Mathematics > Combinatorics

Polytopes, Hopf algebras and Quasi-symmetric functions

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In this paper we use the technique of Hopf algebras and quasi-symmetric functions to study the combinatorial polytopes. Consider the free abelian group $\mathcal{P}$ generated by all combinatorial polytopes. There are two natural bilinear operations on this group defined by a direct product $\times $ and a join $\divideontimes$ of polytopes. $(\mathcal{P},\times)$ is a commutative associative bigraded ring of polynomials, and $\mathcal{RP}=(\mathbb Z\varnothing\oplus\mathcal{P},\divideontimes)$ is a commutative associative threegraded ring of polynomials. The ring $\mathcal{RP}$ has the structure of a graded Hopf algebra. It turns out that $\mathcal {P}$ has a natural Hopf comodule structure over $\mathcal{RP}$. Faces operators d_k that send a polytope to the sum of all its $(n-k)$-dimensional faces define on both rings the Hopf module structures over the universal Leibnitz-Hopf algebra $\mathcal{Z}$. This structure gives a ring homomorphism $\R\to\Qs\otimes\R$, where $\R$ is $\mathcal{P}$ or $\mathcal{RP}$. Composing this homomorphism with the characters $P^n\to\alpha^n$ of $\mathcal{P}$, $P^n\to\alpha^{n+1}$ of $\mathcal {RP}$, and with the counit we obtain the ring homomorphisms $f\colon\mathcal{P}\to\Qs[\alpha]$, $f_{\mathcal{RP}}\colon\mathcal {RP}\to\Qs[\alpha]$, and $\F^*:\mathcal{RP}\to\Qs$, where F is the Ehrenborg transformation. We describe the images of these homomorphisms in terms of functional equations, prove that these images are rings of polynomials over $\mathbb Q$, and find the relations between the images, the homomorphisms and the Hopf comodule structures. For each homomorphism $f,\;f_{\mathcal{RP}}$, and $\F$ the images of two polytopes coincide if and only if they have equal flag f-vectors. Therefore algebraic structures on the images give the information about flag f-vectors of polytopes.

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