J. Symbolic Computation (2001) **31**, 691–716
doi:10.1006/jsco.2001.0461
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Rational Solutions of Riccati-like Partial Differential Equations

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When factoring linear partial differential systems with a finite-dimensional solution space or analysing symmetries of nonlinear ODEs, we need to look for rational solutions of certain nonlinear PDEs. The nonlinear PDEs are called Riccati-like because they arise in a similar way as Riccati ODEs. In this paper we describe the structure of rational solutions of a Riccati-like system, and an algorithm for computing them. The algorithm is also applicable to finding all rational solutions of Lie's system $\{\partial_x u + u^2 + a_1 u + a_2 v + a_3,\ \partial_y u + uv + b_1 u + b_2 v + b_3,\ \partial_x v + uv + c_1 u + c_2 v + c_3,\ \partial_y v + v^2 + d_1 u + d_2 v + d_3\}$, where $a_1, \ldots, d_3$ are rational functions of x and y.

1. Introduction

Riccati's equation is one of the first examples of a nonlinear differential equation that was considered extensively in the literature, shortly after Leibniz and Newton introduced the concept of the derivative of a function at the end of the 17th century. Riccati equations occur in many problems of mathematical physics and pure mathematics. A good survey is given in the book by Reid (1972).

Of particular importance is the relation between Riccati's equation and a linear ODE $y'' + ay' + by = 0$ with $a, b \in \mathbb{C}(x)$, where $\mathbb{C}$ is the field of the complex numbers. For example, solutions h with the property that the quotient $p = h'/h \in \mathbb{C}(x)$ may be represented as $h = \exp\left(\int p\, \mathrm{d}x\right)$ if p satisfies the first-order Riccati equation $p' + p^2 + ap + b = 0$. Equivalently, this linear ODE allows the first-order right factor $y' - qy$ over $\mathbb{C}(x)$ if q obeys the same equation as p. In general, finding the first-order right rational factors of a linear homogeneous ODE is equivalent to finding the rational solutions of its associated Riccati equation (see, for example, Singer, 1991).

It turns out that this correspondence carries over to systems of linear homogeneous partial differential equations with a finite-dimensional solution space. Systems of this kind occur in Lie's symmetry theory for solving nonlinear ODEs and related equivalence problems. For example, Lie studied the coherent nonlinear system

$$\begin{aligned}\{\partial_x u + u^2 + a_1 u + a_2 v + a_3, &\quad \partial_y u + uv + b_1 u + b_2 v + b_3,\\ \partial_x v + uv + c_1 u + c_2 v + c_3, &\quad \partial_y v + v^2 + d_1 u + d_2 v + d_3\}\end{aligned} \tag{1.1}$$

for the first time in connection with the symmetry analysis of second-order ODEs with projective symmetry group (see, for example, Lie 1873, p. 365). It is suggested therefore

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0747–7171/01/060691 + 26 $35.00/0

to call (1.1) *Lie's system*. We will show that Lie's system may be transformed to the Riccati-like system $\mathcal{R}_3^{(2)}$ given in Example 3.1.

Let $\mathbb{Q}$ be the field of the rational numbers and $\bar{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$. This paper solves the following problem. Given a set $\mathcal{L}$ of linear homogeneous PDEs in one unknown function $z(x, y)$ whose coefficients are in $\mathbb{Q}(x, y)$ and whose solution space is of finite dimension, find a solution of $\mathcal{L}$ in the form $\exp\left(\int u\,\mathrm{d}x + v\,\mathrm{d}y\right)$, where u and v are in $\bar{\mathbb{Q}}(x, y)$ with $\partial_y u = \partial_x v$. In other words, we want to find a linear differential ideal over $\bar{\mathbb{Q}}(x, y)$ with one-dimensional solution space containing $\mathcal{L}$. As shown in Section 3, this problem is equivalent to finding a rational solution of the Riccati-like system associated with $\mathcal{L}$. We describe an algorithm for computing all rational solutions of an associated Riccati-like system. The algorithm is being implemented in the ALLTYPES system (Schwarz, 1998b), and applicable to finding rational solutions of Lie's system and hyperexponential solutions of linear homogeneous PDEs with finite-dimensional solution space in several unknowns.

The paper is organized as follows. Section 2 contains necessary preliminaries. Section 3 presents the structure of rational solutions of a Riccati-like system. Section 4 modifies an algorithm for computing rational solutions of Riccati ODEs for our purpose. Section 5 presents an algorithm for computing rational solutions of a Riccati-like system. Applications are given in Section 6.

2. Preliminaries

Throughout the paper, let $\mathbf{C}$ be an algebraically closed constant field of characteristic 0, $\mathbf{K}$ the field of rational functions $\mathbf{C}(x, y)$, and ∂_x, ∂_y the usual partial differential operators acting on $\mathbf{K}$. Let Ω^* be a universal differential field of $\mathbf{K}$. The constant field of Ω^* is denoted by $\mathbf{C}^*$. Remark that the introduction of the universal field of Ω^* is to avoid the logical difficulties involved in the incessant extensions of differential and constant fields. It brings no consequence to the algorithms developed henceforth. An element a of Ω^* is called an x-constant (resp. y-constant) if $\partial_x a = 0$ (resp. $\partial_y a = 0$). An x-derivative (resp. y-derivative) of a means $\partial_x^k a$ (resp. $\partial_y^k a$) for some non-negative integer k. By a system we mean a finite subset of some differential polynomial ring over $\mathbf{K}$. Basic notions related to differential polynomials are used such as: rankings, leaders, autoreduced sets, differential remainders, and characteristic sets for differential ideals. For their definitions, the reader is referred to Ritt (1950), Rosenfeld (1959), and Kolchin (1973). The differential ideal generated by a subset $\mathcal{P}$ of some differential polynomial ring is denoted by $[\mathcal{P}]$.

This section is organized as follows. Section 2.1 describes the elimination property of linear differential ideals with finite linear dimension. Section 2.2 defines the notion of integrable pairs and studies their properties. Section 2.3 reviews the notion of coherent orthonomic systems. Section 2.4 describes two methods for finding solutions (in $\bar{\mathbb{Q}}(t)$) of a zero-dimensional algebraic system over $\bar{\mathbb{Q}}(t)$.

2.1. Linear systems with finite linear dimension

Let z be a differential indeterminate over $\mathbf{K}$ and fix an orderly ranking on the differential polynomial ring $\mathbf{K}\{z\}$. We denote by $\mathbf{L}$ the $\mathbf{K}$-linear space consisting of all linear homogeneous polynomials in $\mathbf{K}\{z\}$.

Given a linear system $\mathcal{L} \subset \mathbf{L}$, the linear dimension of $[\mathcal{L}]$ in $\mathbf{K}\{z\}$ is the codimension of $\mathbf{L} \cap [\mathcal{L}]$ in $\mathbf{L}$ (Kolchin, 1973, p. 151). The solution space of $\mathcal{L}$ is a $\mathbf{C}^*$-vector space. Its dimension is equal to the linear dimension of $[\mathcal{L}]$ if either dimension is finite (Kolchin, 1973, p. 152). To check if the linear dimension of $[\mathcal{L}]$ is finite, we compute a coherent autoreduced set $\mathcal{A}$ Rosenfeld (1959) such that $[\mathcal{L}] = [\mathcal{A}]$. This computation can be done by various methods such as: Janet bases Janet (1920) and Schwarz (1998a), the characteristic set method Wu (1989), Li and Wang (1999) and Gröbner bases for differential operators Kandri-Rody and Weispfenning (1990). The linear dimension of $[\mathcal{L}]$ is finite if and only if an x-derivative and a y-derivative of z appear as leaders of some elements of $\mathcal{A}$.

We denote by $\mathbf{L}_x$ (resp. $\mathbf{L}_y$) the subset of $\mathbf{L}$ consisting of differential polynomials involving only x-derivatives (resp. y-derivatives) of z. The following elimination property is well known.

Lemma 2.1. *Let $\mathcal{L}$ be a finite subset of $\mathbf{L}$. If $[\mathcal{L}]$ is of finite linear dimension, then there is an algorithm for computing two nonzero elements $[\mathcal{L}] \cap \mathbf{L}_x$ and $[\mathcal{L}] \cap \mathbf{L}_y$, respectively.*

Proof. Compute a coherent autoreduced set $\mathcal{A}$ in $\mathbf{L}$ w.r.t. an orderly ranking such that $[\mathcal{L}] = [\mathcal{A}]$. If $\mathcal{A}$ is $\{1\}$, $[\mathcal{L}]$ is trivial. Otherwise, all the differential monomials $\partial_x^i \partial_y^j z$ that cannot be reduced w.r.t. $\mathcal{A}$, form a finite basis $\mathbf{B}$ for the $\mathbf{K}$-vector space $\mathbf{V} = \mathbf{L}/(\mathbf{L} \cap [\mathcal{L}])$. Using $\mathcal{A}$, we can express any element of $\mathbf{V}$ as a $\mathbf{K}$-linear combination of elements of $\mathbf{B}$ by the reduction w.r.t. $\mathcal{A}$ and linear algebra. In particular, we can compute the smallest integer n and $a_0, a_1, \ldots, a_{n-1} \in \mathbf{K}$ such that

$$L_x = a_0 z + a_1 \partial_x z + \cdots + a_{n-1} \partial_x^{n-1} z + \partial_x^n z \in [\mathcal{L}].$$

Similarly, we can find a nonzero linear differential polynomial in $[\mathcal{L}] \cap \mathbf{L}_y$. $\square$

Remark 2.1. As described in the proof of Lemma 2.1, L_x and L_y can be computed by a similar method used in Gröbner bases computation (Faugére *et al.*, 1993).

2.2. integrable pairs

To describe the structure of rational solutions in Section 3, we define a pair of rational functions $(f, g) \in \mathbf{K} \times \mathbf{K}$ to be *integrable* if $\partial_y f = \partial_x g$. For an integrable pair (f, g), the expression $H = \exp\left(\int f \,\mathrm{d}x + g \,\mathrm{d}y\right)$ denotes a nonzero solution (in Ω^*) of the system $\{\partial_x Z - fZ, \partial_y Z - gZ\}$ when the value of the multiplicative constant is irrelevant to our discussion. Two integrable pairs (f, g) and (p, q) are said to be *equivalent*, denoted by "$\sim$", if there exists a nonzero h in $\mathbf{K}$ such that $f - p = \partial_x h/h$ and $g - q = \partial_y h/h$.

Lemma 2.2. *Let (f, g) and (p, q) be two integrable pairs. Then $(f, g) \sim (p, q)$ if and only if the ratio of $A = \exp\left(\int f \,\mathrm{d}x + g \,\mathrm{d}y\right)$ and $B = \exp\left(\int p \,\mathrm{d}x + q \,\mathrm{d}y\right)$ is the product of a nonzero element of $\mathbf{C}^*$ and an element of $\mathbf{K}$.*

Proof. From $\partial_x A = fA$, $\partial_y A = gA$, $\partial_x B = pB$, and $\partial_y B = qB$, it follows that

$$\frac{\partial_x\left(\frac{A}{B}\right)}{\frac{A}{B}} = f - p \qquad \text{and} \qquad \frac{\partial_y\left(\frac{A}{B}\right)}{\frac{A}{B}} = g - q. \tag{2.2}$$

If $A/B = ch$ for some $c \in \mathbf{C}^*$ and $h \in \mathbf{K}$, then (2.2) implies

$$f - p = \frac{\partial_x h}{h} \qquad \text{and} \qquad g - q = \frac{\partial_y h}{h}, \tag{2.3}$$

so that $(f, g) \sim (p, q)$. Conversely, if (2.3) holds for some $h \in \mathbf{K}$, then (2.2) and (2.3) imply that the ratio of A/B and h is a constant in $\mathbf{C}^*$. □

By Lemma 2.2, $\sim$ is an equivalence relation on the set of integrable pairs.

An element a in Ω^* is said to be a *hyperexponential* if $\partial_x a/a$ and $\partial_y a/a$ belong to $\mathbf{K}$. If a is a hyperexponential, $(\partial_x a/a, \partial_y a/a)$ is an integrable pair. Conversely, for an integrable pair (f, g), we may construct a hyperexponential $\exp\left(\int f\,\mathrm{d}x + g\,\mathrm{d}y\right)$ in Ω^*. For an integrable pair (f, g), define $E^{(f,g)}$ to be the $\mathbf{C}^*$-linear space generated by

$$\left\{ h \exp\left(\int f\,\mathrm{d}x + g\,\mathrm{d}y \right) \mid h \in \mathbf{K} \right\}.$$

The following lemma is used to group rational solutions of a Riccati-like system.

Lemma 2.3. *If $(f_1, g_1), (f_2, g_2), \ldots, (f_n, g_n)$ are mutually inequivalent integrable pairs, the sum of $\mathbf{C}^*$-linear spaces $E^{(f_1,g_1)}, E^{(f_2,g_2)}, \ldots, E^{(f_n,g_n)}$ is direct.*

Proof. We proceed by induction. For $n = 2$, if $E^{(f_1,g_1)} \cap E^{(f_2,g_2)}$ contains a nonzero element a, then

$$a = \left(\sum_i c_{i1} h_{i1}\right) \exp\left(\int f_1\,\mathrm{d}x + g_1\,\mathrm{d}y\right) = \left(\sum_j c_{j2} h_{j2}\right) \exp\left(\int f_2\,\mathrm{d}x + g_2\,\mathrm{d}y\right)$$

for some $c_{i1}, c_{j2} \in \mathbf{C}^*$ and $h_{i1}, h_{j2} \in \mathbf{K}$. Denote by $\sum_i c_{i1} h_{i1}$ and $\sum_j c_{j2} h_{j2}$ by H_1 and H_2, respectively. We find

$$f_1 - f_2 = \frac{\partial_x \left(\frac{H_1}{H_2}\right)}{\frac{H_1}{H_2}} \qquad \text{and} \qquad g_1 - g_2 = \frac{\partial_y \left(\frac{H_1}{H_2}\right)}{\frac{H_1}{H_2}}, \tag{2.4}$$

which is equivalent to the algebraic system

$$\begin{aligned} \{H_1 H_2 (f_1 - f_2) &= H_2 \partial_x H_1 - H_1 \partial_x H_2,\ H_1 H_2 (g_1 - g_2) \\ &= H_2 \partial_y H_1 - H_1 \partial_y H_2,\ H_1 H_2 \neq 0\} \end{aligned}$$

in c_{i1}'s and c_{j2}'s over $\mathbf{K}$. This system then has solutions in $\mathbf{C}$ by Lemma 5.1 of Kaplansky (1957). Thus, the c_{i1}'s and c_{j2}'s can be regarded as elements in $\mathbf{C}$. It follows from (2.4) that $(f_1, g_1) \sim (f_2, g_2)$, a contradiction. The sum of $E^{(f_1,g_1)}$ and $E^{(f_2,g_2)}$ is direct.

Assume that the result is proved for lower values of n. If the sum of $E^{(f_1,g_1)}, E^{(f_2,g_2)}, \ldots, E^{(f_n,g_n)}$ is not direct, there are nonzero $z_1 \in E^{(f_1,g_1)}$, $z_2 \in E^{(f_2,g_2)}, \ldots, z_n \in E^{(f_n,g_n)}$ which are $\mathbf{C}^*$-linearly dependent. By a possible rearrangement of indexes, we have

$$z_n = c_1 z_1 + c_2 z_2 + \cdots + c_{n-1} z_{n-1} \tag{2.5}$$

for some $c_1, c_2, \ldots, c_{n-1} \in \mathbf{C}^*$. Since $z_1, z_2, \ldots, z_{n-1}$ are $\mathbf{C}^*$-linearly independent by the induction hypothesis, Theorem 1 in Kolchin (1973, p. 86) implies that there exist derivatives $\theta_1, \theta_2, \ldots, \theta_{n-1}$ such that $W = \det(\theta_i z_j)$ is nonzero, where $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$. Since the z_i's are hyperexponentials, there exists $r_{ij} \in \mathbf{K}$ such that

$\theta_j z_i = r_{ji} z_i$, for each i and each j. Applying $\theta_1, \theta_2, \ldots, \theta_{n-1}$ to (2.5) then yields a linear system

$$\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,n-1} \\ r_{21} & r_{22} & \cdots & r_{2,n-1} \\ \cdot & \cdot & \cdots & \cdot \\ r_{n-1,1} & r_{n-1,2} & \cdots & r_{n-1,n-1} \end{pmatrix} \begin{pmatrix} c_1 z_1 \\ c_2 z_2 \\ \vdots \\ c_{n-1} z_{n-1} \end{pmatrix} = \begin{pmatrix} r_{1n} z_n \\ r_{2n} z_n \\ \vdots \\ r_{n,n-1} z_n \end{pmatrix}$$

whose coefficient matrix (r_{ji}) is of full rank because $W \neq 0$. Solving this system, we get $c_i z_i = s_i z_n$, where $s_i \in \mathbf{K}$. Since $c_k \neq 0$ for some k with $1 \leq k \leq n-1$, so $(f_k, g_k) \sim (f_n, g_n)$ by Lemma 2.2, a contradiction. □

2.3. Coherent orthonomic systems

Let $\mathbf{D}$ be a differential polynomial ring with a preselected ranking. A system $\mathcal{P}$ in $\mathbf{D}$ is orthonomic if it is an autoreduced set and any element of $\mathcal{P}$ is linear w.r.t. its leader and monic. A linear autoreduced set can be considered as an orthonomic system. System (1.1) is orthonomic w.r.t. any orderly ranking on u and v. Systems studied in this paper are always orthonomic.

Let $\mathcal{P}$ be an orthonomic system in $\mathbf{D}$. We recall the notions of Δ-polynomials and coherence in Rosenfeld (1959) for $\mathcal{P}$. Note that these two notions are originally defined for autoreduced sets and the name "Δ-polynomial" is due to Boulier *et al.* (1995). Let P_1 and P_2 belong to $\mathcal{P}$ with the respective leaders $\theta_1 z$ and $\theta_2 z$, where z is a differential indeterminate, and θ_1, θ_2 are derivatives. Then there exist derivatives ϕ_1 and ϕ_2 with minimal orders such that $\phi_1\theta_1 = \phi_2\theta_2$. The Δ-polynomial of P_1 and P_2, denoted by $\Delta(P_1, P_2)$, is $\phi_1 P_1 - \phi_2 P_2$. Note that $\Delta(P_1, P_2)$ is well defined provided the leaders of P_1 and P_2 are derivatives of the same indeterminate. The system $\mathcal{P}$ is coherent if, for every such pair P_1, P_2 in $\mathcal{P}$, $\Delta(P_1, P_2)$ can be written as a $\mathbf{D}$-linear combination of derivatives of elements of $\mathcal{P}$, in which each derivative has its leader lower than $\phi_1\theta_1 z$.

To study orthonomic systems, we need to make sure that they cannot be formally reduced to non-orthonomic or trivial ones. Corollary 3 in Chapter I of Rosenfeld (1959) asserts that an orthonomic system $\mathcal{P}$ is a characteristic set of $[\mathcal{P}]$ if and only if $\mathcal{P}$ is coherent. Hence, $\mathcal{P}$ cannot be formally reduced any further if it is coherent. By the same corollary, one can easily show

Lemma 2.4. *An orthonomic system $\mathcal{P}$ is coherent if and only if all Δ-polynomials (possibly) formed by elements of $\mathcal{P}$ have zero as their differential remainders w.r.t. $\mathcal{P}$.*

This lemma enables us to decide algorithmically if an orthonomic system is coherent.

2.4. Solving zero-dimensional system in $\bar{\mathbb{Q}}(t)$

At a certain point of our algorithm, the following problem has to be considered. Given a zero-dimensional algebraic system $\mathcal{S}$ in $\bar{\mathbb{Q}}(t)[w_1, \ldots, w_n]$ where $t, w_1, \ldots, w_n$ are algebraic indeterminates, find all solutions of $\mathcal{S}$ in $\bar{\mathbb{Q}}(t)$. Since the numerical coefficients of an element of $\bar{\mathbb{Q}}(t)[w_1, \ldots, w_n]$ are considered as algebraic numbers over a finite extension over $\mathbb{Q}$, the (extended) Euclidean algorithm, gcd-calculation, square-free factorization, and Gröbner basis computation can be performed (Trager, 1976).

Computing the Gröbner basis $\mathcal{G}$ for $(\mathcal{S})$ w.r.t. some elimination ordering, say, $w_1 < \cdots < w_n$, we find a univariate polynomial $P(w_1)$ in $(\mathcal{S}) \subset \bar{\mathbb{Q}}(t)[w_1]$. Hence, the problem is reduced to finding all solutions of $P(w_1)$ in $\bar{\mathbb{Q}}(t)$ because the back-substitution of $\tilde{w}_1$ with $P(\tilde{w}_1) = 0$ and $\tilde{w}_1 \in \bar{\mathbb{Q}}(t)$ into $\mathcal{G}$ never extends the coefficient field $\bar{\mathbb{Q}}(t)$. There are at least two ways to find the solutions of $P(w_1)$ in $\bar{\mathbb{Q}}(t)$. One is to factor $P(w_1)$ over $\bar{\mathbb{Q}}(t)$ and consider the linear factors. The methods for absolute factorization are presented by Kaltofen (1985a,b), Bajaj *et al.* (1993) and other researchers, see also Winkler (1996, Section 5.5) and references therein. The other way is a naive undetermined coefficient method described below.

Assume that $w_1 = f(t)/g(t)$ is a solution of $P(w_1) = 0$, where $f, g \in \bar{\mathbb{Q}}[t]$ and $\gcd(f, g) = 1$. Substituting f/g for w_1 in $P(w_1)$, we derive that f and g divide the trailing and leading coefficients of P, respectively. Assume that d and e are the respective degrees of the trailing and leading coefficients in t. Then we make the following ansatz:

$$r_m = \frac{f_d t^d + f_{d-1} t^{d-1} + \cdots f_0}{t^m + g_{m-1} t^{m-1} + \cdots + g_0}, \tag{2.6}$$

where $0 \le m \le e$. For $m = 0, 1, \ldots, e$, (in this order), forcing $P(t, r_m) = 0$ yields an algebraic system $\mathcal{S}_m$ contained in $\bar{\mathbb{Q}}[f_0, \ldots, f_d, g_0, \ldots, g_{m-1}]$. Assume that $\mathcal{S}_m$ is the first system which has a solution. Then $\mathcal{S}_m$ is zero-dimensional, because $P(w_1)$ has finitely many monic linear factors, and g is monic with minimal degree. Each solution of $\mathcal{S}_m$ corresponds to a monic linear factor of $P(w_1)$. Applying this method repeatedly, we find all linear factors of $P(w_1)$.

Example 2.2. Find the linear factors of

$$P = (1 + 2t^2 + t^4) w_1^4 + (t + 2t^3 + t^5 - 2t^2) w_1^2 - 2t^3 \quad \text{over } \bar{\mathbb{Q}}(t).$$

Method 1. Use the Maple function `AFactor` to get

$$P = (w_1^2 + t)(w_1 + w_1 t^2 - \alpha t)(w_1 + w_1 t^2 + \alpha t) \qquad \text{where} \quad \alpha^2 - 2 = 0.$$

Method 2. According to (2.6), set

$$r_m = \frac{f_3 t^3 + f_2 t^2 + f_1 t + f_0}{t^m + g_{m-1} t^{m-1} + \cdots g_0}$$

for $m = 0, \ldots, 4$. $P(r_0) = 0$ and $P(r_1) = 0$ lead to inconsistent systems, while $P(r_2) = 0$ gives rise to a system with solutions $f_3 = f_2 = f_0 = 0$, $f_2 = \pm\sqrt{2}$, $g_1 = 0$, $g_0 = 1$. Therefore, $P(w_1)$ has linear factors $\left(w_1 - \frac{\sqrt{2}t}{t^2+1}\right)$ and $\left(w_1 + \frac{\sqrt{2}t}{t^2+1}\right)$. Dividing out these two factors from P yields a polynomial which has no solutions in $\bar{\mathbb{Q}}(t)$ by the same method. Thus, the set of solutions of P in $\bar{\mathbb{Q}}(t)$ is $\left\{\frac{\sqrt{2}t}{t^2+1}, -\frac{\sqrt{2}t}{t^2+1}\right\}$.

It would be interesting to find a more efficient way to compute linear factors of $P(w_1)$ over $\bar{\mathbb{Q}}(t)$. But discussions along this direction are beyond the scope of this paper.

3. Associated Riccati-like Systems and Their Rational Solutions

Given a linear differential system $\mathcal{L} \subset \mathbf{L}$ with finite linear dimension, we want to compute its hyperexponential solutions. The substitution

$$z \leftarrow \exp\left(\int u \,\mathrm{d}x + v \,\mathrm{d}y\right) \qquad \text{where} \quad \partial_y u = \partial_x v \tag{3.7}$$

transforms $\mathcal{L}$ to a nonlinear system in $\mathbf{K}\{u, v\}$. The union of this system and $\{\partial_y u - \partial_x v\}$ is called the *Riccati-like system associated with* $\mathcal{L}$, or, simply, an *associated Riccati-like system.* Conversely, the substitution $u \leftarrow \partial_x z/z$, $v \leftarrow \partial_y z/z$ transforms an associated Riccati-like system in $\mathbf{K}\{u, v\}$ to a system in $\mathbf{L}$. As in the ordinary case, computing hyperexponential solutions of $\mathcal{L}$ is equivalent to computing rational solutions of its associated Riccati-like system.

Example 3.1. Let $\mathcal{L}_2 = \{\partial_x^2 z + a_1\partial_x z + a_2 z,\ \partial_y z + b_1\partial_x z + b_2 z\}$ be coherent. Then the linear dimension of $\mathcal{L}$ is 2. The Riccati-like system $\mathcal{R}_2$ associated with $\mathcal{L}$ is $\{\partial_x u + u^2 + a_1 u + a_2,\ v + b_1 u + b_2,\ \partial_y u - \partial_x v\}$. Coherent linear systems with dimension 3 may be either $\mathcal{L}_3^{(1)}$ equal to $\{\partial_x^3 z + a_1\partial_x^2 z + a_2\partial_x z + a_3 z,\ \partial_y z + b_1\partial_x^2 z + b_2\partial_x z + b_3 z\}$, or $\mathcal{L}_3^{(2)}$ equal to

$$\{\partial_x^2 z + a_1\partial_x z + a_2\partial_y z + a_3 z,\ \partial_y\partial_x z + b_1\partial_x z + b_2\partial_y z + b_3 z,\ \partial_y^2 z + c_1\partial_x z + c_2\partial_y z + c_3 z\}.$$

Their respective associated Riccati-like systems are $\mathcal{R}_3^{(1)}$ equal to

$$\{\partial_x^2 u + 3u\partial_x u + u^3 + a_1(\partial_x u + u^2) + a_2 u + a_3,\ v + b_1(\partial_x u + u^2) + b_2 u + b_3,\ \partial_y u - \partial_x v\}$$

and $\mathcal{R}_3^{(2)}$ equal to

$$\{\partial_x u + u^2 + a_1 u + a_2 v + a_3,\ \partial_y u + uv + b_1 u + b_2 v + b_3,\ \partial_y v + v^2 + c_1 u + c_2 v + c_3,\ \partial_y u - \partial_x v\}.$$

Notation. In the rest of this section, we fix a system $\mathcal{L}$ in $\mathbf{L}$ of finite linear dimension d. The Riccati-like system associated with $\mathcal{L}$ is denoted by $\mathcal{R}$. The set of the solutions of $\mathcal{R}$ in $\mathbf{K}$ is denoted by $\mathbf{S}$.

Note that all elements of $\mathbf{S}$ are integrable pairs, because $\partial_y u - \partial_x v$ is in $\mathcal{R}$.

To describe rational solutions of Riccati-like systems precisely, we introduce some notation. Let $\mathbf{F}$ be a subfield of $\mathbf{K}$. By an $\mathbf{F}$*-linearly independent set of* $\mathbf{C}[x, y]$, we mean a finite subset of $\mathbf{C}[x, y]$ whose elements are $\mathbf{F}$-linearly independent. Let a, b be in $\mathbf{K}$. Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ be $\mathbf{C}^*(y)$ and $\mathbf{C}^*(x)$-independent sets, respectively. We define

$$X_A^a = \left\{ a + \frac{\partial_x\left(\sum_{i=1}^k c_i a_i\right)}{\sum_{i=1}^k c_i a_i} \mid c_1, \ldots, c_k \text{ are } x\text{-constants of } \Omega^* \right\} \bigcap \mathbf{K}$$

and

$$Y_B^b = \left\{ b + \frac{\partial_y\left(\sum_{i=1}^l c_i b_i\right)}{\sum_{i=1}^l c_i b_i} \mid c_1, \ldots, c_l \text{ are } y\text{-constants of } \Omega^* \right\} \bigcap \mathbf{K}.$$

If, moreover, (a,b) is integrable and $H = \{h_1, \ldots, h_m\}$ is a $\mathbf{C}^*$-linearly independent set, we define

$$S_H^{(a,b)} = \left\{ \left(a + \frac{\partial_x \left(\sum_{i=1}^m c_i h_i\right)}{\sum_{i=1}^m c_i h_i},\ b + \frac{\partial_y \left(\sum_{i=1}^m c_i h_i\right)}{\sum_{i=1}^m c_i h_i} \right) \mid c_1, \ldots, c_m \in \mathbf{C}^* \right\} \bigcap (\mathbf{K} \times \mathbf{K}).$$

The next lemma tells us that the constants appearing in the definition of X_A^a, Y_B^b and $S_H^{(a,b)}$ can be chosen from some special rings.

Lemma 3.1. *Let $r_1, r_2, \ldots, r_k$ be $\mathbf{C}(y)$-linearly independent elements of $\mathbf{C}(y)[x]$ and c_1, $c_2, \ldots, c_k$ be x-constants of Ω^*, not all zero. If*

$$f = \frac{c_1 \partial_x r_1 + c_2 \partial_x r_2 + \cdots + c_k \partial_x r_k}{c_1 r_1 + c_2 r_2 + \cdots + c_k r_k}$$

belongs to $\mathbf{K}$, then there exist $s_1, s_2, \ldots, s_k \in \mathbf{C}[x,y]$ and $d_1, d_2, \ldots, d_k \in \mathbf{C}[y]$ such that

$$f = \frac{d_1 \partial_x s_1 + d_2 \partial_x s_2 + \cdots + d_k \partial_x s_k}{d_1 s_1 + d_2 s_2 + \cdots + d_k s_k}.$$

Proof. Assume that $f = f_1/f_2$, where $f_1, f_2 \in \mathbf{C}[x,y]$. Equating the coefficients of the like powers of x in

$$(c_1 \partial_x r_1 + c_2 \partial_x r_2 + \cdots + c_k \partial_x r_k) f_2 = (c_1 r_1 + c_2 r_2 + \cdots + c_k r_k) f_1,$$

we find that $c_1, c_2, \ldots, c_k$ satisfy a linear homogeneous system over $\mathbf{C}(y)$. It follows that the linear system has a nontrivial solution in $\mathbf{C}(y)$. Hence, $c_1, c_2, \ldots, c_k$ can be chosen as elements in $\mathbf{C}(y)$. Let h be the common denominator of all the $c_i r_i$. Since h is in $\mathbf{C}[y]$,

$$f = \frac{h(c_1 \partial_x r_1 + c_2 \partial_x r_2 + \cdots + c_k \partial_x r_k)}{h(c_1 r_1 + c_2 r_2 + \cdots + c_k r_k)} = \frac{d_1 \partial_x s_1 + d_2 \partial_x s_2 + \cdots + d_k \partial_x s_k}{d_1 s_1 + d_2 s_2 + \cdots + d_k s_k}$$

where $d_1, d_2, \ldots, d_k \in \mathbf{C}[y]$ and $s_1, s_2, \ldots, s_k \in \mathbf{C}[x,y]$. □

By Lemma 3.1 the c_i's in X_A^a and Y_B^b can be chosen as elements in $\mathbf{C}[y]$ and $\mathbf{C}[x]$, respectively. In addition, the c_i's in $S_A^{(a,b)}$ can be chosen as elements in $\mathbf{C}$. The next theorem describes the structure of $\mathbf{S}$.

Theorem 3.2. *There exist mutually inequivalent integrable pairs $(a_1, b_1), \ldots, (a_m, b_m)$, and $\mathbf{C}$-linearly independent sets $H_1, \ldots, H_m$ such that $\mathbf{S}$ is the (disjoint) union of $S_{H_i}^{(a_i,b_i)}$, $i = 1, \ldots, m$. Moreover, $\sum_{i=1}^m |H_i|$ is no greater than d.*

Proof. Let $\mathbf{V}$ be the solution space of $\mathcal{L}$ and $(f_1, g_1), \ldots, (f_m, g_m)$ in $\mathbf{S}$, none of which is equivalent to the other. Then $E^{(f_1,g_1)} \cap \mathbf{V}, \ldots, E^{(f_m,g_m)} \cap \mathbf{V}$ are all nontrivial, and form a direct sum by Lemma 2.3. Thus, m is no greater than d. We may further assume that there are only m equivalence classes (w.r.t. $\sim$) in $\mathbf{S}$.

Let (f,g) be one of the (f_i, g_i)'s and $T = \exp\left(\int f \,\mathrm{d}x + g \,\mathrm{d}y\right)$. Assume that the intersection of V and $E^{(f,g)}$ is of dimension k over $\mathbf{C}^*$. Then there exist $\mathbf{C}^*$-linearly independent $r_1, \ldots, r_k \in \mathbf{K}$ such that $r_1 T, \ldots, r_k T$ form a basis for $E^{(f,g)} \cap V$. If $rT \in E^{(f,g)}$, for some $r \in \mathbf{K}$, then r is a $\mathbf{C}$-linear combination of $r_1, \ldots, r_k$, because the constant coefficients of this linear combination are determined by a non-singular linear system over $\mathbf{K}$. It follows that the subset of $\mathbf{S}$ consisting of integrable pairs equivalent to (f, g) is equal to

$$S_{(f,g)} = \left\{ \left(f + \frac{c_1 \partial_x r_1 + \cdots + c_k \partial_x r_k}{c_1 r_1 + \cdots + c_k r_k},\ g + \frac{c_1 \partial_y r_1 + \cdots + c_k \partial_y r_k}{c_1 r_1 + \cdots + c_k r_k} \right) \mid c_1, \ldots, c_k \in \mathbf{C} \right\}.$$

Assume that $r_j = h_j/h$ where $h, h_j \in \mathbf{C}[x, y]$, for $j = 1, \ldots, k$. Let (a, b) be the integrable pair $(f - \partial_x h/h,\ g - \partial_y h/h)$ and H the $\mathbf{C}^*$-linearly independent set $\{h_1, \ldots, h_k\}$. Then $S_{(f,g)} = S_H^{(a,b)}$. Since $\mathbf{S}$ is the (disjoint) union of $S_{(f_i,g_i)}$, $i = 1, \ldots, m$, there exist integrable pairs (a_i, b_i), with $(a_i, b_i) \sim (f_i, g_i)$, and $\mathbf{C}^*$-linearly independent sets H_i such that $\mathbf{S}$ is the (disjoint) union of $S_{H_i}^{(a_i,b_i)}$, $i = 1, \ldots, m$. Moreover, $\sum_{i=1}^m |H_i| \leq d$, because $\dim_{C^*}(E^{(a_i,b_i)} \cap V) = |H_i|$. □

We call the set $\{S_{H_i}^{(a_i,b_i)} \mid i = 1, \ldots, m\}$ a *representation* of $\mathbf{S}$.

4. Some Special Rational Solutions of Riccati ODEs

Let $\mathbf{F}$ be a constant field of characteristic zero, $\bar{\mathbf{F}}$ the algebraic closure of $\mathbf{F}$, $\mathbf{F}(t)$ the differential field with derivation operator $\frac{\mathrm{d}}{\mathrm{d}t} = {}'$. and z a differential indeterminate w.r.t. $'$. Define the sequence of differential polynomials $(W_i)_{i\geq 0}$ by $W_0 = 1$, and $W_i = W_{i-1}' + zW_{i-1}$, for $i \in \mathbb{Z}^+$. An $(n-1)$th order associated Riccati ODE of z is

$$R_{n-1}(z) = A_n W_n + \cdots + A_1 W_1 + A_0 W_0 = 0, \tag{4.8}$$

where $A_n, \ldots, A_1, A_0 \in \mathbf{F}[t]$. Applying the transformation $z = Z'/Z$ to (4.8) yields the equation $A_n Z^{(n)} + \cdots + A_1 Z' + A_0 Z = 0$ with which (4.8) is associated. We denote by $\mathbf{T}$ and $\bar{\mathbf{T}}$ the solutions of (4.8) in $\mathbf{F}(t)$ and $\bar{\mathbf{F}}(t)$, respectively. Algorithms by Singer (1991) and Bronstein (1992a) are able to compute $\bar{\mathbf{T}}$. We present a theorem to describe the structure of $\mathbf{T}$ and modify Bronstein's algorithm **solve_riccati** to compute $\mathbf{T}$.

For $f, g \in \mathbf{F}(t)$, define that f and g are equivalent if $f - g = h'/h$ for some $h \in \mathbf{F}(t)$. This relation is an equivalence one and denoted by $\sim_t$. By a $\mathbf{F}$-linearly independent set we mean a finite subset of $\mathbf{F}[t]$ whose elements are linearly independent over $\mathbf{F}$. For $a \in \mathbf{F}(t)$ and a linearly independent set $A = \{a_1, \ldots, a_k\}$, we denote by T_A^a the set

$$\left\{ a + \frac{\sum_{i=1}^k c_i a_i'}{\sum_{i=1}^k c_i a_i} \mid c_1, \ldots c_k \in \bar{\mathbf{F}}, \text{not all zero} \right\}.$$

To describe the structure of $\mathbf{T}$, we need a replacement for Lemma 2.3 because $\mathbf{F}$ is not assumed to be algebraically closed.

Lemma 4.1. *Equation* (4.8) *has at most n inequivalent rational solutions in* $\mathbf{F}(t)$.

Proof. We proceed by induction on n. If $n = 2$, then $R_1(z)$ has at most two inequivalent rational solutions a_1 and a_2 because $\exp\left(\int a_1\,\mathrm{d}t\right)$ and $\exp\left(\int a_2\,\mathrm{d}t\right)$ are linearly independent over any constant field containing $\mathbf{F}$. Assume that the lemma holds for lower values of n. If $R_{n-1}(z)$ has $(n+1)$ inequivalent rational solutions $a_1, a_2, \ldots, a_{n+1}$, then

$$\exp\left(\int a_i - a_1 + \frac{(a_i - a_1)'}{(a_i - a_1)}\,\mathrm{d}t \right), \qquad i = 2, 3, \ldots, n+1$$

are n solutions of a linear homogeneous ODE $L_{n-1}(Z)$ of order $(n-1)$ by the transformation trick used in the proof of Lemma 2.4 in Singer (1991). It follows that

$$a_i - a_1 + \frac{(a_i - a_1)'}{(a_i - a_1)}, \qquad i = 2, 3, \ldots, n+1$$

are n inequivalent rational solutions of the Riccati equation associated with $L_{n-1}(Z)$, a contradiction to the induction hypothesis. □

It follows from Lemma 4.1 and the proof of Theorem 3.2 that

THEOREM 4.2. *There exist mutually inequivalent elements $a_1, \ldots, a_m$ in $\mathbf{F}(t)$, and linearly independent subsets $A_1, \ldots, A_m$ such that the set of solutions of* (4.8) *in $\mathbf{F}(t)$ is the disjoint union of $T_{A_i}^{a_i}$, $i = 1, \ldots, m$. Moreover, $\sum_{i=1}^{m} |A_i|$ is no greater than n.*

We call the set $\{T_{A_i}^{a_i} \mid i = 1, \ldots, m\}$ *a representation* of $\mathbf{T}$.

According to Theorem 8.4 in Bronstein (1992b), an element $\tilde{z} \in \mathbf{T}$ can be written as

$$\tilde{z} = P + \frac{R}{G} + \frac{Q'}{Q} \tag{4.9}$$

where $P, R, G, Q \in \mathbf{F}[t]$, G is monic, $\deg R < \deg G$, $\gcd(R, G) = 1$, all the roots of G are the roots of A_n, and $\gcd(A_n, Q) = 1$. We call P the polynomial part, R/G the singular part, and Q'/Q the logarithmic derivative part of $\tilde{z}$.

In the rest of this section, let s be an algebraic indeterminate and $\mathbf{F} = \bar{\mathbb{Q}}(s)$. Now, we modify the algorithm **solve_riccati** to compute $\mathbf{T}$.

The algorithm **solve_riccati** proceeds as follows. First, use algorithms **poly_part** and **singular_part** (Bronstein, 1992a) to compute a finite subset $\mathbf{H}$ of $\bar{\mathbf{F}}(t)$ such that, for every $\tilde{z} \in \bar{\mathbf{T}}$, the sum of its polynomial and singular parts belongs to $\mathbf{H}$. Second, for each $h \in \mathbf{H}$, determine polynomials Q in $\bar{\mathbf{F}}[t]$ such that $h + Q'/Q \in \bar{\mathbf{T}}$ by computing all polynomial solutions of the linear homogeneous ODE $L_h(Q) = R(h + Q'/Q)$.

The algorithm **poly_part** computes a degree bound m for the polynomial part P of any element of $\bar{\mathbf{T}}$, and compute the coefficient p_i of powers of t^i, by solving an univariate polynomial U_i over $\bar{\mathbf{F}}$, for $i = m, m-1, \ldots, 0$. Hence, if p_i does not belong to $\mathbf{F}$, neither does P. The coefficients of U_i are in $\mathbf{F}(p_m, \ldots, p_{i+1})$, because U_i is obtained from the additive change of variables. Consequently, if P belongs to $\mathbf{F}[t]$, its coefficient p_i is a root (in $\mathbf{F}$) of U_i over $\mathbf{F}$. Such roots are computable according to Section 2.4. These considerations lead to a modification that proceeds in the same way as **poly_part** but, for $i = m, m-1, \ldots, 0$, computes the roots of U_i in $\mathbf{F}$, and constructs U_{i-1} by the additive change of variables based on these roots when $i > 0$. The output of this modification is a finite subset of $\mathbf{F}[t]$ containing all polynomial parts of the elements of $\mathbf{T}$.

Let $C_1^{e_1} \cdots C_q^{e_q}$ be a balanced factorization of A_n, where $C_1, \ldots, C_q$ belong to $\mathbf{F}[t]$ and are coprime. The algorithm **padic_part** computes a positive integer m_i for each i with $1 \le i \le q$ such that the singular part S of any element of $\bar{\mathbf{T}}$ can be written as

$$\sum_{i=1}^{q} \sum_{j=1}^{m_i} \frac{B_{ij}}{C_i^j}$$

where B_{ij} is in $\bar{\mathbf{F}}[t]$ and of degree less than $\deg C_i$, for $1 \le i \le q$ and $1 \le j \le m_i$. The algorithm then computes the coefficients of each B_{ij} by finding all solutions of some zero-dimensional algebraic system $\mathcal{S}_{ij}$ over $\bar{\mathbf{F}}$, for $j = m_i, m_i - 1, \ldots, 1$. The uniqueness of partial fraction decomposition w.r.t. $\{C_1, \ldots, C_k\}$ implies that S does not belong to $\mathbf{F}(t)$ if some B_{ij} does not belong to $\mathbf{F}[t]$. The coefficients of $\mathcal{S}_{ij}$ are in the extension field of $\mathbf{F}$ adjoining solutions of the systems $\mathcal{S}_{kl}$, for $k \leq i$ and $l > j$, because $\mathcal{S}_{ij}$ is also obtained from the additive change of variables. Thus, the coefficients of $\mathcal{S}_{ij}$ are in $\mathbf{F}$ when the solutions in $\mathbf{F}$ are adjoined. Similar to the modification of **poly_part**, computing all solutions of $\mathcal{S}_{ij}$ in $\mathbf{F}$ in the equation-solving step of the algorithm **singular_part** yields a finite subset of $\mathbf{F}(t)$, which contains all singular parts of the elements of $\mathbf{T}$.

Assume that $h = P + R/G$ is a candidate in $\mathbf{F}(t)$. The logarithmic derivative part Q'/Q is determined by computing polynomial solutions of $L_h(Q) = 0$. This computation results in a basis $\{Q_1, \ldots, Q_k\}$ for the vector space of the polynomial solutions of $L_h(Q) = 0$ over $\mathbf{F}$, where $Q_1, \ldots, Q_k$ belong to $\mathbf{F}[t]$ because all coefficients of L_h belong to $\mathbf{F}[t]$. Thus, we obtain a family of solutions of (4.8):

$$\tilde{z} = P + \frac{R}{G} + \frac{\sum_{i=1}^{k} c_i Q_i'}{\sum_{i=1}^{k} c_i Q_i}, \tag{4.10}$$

where the c_i's are arbitrary elements (not all zero) in $\bar{\mathbf{F}}$. If $\tilde{z}$ is in $\mathbf{F}(t)$, then the c_i's can be chosen as elements in $\mathbf{F}$ by the similar argument used in the proof of Lemma 3.1.

The discussions in the last three paragraphs lead to the following conclusion: *the algorithm* **solve_riccati** *computes* $\mathbf{T}$ *if the algebraic solver used in the algorithm is able to compute all solutions (in* $\mathbf{F}$*) of a zero-dimensional algebraic system over* $\mathbf{F}$. Such a solver can be constructed by Section 2.4.

Remark 4.1. A solution $\tilde{z}$ given in (4.10) might belong to $\mathbf{T}$, although P or R/G is in $\bar{\mathbf{F}}(t) \setminus \mathbf{F}(t)$. This is because the c_i's can be chosen arbitrarily in $\bar{\mathbf{F}}$. But such $\tilde{z}$ must be contained the output of the algorithm **solve_riccati** with the above modifications.

Example 4.2. Let $\mathbf{F} = \mathbb{Q}(s)$. Compute the set $\mathbf{T}$ of the Riccati equation $W_4 + (1 + s)W_2 + s = 0$. The algorithm **poly_part** computes the degree bound for P to be 0. Hence,

$$P^4 + (s+1)P^2 + s = 0,$$

which has two solutions $\pm\sqrt{-1}$ in $\bar{\mathbb{Q}}(s)$. Since the coefficients of the Riccati equation are constants w.r.t. t, the singular part of any element of $\mathbf{T}$ is equal to 0. After computing the logarithmic derivative parts, we find $\mathbf{T} = \left\{\sqrt{-1}, -\sqrt{-1}\right\}$. The algorithm **solve_riccati** without the modifications yields $\bar{\mathbf{T}} = \left\{\sqrt{-1}, -\sqrt{-1}, \sqrt{-s}, -\sqrt{-s}\right\}$.

The input of the modified algorithm **solve_riccati** is the set $\{A_n, \ldots, A_0\}$ given in (4.8). The output consists of a finite number of sets $T_{B_i}^{b_i}$ whose union is $\mathbf{T}$. Now, we modify the output to get a representation of $\mathbf{T}$.

Lemma 4.3. *Let f, g be in $\mathbf{F}(t)$. Then $f \sim_t g$ if and only if the square-free partial decomposition is of the form $\sum_i (m_i r_i'/r_i)$, for some square-free $r_i \in \mathbf{F}[t]$ and nonzero $m_i \in \mathbb{Z}$.*

Proof. If $f \sim_t g$, there exist $p, q \in \mathbf{F}[t]$ with $\gcd(p,q) = 1$ such that $f - g = (p'/p) - (q'/q)$. Let $p = p_1 p_2^2 \cdots p_d^d$ and $q = q_1 q_2^2 \cdots q_e^e$ be the respective square-free decompositions of p and q. Then

$$f - g = \sum_{i=0}^{d} \frac{ip_i'}{p_i} - \sum_{j=0}^{d} \frac{jq_j'}{q_j}.$$

Conversely, we have $f - g = \left(\prod_i r_i^{m_i}\right)' / \prod_i r_i^{m_i}$. □

If the b_i's obtained from the modified algorithm **solve_riccati** are mutually inequivalent, then we are done. Otherwise, the next lemma is applied.

Lemma 4.4. *If $b_1 \sim_t b_2$, we can compute $b \in \mathbf{F}(t)$ and a linearly independent set B such that $T_{B_1}^{b_1} \cup T_{B_2}^{b_2} \subset T_B^b \subset \mathbf{T}$.*

Proof. Let B_i be $\{h_{i1}, \ldots, h_{ik_i}\}$ for $i = 1, 2$. Compute $g \in \mathbf{F}(t)$ by Lemma 4.3 such that $b_1 = b_2 + g'/g$. Then

$$T_{B_1}^{b_1} = \left\{ b_2 + \frac{c_{11}(h_{11}g)' + \cdots + c_{1k_1}(h_{1k_1}g)'}{c_{11}(h_{11}g) + \cdots + c_{1k_1}(h_{1k_1}g)} \mid c_{11}, \ldots c_{1k_1} \in \mathbf{F} \right\}.$$

From the set $G = \{h_{11}g, \ldots, h_{1k_1}g, h_{21}, \ldots, h_{2k_2}\}$, we pick up a maximally linearly independent set

$$\left\{ \frac{h_1}{h}, \ldots, \frac{h_k}{h} \mid h, h_1, \ldots, h_k, \in \mathbf{F}[t] \right\}.$$

Setting $b = b_2 - h'/h$ and $B = \{h_1, \ldots, h_k\}$, we obtain T_B^b, which contains both $T_{B_1}^{b_1}$ and $T_{B_2}^{b_2}$ because each element of G is an $\mathbf{F}$-linear combination of some elements of B. Note that, for all $c_1, \ldots, c_k \in \mathbf{F}$,

$$(c_1h_1 + \cdots + c_kh_k) \exp\left(\int b \, \mathrm{d}t \right) = \left(c_1 \frac{h_1}{h} + \cdots + c_k \frac{h_k}{h} \right) \exp\left(\int b_2 \, \mathrm{d}t \right),$$

which is contained of S_t. □

Example 4.3. The Riccati equation $t^2(z' + z^2) + tz - 1 = 0$ has rational solutions $1/t$ and $-1/t + 2ct/(1 + ct^2)$, where c is a constant. The set $\mathbf{T}$ is the union of $T_{\{1\}}^{\frac{1}{t}}$ and $T_{\{1,t^2\}}^{\frac{-1}{t}}$. Since $(1/t) \sim_t (-1/t)$ by Lemma 4.3, $T_{\{1\}}^{\frac{1}{t}}$ is contained in $T_{\{1,t^2\}}^{\frac{-1}{t}}$ by Lemma 4.4.

5. Computing Rational Solutions of Associated Riccati-like Systems

In this section, let $\mathbf{C} = \bar{\mathbb{Q}}$ and consider the following problem. *Given a Riccati-like system $\mathcal{R}$ associated with $\mathcal{L}$, compute a representation of the set of its rational solutions* $\mathbf{S}$.

Our idea consists of four steps. First, compute two nonzero ODEs $L_x(z) \in [\mathcal{L}] \cap \mathbf{L}_x$ and $L_y(z) \in [\mathcal{L}] \cap \mathbf{L}_y$ (with lowest order) by Lemma 2.1. Second, translate $L_x(z)$ and $L_y(z)$ to their respective associated Riccati ODEs $R_x(u)$ and $R_y(v)$ by (3.7), and compute respective representations of the solutions of the Riccati ODEs in $\mathbf{K}$ by the modified algorithm **solve_riccati** and Lemma 4.4. We denote by $\mathbf{X}$ and $\mathbf{Y}$ the representations of the solutions of $R_x(u)$ and $R_y(v)$ in $\mathbf{K}$, respectively. Third, from $\mathbf{X}$ and $\mathbf{Y}$, construct a finite number of mutually inequivalent integrable pairs such that an element of $\mathbf{S}$ is equivalent to such a pair. Finally, for each pair obtained from the third step, compute elements of $\mathbf{S}$ that are equivalent to it. Before describing the last two steps, we remind the reader of a useful identity

$$\partial_y\left(\frac{\partial_x a}{a}\right) = \partial_x\left(\frac{\partial_y a}{a}\right) \qquad \text{for all nonzero } a \in \Omega^*. \tag{5.11}$$

LEMMA 5.1. *If (f, g) belongs to $\mathbf{S}$, then there exist unique $X_A^a \in \mathbf{X}$ and $Y_B^b \in \mathbf{Y}$ such that all rational solutions of $\mathcal{R}$ equivalent to (f, g) are contained in $X_A^a \times Y_B^b$.*

PROOF. Since $(f, g) \in \mathbf{S}$, f and g are rational solutions of $R_x(u)$ and $R_y(v)$, respectively. There then exist unique $X_A^a \in \mathbf{X}$ and unique $Y_B^b \in \mathbf{Y}$ such that (f, g) is in $X_A^a \times Y_B^b$. If (p, q) is in $\mathbf{S}$ and equivalent to (f, g), $p \in X_A^a$ and $q \in Y_B^b$ by Theorem 4.2. □

Lemma 5.1 reduces our task to computing $\mathbf{S} \cap (X_A^a \times Y_B^b)$, for all $X_A^a \in \mathbf{X}$ and $Y_B^b \in \mathbf{Y}$. Now, we search for elements in $\mathbf{X} \times \mathbf{Y}$ that have possibly nonempty intersection with $\mathbf{S}$. We call an element $X_A^a \times Y_B^b$ of $\mathbf{X} \times \mathbf{Y}$ a *candidate* if (a, b) is integrable, and try to transform other elements in $\mathbf{X} \times \mathbf{Y}$ to candidates by the following lemma.

LEMMA 5.2. *Let a and b belong to $\mathbf{K}$. There exist two polynomials $p, q \in \mathbf{C}[x, y]$ such that $(a + \partial_x p/p,\ b + \partial_y q/q)$ is integrable if and only if*

$$\partial_x \partial_y (\log z) = \partial_y a - \partial_x b \tag{5.12}$$

has a solution in $\mathbf{K}$.

PROOF. Let r be $\partial_y a - \partial_x b$. Assume that there exist such p and q. Then

$$\begin{aligned} r &= \partial_x\left(\frac{\partial_y q}{q}\right) - \partial_y\left(\frac{\partial_x p}{p}\right) = \partial_x\left(\frac{\partial_y q}{q}\right) - \partial_x\left(\frac{\partial_y p}{p}\right) \quad \text{(by (5.11))} \\ &= \partial_x(\partial_y \log q - \partial_y \log p) = \partial_x \partial_y \left(\log \frac{q}{p}\right). \end{aligned}$$

Conversely, if $z = q/p$ is a solution of (5.12), reversing the above calculation shows that $(a + \partial_x p/p,\ b + \partial_y q/q)$ is integrable. □

Now, we present an algorithm for computing such p and q.

Algorithm IntegrablePair (*Find an integrable pair*). Given a, b in $\mathbf{K}$, the algorithm finds p, q in $\mathbf{C}[x, y]$ such that $(a + \partial_x p/p,\ b + \partial_y q/q)$ is integrable, or determines that no such p and q exist.

I1. [Initialize.] Set $r \leftarrow \partial_y a - \partial_x b$. If $r = 0$, set $p \leftarrow 1, q \leftarrow 1$ and exit.
I2. [Hermite's reduction.] Write

$$r = r_1 + \frac{r_2}{r_3} \qquad \text{where } r_1, r_2, r_3 \in \mathbf{C}(y)[x] \qquad \text{with} \quad \deg_x r_2 < \deg_x r_3.$$

Integrate r_1 w.r.t. x and apply Hermite's reduction to $\frac{r_2}{r_3}$ w.r.t. x to get $f, h \in \mathbf{K}$ such that

$$r = \partial_x f + h. \tag{5.13}$$

If h is nonzero, the algorithm terminates; no such p and q exist.

I3. [Partial fraction.] Compute the square-free partial fraction decomposition of f w.r.t. y over $\mathbf{C}(x)$. If the decomposition is

$$\sum_i m_i \frac{\partial_y q_i}{q_i} - \sum_j n_j \frac{\partial_y p_j}{p_j} + g \tag{5.14}$$

where $p_i, q_j \in \mathbf{C}[x, y]$, $m_i, n_j \in \mathbb{Z}^+ \cup \{0\}$, and $g \in \mathbf{C}(y)$, set $p \leftarrow \prod_j p_j^{n_j}$, $q \leftarrow \prod_i q_i^{m_i}$. Otherwise, no such p and q exist. □

Step I1 is clear. If $h \neq 0$, then $\int r \,\mathrm{d}x$ is not rational by Hermite's reduction (Geddes *et al.*, 1992; Bronstein, 1997). Thus, (5.12) has no rational solution, and such p and q do not exist by Lemma 5.2. Suppose now $h = 0$. Equations (5.12) and (5.13) imply that

$$\frac{\partial_y z}{z} = f + w \tag{5.15}$$

where w is an x-constant. Assume that the square-free partial fraction decomposition of f w.r.t. y is

$$f = \underbrace{\sum_j n_j \frac{\partial_y q_j}{q_j} - \sum_i m_i \frac{\partial_y p_i}{p_i}}_{G} + \underbrace{\sum_k \frac{s_k}{t_k} + r}_{g}, \tag{5.16}$$

where $p_i, q_i, s_k, t_k, r \in \mathbf{C}[x, y]$, p_i, q_i, t_k are square-free over $\mathbf{C}(x)[y]$ and relatively prime to each other. If z is rational, then the partial decomposition of $\partial_y z/z$ is of form G by the proof of Lemma 4.3, so that g must be an x-constant because of (5.15), (5.16) and the uniqueness of partial fraction decomposition. Suppose now that g does belong to $\mathbf{C}(y)$. We compute

$$\begin{aligned}
\partial_y\left(a + \frac{\partial_x p}{p}\right) - \partial_x\left(b + \frac{\partial_y q}{q}\right) &= r + \partial_y\left(\frac{\partial_x p}{p}\right) - \partial_x\left(\frac{\partial_y q}{q}\right) \\
&\overset{(5.11)}{=} r + \partial_x\left(\frac{\partial_y p}{p} - \frac{\partial_y q}{q}\right) \\
&\overset{(5.13)}{=} \partial_x\left(f + \frac{\partial_y p}{p} - \frac{\partial_y q}{q}\right) \overset{(5.14)}{=} \partial_x g = 0.
\end{aligned}$$

IntegrablePair then returns p and q, as desired.

EXAMPLE 5.1. Given

$$a = \frac{y^2x^3 + x^3 + xy^2 + y - x^2y - x}{yx^3 + y^2x^4 - x^5y - x^4} \qquad \text{and} \qquad b = \frac{1}{x-y},$$

IntegrablePair proceeds as follows.

I1. $r = 1/(1 + xy)^2$.
I2. $f = -1/(xy^2 + y)$, $h = 0$.
I3. $f = x/(1 + xy) - 1/y$, $p = 1$, $q = 1 + xy$.

Hence, $\left(a,\, b + \frac{x}{1+xy}\right)$ is integrable.

EXAMPLE 5.2. Apply **IntegrablePair** to

$$a = \frac{y^3 + xy^2 + 2y - x^2y - x}{xy^3 + y^2 - x^2y^2 - xy} \qquad \text{and} \qquad b = \frac{1}{x-y}.$$

We find that

$$r = -\frac{x^2y^2 + 2xy + 1 - y^2}{x^2y^4 + 2xy^3 + x^2y^4 + y^2} \qquad \text{and} \qquad f = -\frac{x^2y + x + y}{y^2(xy+1)}.$$

In step I3 f decomposes into

$$\frac{x}{1+xy} - \frac{1}{y} - \frac{x}{y^2}.$$

Since $g = -x/y^2$ is not in $\mathbf{C}(y)$, no such p and q exist.

EXAMPLE 5.3. Apply **IntegrablePair** to $\left(\frac{-1}{(x-\sqrt{2}y)^2}, \frac{-\sqrt{2}}{(x+\sqrt{2}y)^2}\right)$. We get

I1. $r = \frac{-2\sqrt{2}}{(x-\sqrt{2}y)^3} - \frac{2\sqrt{2}}{(x+\sqrt{2}y)^3}$.
I2. $f = \frac{\sqrt{2}}{(x-\sqrt{2}y)^2} + \frac{\sqrt{2}}{(x+\sqrt{2}y)^2}$.
I3. $f = \frac{\sqrt{2}}{(x-\sqrt{2}y)^2} + \frac{\sqrt{2}}{(x+\sqrt{2}y)^2}$.

Since $g = f$ is not in $\mathbf{C}(y)$, the pair is not integrable. In the same vein, one can show that $\left(\frac{-1}{(x+\sqrt{2}y)^2}, \frac{\sqrt{2}}{(x-\sqrt{2}y)^2}\right)$ is not integrable.

In what follows, by "given $X_A^a \times Y_B^b$", we mean that we are given $a, b \in \mathbf{K}$, a $\mathbf{C}(y)$-linearly independent set A, and a $\mathbf{C}(x)$-linearly independent set B.

The set $X_A^a \times Y_B^b$ contains no rational solutions of $\mathcal{R}$ if **IntegrablePair** confirms that no $p, q \in \mathbf{C}[x, y]$ are such that $(a + \partial_x p/p,\, b + \partial_y q/q)$ is integrable, because a rational solution of $\mathcal{R}$ must be integrable. Otherwise, we construct a candidate described below.

Algorithm Candidate (*Find a solution candidate*). Given $X_A^a \times Y_B^b$, the algorithm finds $X_F^f \times Y_G^g$ such that (f, g) is an integrable pair and $X_F^f \times Y_G^g = X_A^a \times Y_B^b$, or determines that $X_A^a \times Y_B^b$ contains no integrable pairs.

C1. [Construct f and g.] If **IntegrablePair**(a, b) returns $p, q \in \mathbf{C}[x, y]$, set

$$f \leftarrow a - \frac{\partial_x q}{q}, \qquad g \leftarrow b - \frac{\partial_y p}{p}.$$

Otherwise, the algorithm terminates; $X_A^a \times Y_B^b$ contains no integrable pairs.
C2. [Construct F and G.] Set $F \leftarrow \{qh \mid h \in A\}$, $G \leftarrow \{ph \mid h \in B\}$. $\square$

In Step C1 we set the integrable pair (f, g) to be $(a - \partial_x q/q,\, b - \partial_y p/p)$ instead of $(a + \partial_x p/p,\, b + \partial_y q/q)$, because the former construction makes step C2 simpler. Equation (5.11) implies that (f, g) is integrable if $(a + \partial_x p/p,\, b + \partial_y q/q)$ is. To see $X_F^f \times Y_G^g = X_A^a \times Y_B^b$, we let $\alpha = \sum_{s \in A} c_s s$, where the c_s's are in $\mathbf{C}[y]$, not all zero. Since $a + \partial_x \alpha/\alpha = f + \partial_x(q\alpha)/q\alpha$, $X_F^f = X_A^a$. Likewise, $Y_G^g = Y_B^b$. The algorithm **Candidate** is correct.

EXAMPLE 5.4. Let a and b be the same as those in Example 5.1. Let $A = \{1, x\}$ and $B = \{1\}$. **IntegrablePair**(a, b) returns $p = 1$ and $q = 1 + xy$. **Candidate** then returns $f = a - y/(1 + xy)$, $g = b$, $F = \{1 + xy,\ x(1 + xy)\}$ and $G = B$.

Applying the above algorithm to each member of $\mathbf{X} \times \mathbf{Y}$, we obtain a set of disjoint candidates

$$\mathcal{C} = \{X_{F_1}^{f_1} \times Y_{G_1}^{g_1}, \ldots, X_{F_k}^{f_k} \times Y_{G_k}^{g_k}\}$$

such that $\mathbf{S}$ is contained in $(X_{F_1}^{f_1} \times Y_{G_1}^{g_1}) \bigcup \cdots \bigcup (X_{F_k}^{f_k} \times Y_{G_k}^{g_k})$, and a rational solution of $\mathcal{R}$ can belong to only one member in $\mathcal{C}$.

LEMMA 5.3. *Let $X_F^f \times Y_G^g$ be one of the sets in $\mathcal{C}$. Let*

$$e_x = \max_{p \in F} \deg_x p \qquad \text{and} \qquad e_y = \max_{q \in G} \deg_y q.$$

If an integrable pair (a, b) belongs to $X_F^f \times Y_G^g$, then there exists a polynomial $h \in \mathbf{C}[x, y]$ with $\deg_x h \le e_x$ and $\deg_y h \le e_y$ such that

$$(a, b) = \left(f + \frac{\partial_x h}{h},\ g + \frac{\partial_y h}{h}\right). \tag{5.17}$$

PROOF. Let $(a, b) = (f + \partial_x s/s,\ g + \partial_y t/t)$, where s and t are, respectively, $\mathbf{C}(y)$- and $\mathbf{C}(x)$-linear combinations of elements in F and G. Since (a, b) and (f, g) are integrable pairs, so is $(\partial_x s/s, \partial_y t/t)$. The function

$$h = \exp\left(\int \frac{\partial_x s}{s}\, \mathrm{d}x + \frac{\partial_y t}{t}\, \mathrm{d}y\right)$$

is well defined and has the property $\partial_x h/h = \partial_x s/s$ and $\partial_y h/h = \partial_y t/t$. It follows that

$$s = c_1 h \qquad \text{and} \qquad t = c_2 h \tag{5.18}$$

for some x-constant c_1 and y-constant c_2. Consequently, $sc_2 = tc_1$. Let α be an element of $\mathbf{C}$ such that $t(\alpha, y) \neq 0$. Then $c_1 = s(\alpha, y)b(\alpha)/t(\alpha, y)$, which implies that c_1 belongs to $\mathbf{C}(y)$, and, therefore, h is in $\mathbf{C}(y)[x]$ by (5.18). Likewise, h is in $\mathbf{C}(x)[y]$. Hence, h is in $\mathbf{C}[x, y]$. Accordingly, $\deg_x h = \deg_x s \le e_x$ and $\deg_y h = \deg_y t \le e_y$ by (5.18). □

Suppose that $(a, b) \in \mathbf{S}$ is contained in $X_F^f \times Y_G^g$. By Lemma 5.3 there exists $h \in \mathbf{C}[x, y]$ such that (5.17) holds. Moreover, respective degree bounds for h in x and y are known. The next algorithm **PolynomialPart** determines h.

Algorithm PolynomialPart (*Find polynomial part*). Given $\mathcal{R}$ and a candidate $X_F^f \times Y_G^g$, the algorithm computes a $\mathbf{C}$-linearly independent set H such that $S_H^{(f,g)}$ is equal to the subset of $\mathbf{S}$ whose elements are equivalent to (f, g). If H is empty, then such rational solutions do not exist.

P1. [Bound degrees.] Set $e_x \leftarrow \max_{p \in F} \deg_x p$, $e_y \leftarrow \max_{q \in G} \deg_y q$.

P2. [$e_x = e_y = 0$.] If both e_x and e_y are zero, check whether (f, g) satisfies all the equations in $\mathcal{R}$; if the answer is affirmative, set $H \leftarrow \{1\}$, otherwise, set $H \leftarrow \emptyset$; the algorithm terminates.

P3. [Form a linear algebraic system.] Set $h \leftarrow \sum_{i=0}^{e_x} \sum_{j=0}^{e_y} c_{ij} x^i y^j$, where the c_{ij} are unspecified constants. Substitute $a + \partial_x h/h$ and $b + \partial_y h/h$ for u and v in each equation of $\mathcal{R}$, respectively. Set L be the result, which is a linear homogeneous algebraic system in the c_{ij}'s.

P4. [Compute H.] Calculate a basis B for the solution space of L. If B consists of only zero vector, then set $H \leftarrow \emptyset$. Otherwise, set H to be the set consisting of polynomials corresponding to vectors of B. $\square$

In Step P3 substituting $u \leftarrow a + \partial_x h/h$, $v \leftarrow b + \partial_y h/h$ into $\mathcal{R}$ is equivalent to substituting $z \leftarrow h \exp(f\,\mathrm{d}x + g\,\mathrm{d}y)$ into $\mathcal{L}$; the latter yields a linear homogeneous system in the unspecified constants c_{ij}'s, because $\exp(f\,\mathrm{d}x + g\,\mathrm{d}y)$ is a nonzero overall factor. Thus, L obtained in step P3 is a linear homogeneous algebraic system. The correctness of **PolynomialPart** then follows from Lemma 5.3.

Example 5.5. Determine the rational solutions of

$$\mathcal{R} = \left\{ \partial_x u + u^2 - \frac{2}{x} u + \frac{2y^2 - 2y}{x^2} v + \frac{2}{x^2},\ \partial_y u + uv,\ \partial_y v + v^2 + \frac{2}{y-1} v,\ \partial_y u - \partial_x v \right\}$$

in $S_1 = X^0_{\{1,\,x\,x^2\}} \times Y^{\frac{-1}{y-1}}_{\{y-1,\,1\}}$ and $S_2 = X^y_{\{1,\,x\}} \times Y^x_{\{1,\,y\}}$, respectively. Applying **PolynomialPart** to S_1 yields

P1. $e_x = 2$ and $e_y = 1$.
P2. Skipped.
P3. $h = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 x^2 y$, $u \leftarrow \partial_x h/h$, $v \leftarrow -1/(2y) + \partial_y h/h$, $L = \{c_0, c_4 + c_1, c_5 + c_3\}$.
P4. $H = \{y,\, x - xy,\, x^2 - x^2 y\}$.

The rational solutions in $X^0_{\{1,\,x\,x^2\}} \times Y^{\frac{-1}{y-1}}_{\{y-1,\,1\}}$ are

$$\left(\frac{c_2(1-y) + 2c_3 x(1-y)}{c_1 y + c_2(x - xy) + c_3(x^2 - x^2 y)},\ -\frac{1}{y-1} + \frac{c_1 - c_2 x - c_3 x^2}{c_1 y + c_2(x - xy) + c_3(x^2 - x^2 y)} \right)$$

where $c_1, c_2, c_3 \in \mathbf{C}$. Applying **PolynomialPart** to S_2 yields

P1. $e_x = 1$ and $e_y = 1$.
P2. Skipped.
P3. $h = c_0 + c_1 x + c_2 y + c_3 xy$, $u \leftarrow y + \partial_x h/h$, $v \leftarrow x + \partial_y h/h$, $L = \{c_0, c_1, c_2, c_3\}$.
P4. $H = \emptyset$.

Now, we present the main algorithm.

Algorithm RationalSolution (*Find rational solutions of an associated Riccati-like system*). Given an associated Riccati-like system $\mathcal{R}$, the algorithm computes a representation of all rational solutions of $\mathcal{R}$.

R1. [Compute a coherent autoreduced set.] Transform $\mathcal{R}$ to the linear system $\mathcal{L}$ by the substitution $u \leftarrow \partial_x z/z$, $v \leftarrow \partial_y z/z$. Compute a coherent autoreduced set $\mathcal{A}$ such that $[\mathcal{A}] = [\mathcal{L}]$. If $\mathcal{A} = \{1\}$, the algorithm terminates; no solution exists.

R2. [Eliminate.] Use $\mathcal{A}$ and Lemma 2.1 to compute two linear ODEs $L_x(z)$ (w.r.t. x) and $L_y(z)$ (w.r.t. y). Transform $L_x(z)$ and $L_y(z)$ to their associated Riccati ODEs $R_x(u)$ and $R_y(v)$, respectively.

R3. [Find rational solutions of the Riccati ODEs.] Compute respective representations $\mathbf{X}$ and $\mathbf{Y}$ of rational solutions of $R_x(u)$ and $R_y(v)$. If either $\mathbf{X}$ or $\mathbf{Y}$ is empty, the algorithm terminates; no rational solution exists.

R4. [Construct candidates.] Apply **Candidate** to each member of $\mathbf{X} \times \mathbf{Y}$ to get a set of candidates $\mathcal{C} = \{X_{F_1}^{f_1} \times Y_{G_1}^{g_1}, \ldots, X_{F_k}^{f_k} \times Y_{G_k}^{g_k}\}$. If $\mathcal{C}$ is empty, the algorithm terminates; no rational solution exists.

R5. [Compute polynomial parts.] Apply **PolynomialPart** to each member of $\mathcal{C}$ and collect all nonempty results to construct a representation of $\mathbf{S}$. $\square$

A few words need to be said about the correctness of **RationalSolution**. The set $\mathbf{S}$ is the set of rational solutions of the Riccati-like system associated with $\mathcal{A}$ because $[\mathcal{L}] = [\mathcal{A}]$. The set $\mathbf{S}$ is contained in the union of members of $\mathbf{X} \times \mathbf{Y}$ by step R3, and in the union of members in $\mathcal{C}$ by step R4. Hence, the algorithm returns a representation of $\mathbf{S}$ in step R5. Step R1 is a necessary preparation for step R2, because the operations given in Lemma 2.1 are based on a characteristic set (Janet basis) of $[\mathcal{L}]$. If the coefficients of equations in $\mathcal{R}$ belong to $\mathbb{Q}(x, y)$, step R3 may require to introduce a finite algebraic extension of $\mathbb{Q}$. Nevertheless, steps R4 and R4 can be performed.

A few examples illustrate how **RationalSolution** works.

Example 5.6. Find rational solutions of the Riccati-like system

$$\mathcal{R} = \left\{ \partial_x u + u^2 - \frac{2}{x} u + \frac{2y^2 - 2y}{x^2} v + \frac{2}{x^2},\ \partial_y u + uv,\ \partial_y v + v^2 + \frac{2}{y-1} v,\ \partial_y u - \partial_x v \right\}.$$

Apply **RationalSolution** to get

R1. $\mathcal{L} = \mathcal{A} = \left\{ \partial_x^2 z - \frac{2}{x} \partial_x z + \frac{2y^2 - 2y}{x^2} \partial_y z + \frac{2}{x^2} z,\ \partial_x \partial_y z,\ \partial_y^2 z + \frac{2}{y-1} \partial_y z \right\}$.

R2. $L_x(z) = \partial_x^3 z,\ L_y(z) = \partial_y^2 z + 2\partial_y z/(y-1),\ R_x(u) = \partial_x^2 u + 3u\partial_x u + u^3,\ R_y(v) = \partial_y v + v^2 + 2v/(y-1)$.

R3. $\mathbf{X} = \{X_A^a \mid a = 0,\ A = \{1, x, x^2\}\},\ \mathbf{Y} = \{Y_B^b \mid b = -1/(y-1),\ B = \{y-1, 1\}\}$.

R4. $\mathcal{C} = \{X_A^a \times Y_B^b\}$

R5. A representation of $\mathbf{S}$ is $\{S_H^{(a,b)} \mid H = \{y,\ x - xy,\ x^2 - x^2 y\}\}$ (see Example 5.5).

In other words, the hyperexponential solutions of $\mathcal{L}$ are

$$(c_1 y + c_2(x - xy) + c_3(x^2 - x^2 y)) \exp\left(\int 0 \, \mathrm{d}x - \frac{1}{y-1} \, \mathrm{d}y \right),$$

where c_1, c_2, c_3 are arbitrary elements of $\mathbf{C}$.

Example 5.7. Compute rational solutions of the Riccati-like system

$$\mathcal{R} = \left\{ \partial_x^2 u + 3u\partial_x u + u^3 + \frac{6x^2 - 6xy + y^2}{x^2(2x - y)} (\partial_x u + u^2), \right.$$
$$\left. \partial_y^2 v + 3v\partial_y v + v^3 + \frac{2y - 3x}{x(y - x)} (\partial_y v + v^2) + \frac{y - 2x}{x^2(y - x)} v, \partial_x v - \partial_y u \right\}.$$

Steps R1 and R2 yield the first and second equations in $\mathcal{R}$. In step R3 we compute $\mathbf{X} = \{X_{A_1}^{a_1}, X_{A_2}^{a_2}\}$ and $\mathbf{Y} = \{Y_{B_1}^{b_1}, Y_{B_2}^{b_2}\}$, where $a_1 = 0$, $A_1 = \{1, x\}$, $a_2 = y/x^2$, $A_2 = \{1\}$, $b_1 = 0$, $B_1 = \{1\}$, $b_2 = -1/x$, and $B_2 = \{1, y^2\}$. Step R4 finds two candidates $X_{A_1}^{a_1} \times Y_{B_1}^{b_1}$ and $X_{A_2}^{a_2} \times Y_{B_2}^{b_2}$. Step R5 finds a representation of $\mathbf{S}$ as the union of

$$S_{\{1,x\}}^{(0,0)} = \left\{ \left(\frac{c_2}{c_1 + c_2 x}, 0 \right) \mid c_1, c_2 \in \mathbf{C} \right\}$$

and

$$S_{\{1,\, y^2\}}^{(a_2,b_2)} = \left\{ \left(\frac{y}{x^2}, -\frac{1}{x} + \frac{2c_4 y}{c_3 + c_4 y^2} \right) \mid c_3, c_4 \in \mathbf{C} \right\}.$$

The linear system with which $\mathcal{R}$ is associated, is

$$\left\{ \partial_x^3 z + \frac{6x^2 - 6xy + y^2}{x^2(2x - y)} \partial_x^2 z,\ \partial_y^3 z + \frac{2y - 3x}{x(y - x)} \partial_y^2 z + \frac{y - 2x}{x^2(y - x)} \partial_y z \right\}.$$

Its hyperexponential solutions are $c_1 + c_2 x$ and $(c_3 + c_4 y^2) \exp\left(\int \frac{y}{x^2}\, \mathrm{d}x - \frac{1}{x}\, \mathrm{d}y\right)$ where c_1, c_2, c_3, c_4 are constants.

Example 5.8. Compute hyperexponential solutions of the system

$$\{L_1 = \partial_x^4 Z + 2\partial_x^2 Z + Z, \quad L_2 = \partial_y^4 Z + 2\partial_y^2 Z + Z\}.$$

Steps R1 and R2 find two Riccati ODEs $R_1(u)$ and $R_2(v)$ associated with L_1 and L_2, respectively. Step R3 yields

$$\mathbf{X} = \left\{ X_{\{1,x\}}^{\sqrt{-1}}, X_{\{1,x\}}^{-\sqrt{-1}} \right\} \qquad \text{and} \qquad \mathbf{Y} = \left\{ Y_{\{1,y\}}^{\sqrt{-1}}, Y_{\{1,y\}}^{-\sqrt{-1}} \right\}.$$

Step R4 yields four candidates $X_{\{1,x\}}^{\pm\sqrt{-1}} \times Y_{\{1,y\}}^{\pm\sqrt{-1}}$. Step R4 computes a representation consisting of $S_{\{1,x,y\}}^{(\pm\sqrt{-1},\pm\sqrt{-1})}$. Hence, the hyperexponential solutions of $\{L_1, L_2\}$ are

$$(c_1 + c_2 x + c_3 y)\exp(\sqrt{-1}x + \sqrt{-1}y), \qquad (c_4 + c_5 x + c_6 y)\exp(-\sqrt{-1}x + \sqrt{-1}y),$$

$$(c_7 + c_8 x + c_9 y)\exp(\sqrt{-1}x - \sqrt{-1}y), \qquad (c_{10} + c_{11} x + c_{12} y)\exp(-\sqrt{-1}x - \sqrt{-1}y),$$

where the c's are constants.

Example 5.9. Compute the hyperexponential solutions of the system consisting of

$$E_1 = \partial_x^2 Z + \left(\frac{-1}{x} + \frac{2(1 + 2x)}{a} + \frac{8y^2}{a^2} \right) \partial_x Z + \left(\frac{-1}{2y^2 x} + \frac{x}{2y^2 a} + \frac{1}{a^2} \right) Z, \qquad \text{and}$$

$$E_2 = \partial_y^2 Z + \left(\frac{-4y}{x^2 + 2y^2} - \frac{8y}{a} - \frac{8xy}{a^2} \right) \partial_y Z + \left(\frac{-2}{x(x^2 + 2y^2)} - \frac{2}{xa} - \frac{2}{a^2} \right) Z,$$

where $a = x^2 - 2y^2$.

Let $\mathcal{R}$ be the Riccati-like system associated with $\{E_1, E_2\}$. Step R1 yields an autoreduced set consisting of E_1 and $E_3 = 2xy\partial_y Z + (x^2 + 2y^2)\partial_x Z + Z$ w.r.t. the orderly ranking $Z < \partial_x Z < \partial_y Z < \partial_x^2 Z < \partial_x \partial_y Z < \partial_y^2 Z < \cdots$. Step R2 yields E_1 and E_2. Step R3 produces $\mathbf{X}_{\{1\}}^{-b_1}$, $\mathbf{X}_{\{1\}}^{-b_2}$ $\mathbf{Y}_{\{1\}}^{\sqrt{2}b_1}$ and $\mathbf{Y}_{\{1\}}^{-\sqrt{2}b_2}$, where $b_1 = 1/(x - \sqrt{2}y)^2$ and $b_2 = 1/(x + \sqrt{2}y)^2$. Step R4 gives rise to two candidates $\mathbf{X}_{\{1\}}^{-b_1} \times \mathbf{Y}_{\{1\}}^{\sqrt{2}b_1}$ and $\mathbf{X}_{\{1\}}^{-b_2} \times \mathbf{Y}_{\{1\}}^{-\sqrt{2}b_2}$ (see Example 5.3).

Step R5 produces two rational solutions $(-b_1, \sqrt{2}b_1)$ and $(-b_2, -\sqrt{2}b_2)$ of $\mathcal{R}$. Hence, the hyperexponential solutions of $\{E_1, E_2\}$ are $\exp\left(\frac{1}{x\pm\sqrt{2}y}\right)$.

6. Applications

In this section we apply the algorithm **RationalSolution** to finding rational solutions of Lie's system and hyperexponential solutions of linear homogeneous differential systems with finite linear dimension in several unknowns.

Lie's system (1.1) occurred originally in his investigation of certain second-order ODEs that were based on its symmetries (Lie, 1873). It turned out to be almost as ubiquitous as the Riccati ODEs, e.g. in decomposing systems of linear PDEs into smaller components.

To extend the applicability of the algorithm, we consider the following more general orthonomic system:

$$\begin{aligned}\{P_1 &= \partial_x u + a_0 u^2 + a_1 u + a_2 v + a_3, \quad P_2 = \partial_y u + b_0 uv + b_1 u + b_2 v + b_3,\\ P_3 &= \partial_x v + c_0 uv + c_1 u + c_2 v + c_3, \quad P_4 = \partial_y v + d_0 v^2 + d_1 u + d_2 v + d_3\}\end{aligned} \tag{6.19}$$

in $\mathbf{K}\{u, v\}$ w.r.t. the ranking $1 < v < u < \partial_y v < \partial_x v < \partial_y u < \partial_x u < \cdots$. Moreover, we assume that both a_0 and d_0 are nonzero.

The differential remainders of $\Delta(P_1, P_2)$ and $\Delta(P_3, P_4)$ are, respectively,

$$R_{12} = b_0(c_0 - a_0)vu^2 + a_2(b_0 - d_0)v^2 + \text{terms involving } u^2, uv, u, v, 1$$

and

$$R_{34} = c_0(d_0 - b_0)v^2 u + d_1(a_0 - c_0)u^2 + \text{terms involving } v^2, uv, u, v, 1.$$

Observe that R_{12} and R_{34} are in $\mathbf{K}[u, v]$. Hence, if either R_{12} or R_{34} is nonzero, all solutions of (6.19) would be solutions of some polynomials in $\mathbf{K}[u, v]$. We will not consider this degenerating case.

Lemma 6.1. *If* (6.19) *is coherent, either* $a_0 = c_0$ *and* $b_0 = d_0$*, or* $b_0 = c_0 = a_2 = d_1 = 0$.

Proof. If (6.19) is coherent, $R_{12} = R_{34} = 0$ by Lemma 2.4. Hence

$$b_0(c_0 - a_0) = a_2(b_0 - d_0) = c_0(d_0 - b_0) = d_1(a_0 - c_0) = 0.$$

Since $a_0 d_0 \neq 0$ in (6.19), either $a_0 = c_0$ and $b_0 = d_0$, or $b_0 = c_0 = a_2 = d_1 = 0$. □

Lemma 6.1 splits (6.19) into two systems

$$\begin{aligned}\{F_1 &= \partial_x u + a_0 u^2 + a_1 u + a_2 v + a_3, \quad F_2 = \partial_y u + d_0 uv + b_1 u + b_2 v + b_3,\\ F_3 &= \partial_x v + a_0 uv + c_1 u + c_2 v + c_3, \quad F_4 = \partial_y v + d_0 v^2 + d_1 u + d_2 v + d_3\}\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}\{G_1 &= \partial_x u + a_0 u^2 + a_1 u + a_3, \quad G_2 = \partial_y u + b_1 u + b_2 v + b_3,\\ G_3 &= \partial_x v + c_1 u + c_2 v + c_3, \quad G_4 = \partial_y v + d_0 v^2 + d_2 v + d_3\}.\end{aligned} \tag{6.21}$$

Note that Lie's system is a special case of (6.20). Clearly, the coherence of (6.19) implies the coherence of (6.20) and (6.21). We solve (6.20) and (6.21) separately.

THEOREM 6.2. *If* (6.20) *is coherent, then the substitution*

$$u \leftarrow \frac{1}{a_0}U - \frac{1}{3a_0}\left(a_1 + c_2 - \frac{\partial_x(a_0d_0)}{a_0d_0}\right), \qquad v \leftarrow \frac{1}{d_0}V - \frac{1}{3d_0}\left(b_1 + d_2 - \frac{\partial_y(a_0d_0)}{a_0d_0}\right) \tag{6.22}$$

transforms (6.20) *into an associated Riccati-like system (in U and V) of type $\mathcal{R}_3^{(2)}$ in Example* 3.1

PROOF. Normalizing (6.20) by the substitution

$$u \leftarrow \frac{\bar{u}}{a_0}, \qquad v \leftarrow \frac{\bar{v}}{d_0}, \tag{6.23}$$

we transform (6.20) into the coherent system

$$\begin{aligned}\{\bar{F}_1 = \partial_x\bar{u} + \bar{u}^2 + \bar{a}_1\bar{u} + \bar{a}_2\bar{v} + \bar{a}_3, \quad \bar{F}_2 = \partial_y\bar{u} + \bar{u}\bar{v} + \bar{b}_1\bar{u} + \bar{b}_2\bar{v} + \bar{b}_3,\\ \bar{F}_3 = \partial_x\bar{v} + \bar{u}\bar{v} + \bar{c}_1\bar{u} + \bar{c}_2\bar{v} + \bar{c}_3, \quad \bar{F}_4 = \partial_y\bar{v} + \bar{v}^2 + \bar{d}_1\bar{u} + \bar{d}_2\bar{v} + \bar{d}_3\},\end{aligned} \tag{6.24}$$

where $\bar{a}_1, \ldots, \bar{d}_3 \in \mathbf{K}$. Since the differential remainders of $\Delta(\bar{F}_1, \bar{F}_2)$ and $\Delta(\bar{F}_3, \bar{F}_4)$ are, respectively,

$$\begin{aligned}\bar{R}_{12} = (\bar{c}_1 - \bar{b}_1)\bar{u}^2 + (\bar{c}_2 - \bar{b}_2)\bar{u}\bar{v} + \underbrace{(\bar{c}_3 + \bar{b}_2\bar{c}_1 + \partial_y\bar{a}_1 - \partial_x\bar{b}_1 - 2\bar{b}_3 - \bar{a}_2\bar{d}_1)}_{p}\,\bar{u}\\ + \text{ terms involving } \bar{v} \text{ and } 1,\end{aligned}$$

and

$$\begin{aligned}\bar{R}_{34} = (\bar{c}_1 - \bar{b}_1)\bar{u}\bar{v} + (\bar{c}_2 - \bar{b}_2)\bar{v}^2 + \underbrace{(2\bar{c}_3 - \bar{b}_2\bar{c}_1 - \bar{b}_3 + \bar{a}_2\bar{d}_1 - \partial_x\bar{d}_2 + \partial_y\bar{c}_2)}_{q}\,\bar{v}\\ + \text{ terms involving } \bar{u} \text{ and } 1,\end{aligned}$$

we deduce $\bar{b}_1 = \bar{c}_1$, $\bar{b}_2 = \bar{c}_2$ and $p = q = 0$. It follows that

$$p + q = 3\bar{c}_3 + \partial_y(\bar{a}_1 + \bar{c}_2) - 3\bar{b}_3 - \partial_x(\bar{b}_1 + \bar{d}_2) = 0. \tag{6.25}$$

Applying the substitution

$$\bar{u} \leftarrow U - s, \qquad \bar{v} \leftarrow V - t, \qquad \text{where} \quad s = \tfrac{1}{3}(\bar{a}_1 + \bar{c}_2) \qquad \text{and} \qquad t = \tfrac{1}{3}(\bar{b}_1 + \bar{d}_2) \tag{6.26}$$

to (6.24), we get

$$\begin{aligned}\{f_1 = \partial_xU + U^2 + A_1U + A_2V + A_3, \quad f_2 = \partial_yU + UV + B_1U + B_2V + B_3,\\ f_3 = \partial_xV + UV + C_1U + C_2V + C_3, \quad f_4 = \partial_yV + V^2 + D_1U + D_2V + D_3\}\end{aligned} \tag{6.27}$$

where $A_1, \ldots, D_3 \in \mathbf{K}$. Since $\bar{b}_1 = \bar{c}_1$ and $\bar{b}_2 = \bar{c}_2$, $B_1 = C_1$ and $B_2 = C_2$. We compute

$$\begin{aligned}B_3 - C_3 &= (-\partial_ys - \bar{b}_1s - \bar{b}_2t + st + \bar{b}_3) - (-\partial_xt - \bar{c}_1s - \bar{c}_2t + st + \bar{c}_3)\\ &= \bar{b}_3 + \partial_xt - \bar{c}_3 - \partial_ys \quad (\text{since } \bar{b}_1 = \bar{c}_1 \text{ and } \bar{b}_2 = \bar{c}_2)\\ &= \frac{1}{3}(3\bar{b}_3 + \partial_x(\bar{b}_1 + \bar{d}_2) - 3\bar{c}_3 - \partial_y(\bar{a}_1 + \bar{c}_2)) \quad (\text{by } (6.26))\\ &= 0 \quad (\text{by } (6.25)).\end{aligned}$$

Therefore, $f_2 - f_3 = \partial_y U - \partial_x V$. This implies that $\{f_1, f_2, f_2 - f_3, f_4\}$ is the associated Riccati-like system $\mathcal{R}_3^{(2)}$, so is (6.24). Since

$$\bar{a}_1 = a_1 - \partial_x a_0/a_0, \quad \bar{c}_2 = c_2 - \partial_x d_0/d_0, \quad \bar{b}_1 = b_1 - \partial_y a_0/a_0, \quad \bar{d}_2 = d_2 - \partial_x d_0/d_0,$$

substitution (6.22) is the result of the composition of (6.23) and (6.26). □

EXAMPLE 6.1. Consider the coherent Lie's system

$$\begin{aligned}
&\partial_x u + u^2 + \frac{y+x^2}{x(y-x^2)}u + \frac{4y(y+x^2)}{x^2(y-x^2)}v + \frac{6yx^2+x^4+4y^2}{x^2(y^2-2yx^2+x^4)} = 0,\\
&\partial_y u + uv + \frac{1}{y-x^2}u - \frac{2y+x^2}{x(y-x^2)}v - \frac{2(y+x^2)}{x(y^2-2yx^2+x^4)} = 0,\\
&\partial_x v + uv + \frac{1}{y-x^2}u - \frac{2y+x^2}{x(y-x^2)}v + \frac{x^2-2y}{x(y^2-2yx^2+x^4)} = 0,\\
&\partial_y v + v^2 + \frac{4}{y-x^2}v + \frac{2}{y^2-2yx^2+x^4} = 0.
\end{aligned}$$

By (6.22) we apply the transformation

$$u \leftarrow U + \frac{y}{3x(y-x^2)}, \qquad v \leftarrow V - \frac{5}{3(y-x^2)}$$

to get

$$\begin{aligned}
&\partial_x U + U^2 + \frac{5y+3x^2}{3x(y-x^2)}U + \frac{4y(y+x^2)}{x^2(y-x^2)}V + \frac{9x^4+6yx^2-23y^2}{9x^2(y^2-2yx^2+x^4)} = 0,\\
&\partial_y U + UV - \frac{2}{3(y-x^2)}U - \frac{5y+3x^2}{3x(y-x^2)}V + \frac{2(5y-3x^2)}{9x(y^2-2yx^2+x^4)} = 0,\\
&\partial_x V + UV - \frac{2}{3(y-x^2)}U - \frac{5y+3x^2}{3x(y-x^2)}V + \frac{2(5y-3x^2)}{9x(y^2-2yx^2+x^4)} = 0,\\
&\partial_y V + V^2 + \frac{2}{3(y-x^2)}V - \frac{2}{9(y^2-2yx^2+x^4)} = 0,
\end{aligned}$$

which is equivalent to the Riccati-like system $\mathcal{R}_3^{(2)}$ in Example 3.1, because the difference between the second and third equations is $\partial_y U - \partial_x V$. Apply **RationalSolution** to this system to find the rational solutions:

$$U = \frac{-1}{3x} + \frac{2x}{3(y-x^2)} + \frac{2c_2 x}{c_1 y + c_2 x^2}, \qquad V = \frac{-1}{3(y-x^2)} + \frac{c_1}{c_1 y + c_2 x^2}, \qquad c_1, c_2 \in \mathbf{C}.$$

Hence, the rational solutions of the original system are

$$u = \frac{x}{y-x^2} + \frac{2c_2 x}{c_1 y + c_2 x^2}, \qquad v = \frac{-2}{y-x^2} + \frac{c_1}{c_1 y + c_2 x^2}, \qquad c_1, c_2 \in \mathbf{C}.$$

We turn our attention to (6.21).

THEOREM 6.3. *If* (6.21) *is coherent, it decouples into two individual coherent systems*

$$\{\partial_x u + a_0 u^2 + a_1 u + a_3, \partial_y u + b_1 u + b_3\} \quad \{\partial_y v + d_0 v^2 + d_2 v + d_3, \partial_x v + c_2 v + c_3\} \tag{6.28}$$

for u and v, respectively.

PROOF. The differential remainders of $\Delta(G_1, G_2)$ and $\Delta(G_3, G_4)$ are, respectively

$$R_{12} = -2b_2a_0uv \; + \; \text{ terms involving } u^2,\ u,\ v \text{ and } 1$$

and

$$R_{34} = 2c_1d_0uv \; + \; \text{ terms involving } v^2,\ u,\ v \text{ and } 1.$$

Since (6.21) is coherent, $b_2 = c_1 = 0$. The theorem follows. □

The following system also appears frequently in symmetry analysis.

THEOREM 6.4. *Let $a_1, \ldots, b_3 \in \mathbf{K}$ and $a_1b_1 \neq 0$. The first-order Riccati-like system*

$$\{F = \partial_x z + a_1z^2 + a_2z + a_3,\ G = \partial_y z + b_1z^2 + b_2z + b_3\} \tag{6.29}$$

is coherent if and only if its general solution depends on a single constant. If (6.29) has a rational solution, one of the following alternatives applies.

(1) The general solution is rational and has the form

$$\frac{1}{a_1}\frac{\partial_x r}{r+c} + p = \frac{1}{b_1}\frac{\partial_y r}{r+c} + p$$

with $p, r \in \mathbf{K}$ and $c \in \mathbf{C} \cup \{\infty\}$.

(2) There are at most two special rational solutions not involving unspecified constants.

PROOF. Since the differential remainder of $\Delta(F, G)$ is an algebraic polynomial in $\mathbf{K}[z]$ of degree no greater than 2, (6.29) has at most two solutions if it is not coherent.

Assume that (6.29) is coherent. We show that it is the system $\mathcal{R}_2$ in disguise (see Example 3.1). The substitution

$$z \leftarrow \frac{1}{a_1}u - \frac{1}{2a_1}\left(a_2 - \frac{\partial_x a_1}{a_1}\right) \tag{6.30}$$

transforms (6.29) into the coherent system

$$\{f = \partial_x u + u^2 + A_3,\ g = \partial_y u + B_1u^2 + B_2u + B_3\}, \tag{6.31}$$

where $A_3, B_1, B_2, B_3 \in \mathbf{K}$. Since the differential remainder of $\Delta(f, g)$ is zero,

$$B_2 = -\partial_x B_1, \qquad \partial_x B_2 = 2A_3B_1 - 2B_3,$$

which implies

$$g - B_1 f = \partial_y u - B_1\partial_x u - (\partial_x B_1)u + \frac{1}{2}\partial_x^2 B_1 = \partial_y u - \partial_x\left(B_1u - \frac{1}{2}\partial_x B_1\right).$$

Hence, (6.31) is equivalent to the system

$$\left\{\partial_x u + u^2 + A_3,\ \partial_y u - \partial_x\left(B_1u - \frac{1}{2}\partial_x B_1\right)\right\}.$$

Set $v = \left(B_1u - \frac{1}{2}\partial_x B_1\right)$. The above system becomes

$$\left\{\partial_x u + u^2 + A_3,\ v - B_1u + \frac{1}{2}\partial_x B_1,\ \partial_y u - \partial_x v\right\} \tag{6.32}$$

which is of type $\mathcal{R}_2$ in Example (3.1). It follows from (6.30) and the definition of v that (u, v) is a solution of (6.32) if and only if z given in (6.30) is a solution of (6.29). Since (6.32) is associated with a coherent linear system with linear dimension two, the general solution of (6.32) can be written as

$$(u, v) = \left(\frac{c_1 \partial_x s_1 + c_2 \partial_x s_2}{c_1 s_1 + c_2 s_2}, \frac{c_1 \partial_y s_1 + c_2 \partial_y s_2}{c_1 s_1 + c_2 s_2} \right),$$

where s_1 and s_2 are in some differential extension $\mathbf{F}$ of $\mathbf{K}$, linearly independent over the constant field of $\mathbf{F}$, and c_1 and c_2 are in the same constant field. Therefore, (6.30) implies that the general solution of (6.29) can be written as

$$z = \frac{1}{a_1} \frac{c_1 \partial_x s_1 + c_2 \partial_x s_2}{c_1 s_1 + c_2 s_2} - \frac{1}{2a_1} \left(a_2 - \frac{\partial_x a_1}{a_1} \right).$$

Setting $c = c_1/c_2$, we prove that the general solution of (6.29) depends on one constant.

We now consider the rational solutions of (6.29). By (6.30) and the second equation in (6.31), z is a rational solution of (6.29) if and only if (u, v) is a rational solution of (6.32). By Theorem 3.2 (6.32) has either infinitely many rational solutions or at most two inequivalent rational solutions. The former case corresponds to the first alternative, and the latter to the second. Assume that (6.32) has infinitely many rational solutions. Then, by Theorem 3.2,

$$(u, v) = \left(\frac{c_1 \partial_x h_1 + c_2 \partial_x h_2}{c_1 h_1 + c_2 h_2} + a, \frac{c_1 \partial_y h_1 + c_2 \partial_y h_2}{c_1 h_1 + c_2 h_2} + b \right)$$

where $h_1, h_2, a, b \in \mathbf{K}$ and $c_1, c_2 \in \mathbf{C}$. Setting $r = h_2/h_1$, $c = c_1/c_2$, $f = \partial_x h_1/h_1 + a$ and $g = \partial_y h_1/h_1 + b$, we get

$$(u, v) = \left(\frac{\partial_x r}{c + r} + f, \frac{\partial_y r}{c + r} + g \right).$$

Transformation (6.30) implies that

$$z = \frac{1}{a_1} \frac{\partial_x r}{c + r} + p$$

for some $p \in \mathbf{K}$. Substituting $\partial_y r/(c + r) + g$ for v in the second equation of (6.32) yields

$$z = \frac{1}{a_1 B_1} \frac{\partial_y r}{c + r} + q$$

for some $q \in \mathbf{K}$. Hence, $p = q$ because we may set $c = \infty$, i.e. $c_2 = 0$. The theorem is then proved by noting that $a_1 B_1 = b_1$. □

According to the proof of Theorem 6.4, the rational solutions of (6.29) can be computed by the algorithm **RationalSolution**. We may also proceed as follows. Compute the rational solutions of F. If there are only a finite number of solutions, we need only check if they satisfy G. Otherwise, the rational solutions of F are given by

$$\frac{\partial_x r}{C + r} + f$$

where $r, f \in \mathbf{K}$ and $C \in \mathbf{C}(y)$. Substituting this expression for z in G yields

$$H = \partial_y C + B_1 C^2 + B_2 C + B_3,$$

for some $B_1, B_2, B_3 \in \mathbf{K}$. Collecting coefficients of H w.r.t. the powers of x yields a system in $\mathbf{C}(y)\{C\}$ consisting possibly of first-order Riccati ODEs, first-order linear ODEs and algebraic equations, whose rational solutions can be easily found.

Example 6.2. Compute the rational solutions of

$$\{\partial_x z + z^2,\ \partial_y z + (1 - x^2 - 2xy - y^2)z^2 + (2x + 2y)z - 1 = 0\}.$$

The rational solutions of the first equation are $1/(C(y) + x)$, where $C(y)$ is an x-constant. Substituting the expression into the second equation yields

$$\partial_y C(y) + C(y)^2 - 2yC(y) + y^2 - 1 = 0,$$

so that $C(y) = y + 1/(y + c)$, where c is a constant. This system has the rational solutions

$$z = \frac{1}{x + y + \frac{1}{y+c}} = \frac{\partial_x\left(\frac{xy+y^2+1}{x+y}\right)}{c + \frac{xy+y^2+1}{x+y}} + \frac{1}{x+y}.$$

At last, the following problem is considered. Let $\mathbf{D}$ be the differential polynomial ring $\mathbf{K}\{z_1, \ldots, z_n\}$. Given a linear system $\mathcal{L} \subset \mathbf{D}$ with finite linear dimension, find all hyperexponential solutions of $\mathcal{L}$. By general elimination procedures (Janet, 1920; Wu, 1989; Kandri-Rody and Weispfenning, 1990; Boulier *et al.*, 1995; Schwarz, 1998a; Li and Wang, 1999) we compute a linear characteristic set $\mathcal{L}_i$ for $[\mathcal{L}] \cap \mathbf{K}\{z_i\}$, for $i = 1, \ldots, n$. Since each $[\mathcal{L}_i]$ is also of finite linear dimension, the algorithm **RationalSolution** computes a representation $\mathbf{S}_i$ of the rational solutions of the Riccati-like system associated with $\mathcal{L}_i$. Assume that

$$\mathbf{S}_i = \{S_{H_{ij}}^{(f_{ij}, g_{ij})} \mid j = 1, \ldots, m_i\}.$$

The hyperexponential solutions of $\mathcal{L}_i$ are then expressed as $E_i = \bigcup_{j=1}^{m_i} V_{ij}$, where

$$V_{ij} = \left\{ \left(\sum_{h \in H_{ij}} c_h h \right) \exp\left(\int f_{ij}\, \mathrm{d}x + g_{ij}\, \mathrm{d}y \right) \mid c_h \in \mathbf{C} \right\}.$$

The problem is thus reduced to computing hyperexponential solutions of $\mathcal{L}$ contained in $V_{1j_1} \times \cdots \times V_{nj_n}$ for $1 \le j_1 \le m_1, \ldots, 1 \le j_n \le m_n$. Substituting

$$\left(\sum_{h \in H_{ij_i}} c_h h \right) \exp\left(\int f_{ij_i}\, \mathrm{d}x + g_{ij_i}\, \mathrm{d}y \right)$$

for z_i in $\mathcal{L}$ yields a linear algebraic system $\mathcal{A}$ in the unspecified constants c's. Notice that the coefficients of $\mathcal{A}$ may be hyperexponential. Nevertheless, the constant solutions of $\mathcal{A}$ gives us the hyperexponential solutions of $\mathcal{L}$ in $V_{1j_1} \times \cdots \times V_{nj_n}$.

Example 6.3. Consider the system $\mathcal{L}$

$$\begin{aligned} \{x^2\partial_x z_2 - xy\partial_x z_1 + yz_1, \qquad & x^2\partial_x^2 z_1 - x\partial_x z_1 + z_1, \\ y\partial_y z_1 - x\partial_x z_1 + z_1, \qquad & xy\partial_y z_2 + xy\partial_x z_1 - xz_2 + yz_1\}. \end{aligned}$$

By elimination we get

$$\mathcal{L}_1 = \{y\partial_y z_1 - x\partial_x z_1 + z_1,\ x^2\partial_x^2 z_1 - x\partial_x z_1 + z_1\}$$

and

$$\mathcal{L}_2 = \{y\partial_y z_2 - xy^2\partial_x z_2 - z_2,\ x^2\partial_x^3 z_2 + 3x\partial_x^2 z_2 + \partial_x z_2\}.$$

By the algorithm **RationalSolution** we find that respective hyperexponential solutions of $\mathcal{L}_1$ and $\mathcal{L}_2$ are $c_1 x$ and $c_2 y$, where $c_1, c_2 \in \mathbf{C}$. Substituting $c_1 x$ for z_1 and $c_2 y$ for z_2 into $\mathcal{L}$ yields the linear system $\{c_1 = 0\}$. Hence, the hyperexponential solutions of $\mathcal{L}$ are $(0, c_2 y)$, where $c_2 \in \mathbf{C}$.

Acknowledgements

We thank the anonymous referees for many helpful comments and careful work.

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Originally Received 7 December 1999
Accepted 22 March 2001