ON THE LYAPOUNOV EXPONENTS OF SCHRÖDINGER OPERATORS ASSOCIATED WITH THE STANDARD MAP

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Abstract. It is shown that Schrödinger operators defined from the standard map have positive (mean) Lyapounov exponents for almost all energies

1. Definitions

(1.1) Let V be a bounded potential on $\mathbb{Z}_+$. Define

$\omega(V) = \{W = (W_n)_{n\in\mathbb{Z}};$ there is a sequence $n_j \to \infty$ such that $d(S^{n_j} V, W) \to 0\}$
where

$d(V, V') = \sum 2^{-n} |V_n - V_n'|$
and

$(S^k V)(n) = V(n+k)$.

(1.2) Let $A \subset \mathbb{R}$, $\text{mes}\,(A) > 0$. Denote

$$\mathcal{R}(A) = \{\text{bounded potentials } W = (W_n)_{n\in\mathbb{Z}} \text{ that are reflectionless on } A\}$$

and $\mathcal{R}^C(A) = \{W \in \mathbb{R}(A); \sup_n |W_n| \leq C\}$

$W \to H = W + \Delta$ and let $G(z)$ be the Green's function of H.

Recall that W is reflectionless on A if

$$Re G(t)(n) = 0 \text{ for a.e. } t \in A \text{ and every } n \in \mathbb{Z}.$$

2. Remling Theorems (see [R]).

(2.1) $\omega(V) \subset \mathcal{R}\big(\sum_{ac}(V)\big)$ ([R] Theorem 1.4)

(2.2) $R^C(A)$ is compact ([R] Proposition 4.1, (d))

(2.3) The restriction maps $\mathcal{R}(A) \to \mathcal{R}_\pm(A)$ are injective ($\pm$ refers to $\pi_{\mathbb{Z}_\pm}$).

and for any constant C, the map

$$\mathcal{R}^C(A) \to \mathcal{R}^C_+(A) \text{ is uniformly continuous}$$

([R] Proposition 4.1 (c), (e))

(2.4) $W \in \mathcal{R}(A) \Rightarrow A \subset \sum_{ac}(W_\pm)$

3. Lyapounov Exponents

Let T be a measure preserving homeomorphism of a compact metric space Ω endowed with a probability measure μ that changes any non-empty open subset (we do not assume T ergodic).

Let $\varphi \in \mathcal{C}(\Omega)$.

Denote half-line SO

$$H_x = \varphi(T^n x)\delta_n + \Delta$$

and

$$M_N(E,x) = \prod_N^1 \begin{pmatrix} E - \varphi(T^n x) & -1 \\ 1 & 0 \end{pmatrix}$$

$$L_N(E) = \frac{1}{N} \int \log \|M_N(E,x)\| \mu(dx)$$

Let $A \subset \mathbb{R}, |A| > 0$ and assume

$$\underline{\lim}_N L_N(E) = 0 \text{ on } A. \tag{3.1}$$

Then

$$A \subset \sum\nolimits_{a.c}(V_x) \text{ for } \mu \ a.e.x \tag{3.2}$$

Proof.

Let

$$\mu = \int \beta \, \alpha(d\beta)$$

be the ergodic decomposition of μ.

By Fubini, for $E \in A$

$$L_N(E) = \int \Big\{ \frac{1}{N} \int \log \|M_N(E,x)\| \beta(dx) \Big\} \alpha(d\beta)$$

and

$$\int \Big\{ \frac{\lim}{N} \Big(\frac{1}{N} \int \log \|M_N(E,x)\| \beta(dx) \Big) \Big\} \alpha(d\beta).$$

Since β is ergodic, it follows that for α - a.e. β

$$\frac{1}{N} \log \|M_N(E,x)\| \to 0 \text{ for } \beta \text{ - a.e.} \, x \in \Omega \tag{3.3}$$

Again by Fubini, (3.3) holds for a.e. $E \in A$ and β - a.e. $x \in \Omega$

By Kotani theory, this implies that

$$A \subset \sum\nolimits_{ac} (H_x) \text{ for } \beta - a.e.\, x \in \Omega. \tag{3.4}$$

Since (3.4) is valid for α - a.e. β, (3.2) follows.

4. Use of recurrence

Let T be as in §3.

Let $V_x = \big(\varphi(T^n x)\big)_{n \in \mathbb{Z}_+}$

Then

$$V_x \in \pi_{\mathbb{Z}_+}\big(\omega(V_x)\big) \text{ for } \mu \text{ - a.e. } x \in \Omega. \tag{4.1}$$

Proof.

By Poincaré recurrence lemma, for μ - a.e. $x \in \Omega$, there is a sequence $n_j \to \infty$ such that

$$T^{n_j} x \to x. \tag{4.2}$$

Hence

$$\varphi(T^{n_j + k} x) \to \varphi(T^k x) \equiv W_x(k)$$

for all $k \in \mathbb{Z}$ and

$$d(W_x, V_{T^{n_j} x}) \to 0$$

It follows that

$$W_x \in \omega(V_x)$$

and obviously $V_x = \pi_{\mathbb{Z}_+}(W_x)$.

By (2.1), (4.1) implies

$$V_x \in \mathcal{R}_+\Big(\sum\nolimits_{ac} (V_x)\Big) \text{ for } \mu \; a.e \; x. \tag{4.3}$$

From (4.3), (3.2)

$$V_x \in \mathcal{R}_+(A) \text{ for } \mu \text{ } a.e. \text{ } x \tag{4.4}$$

thus $V_x \in \mathcal{R}_+^C(A)$ for μ a.e. x and since $\mathcal{R}_+^C(A)$ is compact by (2.2) and V_x depends continuously on x, we conclude that

$$V_x \in \mathcal{R}_+(A) \text{ for all } x \in \Omega \tag{4.5}$$

(Ω = closure of any subset of full measure).

Finally, by (2.4)

$$A \subset \sum\nolimits_{ac}(V_x) \text{ for all } x \in \Omega. \tag{4.6}$$

5. Application to standard map

Let $T = T_\lambda$ be the standard map on the torus $\mathbb{T}^2 = \Omega$ with sufficiently large λ. Thus

$$f_\lambda(x, y) = (-y + 2x + \lambda \sin 2\pi x, x). \tag{5.1}$$

Let $\varphi \in \mathcal{C}^1(\mathbb{T}^2)$ be a fixed not-constant function.

If the corresponding SO has vanishing Lyapounov exponents for $E \in A$, (4.6) implies

$$A \subset \sum\nolimits_{ac}(\{\varphi(T^n x)\}_{n\in\mathbb{Z}_+} \tag{5.2}$$

for all $x \in \Omega$.

By Duarte's work (theorem A in [D]), there is an invariant hyperbolic set $\Lambda = \Lambda_\lambda \subset \Omega$ such that $T|_\Lambda$ is conjugate to a Bernoulli shift. In particular for $x \in \Lambda, (5.2) = \phi$. Furthermore, Duarte's result asserts that each point in $\mathbb{T}^2$ is within a $4\lambda^{-\frac{1}{3}}$-neighborhood of Λ, so that φ will not be constant on Λ for λ large enough. Hence (5.2) restricted to Λ is non-deterministic and Kotani's theorem implies that A is of zero-measure (contradiction).

Hence we proved

Proposition. *For $\lambda > \lambda_0$, the SO (5.2) associated to the standard map T_λ has positive (mean) Lyapounov exponents for a.e. $E \in \mathbb{R}$.*

Acknowledgement: The author is grateful to A. Avila, S. Sodin and T. Spencer for some discussions on this topic.

References

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[R]. C. Remling, *The absolutely continuous spectrum of Jacobi matrices*, arXiv:1007.5033.