[10] H. Edelsbrunner, J. Harer *Persistent homology – a survey*. Surveys on Discrete and Computational Geometry. Twenty Years Later, 257-282, eds. J. E. Goodman, J. Pach and R. Pollack, Contemporary Mathematics **453**, Amer. Math. Soc., Providence, Rhode Island, 2008.

[11] H. Edelsbrunner, *The union of balls and its dual shape*. Discrete & Computational Geometry **13**, 3-4 (1995), 415-440.

[12] M. Fraser. *Two extensions to Manifold Learning Algorithms using* α-*Complexes*. University of Chicago Dept. of Computer Science Technical Report, TR-2010-07, (2010).

[13] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[14] P. Niyogi , S. Smale , S. Weinberger, *Finding the Homology of Submanifolds with High Confidence from Random Samples*. Discrete & Computational Geometry, **39** (2008), 419-441.

[15] P. Niyogi , S. Smale , S. Weinberger, *A Topological View of Unsupervised Learning and Clustering*. SIAM J. Comput. **40**, 3 (2011), 646663.

[16] S. Oudot, L. Guibas, J.-D. Boissonat, *Manifold Reconstruction in Arbitrary Dimensions Using Witness Complexes*. SCG-07, June 6-8, Gyeongju, South Korea (2007).

[17] S. Weinberger, *What is persistent homology?*. Notices AMS, **58**, 01 (2011), 36-39.

[18] A. Zomorodian, *Computing and Comprehending Topology: Persistence and Hierarchical Morse Complexes*. PhD thesis, Duke University (advisor H. Edelsbrunner), 2001.

[19] A. Zomorodian, G. Carlsson, *Computing persistent homology*. Symposium on Computational Geometry (2004), 347-356.