# Cartan MASAs in von Neumann Algebras are Norming and a New Proof of Mercer's Theorem

**Jan Cameron, David R. Pitts and Vrej Zarikian**

Abstract. In this note we observe that a result of Sinclair and Smith together with the Feldman-Moore description of a von Neumann algebra with a Cartan MASA shows that Cartan MASAs are norming. We also use the outline in [8, Remark 2.17] to give a new proof of Mercer's Theorem [6, Corollary 4.3].

In a 1991 paper, Mercer proves the following result.

**Theorem 1** ([6, Corollary 4.3])**.** *For $i = 1, 2$, let $\mathcal{M}_i$ be a von Neumann algebra with separable predual and let $\mathcal{D}_i \subseteq \mathcal{M}_i$ be a Cartan MASA. Suppose $\mathcal{A}_i$ is a $\sigma$-weakly closed subalgebra of $\mathcal{M}_i$ which contains $\mathcal{D}_i$ and which generates $\mathcal{M}_i$ as a von Neumann algebra.*

*If $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric algebra isomorphism such that $\theta(\mathcal{D}_1) = \mathcal{D}_2$, then $\theta$ extends to a von Neumann algebra isomorphism $\overline{\theta} : \mathcal{M}_1 \to \mathcal{M}_2$. Furthermore, if $\mathcal{M}_i$ is identified with its Feldman-Moore representation, so $\mathcal{M}_i \subseteq \mathcal{B}(L^2(R_i))$, then $\overline{\theta}$ may be taken to be a spatial isomorphism.*

Our purpose in this note is to give a new proof of this result based on the conceptual outline for $C^*$-diagonals in [8]. Our proof seems more transparent, and avoids the technicalities involving chains and loops found in Mercer's original proof. Furthermore, the outline can be adapted to other contexts as well, see [9].

We begin with an observation, of independent interest, regarding norming algebras. While we do not directly use the full strength of this result, we certainly use the fact that the MASA $\mathcal{D}$, in the notation below, norms a certain $C^*$-subalgebra $\mathcal{C}$ of the von Neumann algebra $\mathcal{M}$. Note that the result below is implicit in [11] for the case when $\mathcal{M}$ is a finite von Neumann algebra.

**Proposition 2.** *Let $\mathcal{M}$ be a von Neumann algebra with separable predual, and suppose that $\mathcal{D} \subseteq \mathcal{M}$ is a Cartan MASA. Then $\mathcal{D}$ is norming for $\mathcal{M}$ in the sense of Pop-Sinclair-Smith* [10].

**Proof.** We will use the notation found in [3]. By [3, Theorem 1], there exists a countable, standard equivalence relation $R$ on a finite measure space $(X, \mathcal{B}, \mu)$, a cocycle $\sigma \in H^2(R, \mathbb{T})$, and an isomorphism of $\mathcal{M}$ onto $\mathbf{M}(R, \sigma)$ which carries $\mathcal{D}$ onto the diagonal subalgebra $\mathbf{A}(R, \sigma)$ of $\mathbf{M}(R, \sigma)$. We may therefore assume that $\mathcal{M} = \mathbf{M}(R, \sigma)$ and that $\mathcal{D} = \mathbf{A}(R, \sigma)$. With this identification, $\mathcal{M}$ acts on the separable Hilbert space $L^2(R, \nu)$, where $\nu$ is the right counting measure associated with $\mu$. By [3, Proposition 2.9], $J\mathcal{D}J$ is an abelian subalgebra of $\mathcal{M}'$ and $(J\mathcal{D}J \vee \mathcal{D})''$ is a MASA in $\mathcal{B}(L^2(R, \nu))$. Therefore, the $C^*$-subalgebra of $\mathcal{B}(L^2(R, \nu))$ generated by $J\mathcal{D}J$ and $\mathcal{D}$ has a cyclic vector. An application of [11, Proposition 4.1] completes the proof. $\square$

Let $(\mathcal{M}, \mathcal{D})$ be a pair consisting of a von Neumann algebra $\mathcal{M}$ with separable predual and a Cartan MASA $\mathcal{D}$ in $\mathcal{M}$. A *Cartan bimodule algebra* is a $\sigma$-weakly closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ which contains $\mathcal{D}$ and which generates $\mathcal{M}$ as a von Neumann algebra. As in the proof of Proposition 2, we may identify $\mathcal{M}$ with $\mathbf{M}(R, \sigma)$ and $\mathcal{D}$ with $\mathbf{A}(R, \sigma)$. By the Spectral Theorem for Bimodules [7, Theorem 2.5], there exists an essentially unique Borel set $\Gamma(\mathcal{A}) \subseteq R$ such that

$$\mathcal{A} = \{a \in \mathcal{M} : a(x, y) = 0 \text{ for all } (x, y) \notin \Gamma(\mathcal{A})\}. \tag{1}$$

In fact, $\Gamma(\mathcal{A})$ is a reflexive and transitive relation on $X$, and $R$ equals the equivalence relation generated by $\Gamma(\mathcal{A})$ [7, Theorem 3.2].

2000 *Mathematics Subject Classification.* 47L30, 46L10, 46L07.
*Key words and phrases.* Norming algebra, Cartan MASA, $C^*$-diagonal.
Zarikian was partially supported by Nebraska IMMERSE.

It follows from (1) that $\mathcal{A}$ contains an abundance of (groupoid) normalizers of $\mathcal{D}$. Recall that $v \in \mathcal{M}$ is a *normalizer* of $\mathcal{D}$ if $v\mathcal{D}v^*, v^*\mathcal{D}v \subseteq \mathcal{D}$. If, in addition, $v$ is a partial isometry, then we say that $v$ is a *groupoid normalizer* of $\mathcal{D}$, and write $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Now suppose $\phi$ is a Borel isomorphism between Borel sets $\mathrm{dom}(\phi) \subseteq X$ and $\mathrm{range}(\phi) \subseteq X$ whose graph $\Gamma(\phi) = \{(\phi(x), x) : x \in \mathrm{dom}(\phi)\}$ is contained in $R$. Such a $\phi$ will be called a *partial $R$-isomorphism*. Then $F(\phi) := \chi_{\Gamma(\phi)} \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Indeed,

$$F(\phi)dF(\phi)^* = (d \circ \phi^{-1})\chi_{\mathrm{range}(\phi)} \text{ and } F(\phi)^*dF(\phi) = (d \circ \phi)\chi_{\mathrm{dom}(\phi)}$$

for all $d \in \mathcal{D}$. In general $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ may be written $v = F(\phi)u$, where $\phi$ is a partial $R$-isomorphism and $u \in \mathcal{D}$ is unitary [5, Proposition 2.2]. It follows from this discussion that

$$\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A} = \{F(\phi)u : \phi \text{ is a partial } R\text{-isomorphism such that } \Gamma(\phi) \subseteq \Gamma(\mathcal{A}), \text{ and } u \in \mathcal{D} \text{ is unitary}\}.$$

By [7, Corollary 2.7], we have that

$$\mathcal{A} = \overline{\mathrm{span}}^{\sigma}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A}), \tag{2}$$

justifying the opening statement of this paragraph.

We say that a map $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ between Cartan bimodule algebras is a *Cartan bimodule algebra isomorphism* if it is an isometric algebra isomorphism such that $\theta(\mathcal{D}_1) = \mathcal{D}_2$. With this terminology, Theorem 1 says that a Cartan bimodule algebra isomorphism extends to a von Neumann algebra isomorphism of the generated von Neumann algebras. The starting point for our proof of Theorem 1 is Mercer's structure theorem for Cartan bimodule algebra isomorphisms:

**Theorem 3** ([6], Propositions 2.1 and 2.2)**.** *Let $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ be a Cartan bimodule algebra isomorphism. Then there exists a Borel isomorphism $\tau : X_1 \to X_2$ and a Borel function $c : \Gamma(\mathcal{A}_2) \to \mathbb{T}$ such that*

i) *$\mu_2$ and $\mu_1 \circ \tau^{-1}$ are mutually absolutely continuous.*
ii) *$(\tau \times \tau)(R_1) = R_2$ and $(\tau \times \tau)(\Gamma(\mathcal{A}_1)) = \Gamma(\mathcal{A}_2)$.*
iii) *$\theta(a)(x, y) = c(x, y)a(\tau^{-1}(x), \tau^{-1}(y))$ for all $a \in \mathcal{A}_1$ and all $(x, y) \in \Gamma(\mathcal{A}_2)$.*

*Furthermore,*

iv) *$c(x, x) = 1$ for all $x \in X_2$.*
v) *If $(x, y), (y, x) \in \Gamma(\mathcal{A}_2)$, then $c(y, x) = \overline{c(x, y)}$.*
vi) *If $(x, y), (y, z) \in \Gamma(\mathcal{A}_2)$, then*

$$c(x, z)\sigma_1(\tau^{-1}(x), \tau^{-1}(y), \tau^{-1}(z)) = c(x, y)c(y, z)\sigma_2(x, y, z).$$

*In particular,*

vii) *If $d \in \mathcal{D}_1$, then $\theta(d) = d \circ \tau^{-1}$.*
viii) *If $\phi$ is a partial $R_1$-isomorphism such that $\Gamma(\phi) \subseteq \Gamma(\mathcal{A}_1)$, then there exists a unitary $u \in \mathcal{D}_2$ such that $\theta(F(\phi)) = F(\tau \circ \phi \circ \tau^{-1})u$.*

In order to prove Theorem 1, Mercer extends the function $c$ in Theorem 3 above from $\Gamma(\mathcal{A}_2)$ to all of $R_2$, such that (v) and (vi) still hold. This requires an intricate analysis of $R_2$ and its finite subequivalence relations. Our proof, on the other hand, extends $\theta$ directly, without extending $c$ first.

Before embarking on our proof, we draw two corollaries of Theorem 3.

**Corollary 4.** *Let $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ be a Cartan bimodule algebra isomorphism. Then $\theta(\mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{A}_1) = \mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2) \cap \mathcal{A}_2$.*

**Proof.** Immediate from Theorem 3 (viii). □

**Corollary 5.** *Let $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ be a Cartan bimodule algebra isomorphism. Then $\theta$ is $\sigma$-weakly continuous.*

**Proof (sketch).** By the Krein-Smulian Theorem, it suffices to show that if $\{a_i : i \in I\} \subseteq \mathcal{A}_1$ is a bounded net and $a_i \to a$ WOT, then

$$\langle \theta(a_i)\chi_{\Gamma(\phi)}, \chi_{\Gamma(\psi)} \rangle \to \langle \theta(a)\chi_{\Gamma(\phi)}, \chi_{\Gamma(\psi)} \rangle \tag{3}$$

for all partial $R_2$-isomorphisms $\phi, \psi$. Let $g = \left[\frac{d(\mu_1 \circ \tau^{-1})}{d\mu_2}\right] \in L^1(X_2, \mu_2)$ (Theorem 3 (i)). Straightforward calculations using Theorem 3 (iii) show that (3) is equivalent to

$$(4) \qquad \int_{\mathrm{dom}(\phi') \cap \mathrm{dom}(\psi')} a_i(\psi'(y), \phi'(y)) h(y) d\mu_1(y) \to \int_{\mathrm{dom}(\phi') \cap \mathrm{dom}(\psi')} a(\psi'(y), \phi'(y)) h(y) d\mu_1(y),$$

where $\phi' = \tau^{-1} \circ \phi \circ \tau$, $\psi' = \tau^{-1} \circ \psi \circ \tau$, and

$$h(y) = c(\tau(\psi'(y)), \tau(\phi'(y)))\sigma_2(\tau(\psi'(y)), \tau(\phi'(y)), \tau(y)) g^{-1}(\tau(y))$$

for all $y \in \mathrm{dom}(\phi') \cap \mathrm{dom}(\psi')$. But the assumption $a_i \to a$ WOT forces (4) to hold for any $h \in L^1(X_1, \mu_1)$ and any partial $R_1$-isomorphisms $\phi', \psi'$. $\square$

We now begin our proof in earnest. The first goal is to extend a restriction of $\theta$ to a *-isomorphism $\Theta$ of certain $\sigma$-weakly dense $C^*$-algebras (Corollary 7).

**Proposition 6.** *Let $\mathcal{A}$ be a Cartan bimodule subalgebra of $(\mathcal{M}, \mathcal{D})$. Define $\mathcal{A}^0 = \overline{\mathrm{span}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A})$ (norm closure) and $\mathcal{C} = C^*(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A})$. Then*

i) *$\mathcal{C} = C^*(\mathcal{A}^0)$ and $\mathcal{D} \subseteq \mathcal{A}^0 \subseteq \mathcal{C}$;*
ii) *$\mathcal{C} = \overline{\mathrm{span}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{C})$;*
iii) *$\overline{\mathcal{A}^0}^\sigma = \mathcal{A}$; and*
iv) *$\overline{\mathcal{C}}^\sigma = \mathcal{M}$.*

*In particular, the pair $(\mathcal{C}, \mathcal{D})$ is a $C^*$-diagonal in the sense of Kumjian [4].*

**Proof.** (i) and (ii) are routine. (iii) is just (2) above. (iv) follows from the calculation

$$\overline{\mathcal{C}}^\sigma = \overline{C^*(\mathcal{A}^0)}^\sigma = W^*(\mathcal{A}^0) = W^*(\overline{\mathcal{A}^0}^\sigma) = W^*(\mathcal{A}) = \mathcal{M}.$$

Now (ii) says that $(\mathcal{C}, \mathcal{D})$ is a regular inclusion. Moreover, as $\mathcal{D}$ is a MASA in $\mathcal{M}$, it is a MASA in $\mathcal{C}$. Since $\mathcal{D}$ is injective, the pair $(\mathcal{C}, \mathcal{D})$ is a $C^*$-diagonal [9, Section 2]. $\square$

**Corollary 7.** *Let $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ be a Cartan bimodule algebra isomorphism. Then*

i) *there exists a *-isomorphism $\Theta : \mathcal{C}_1 \to \mathcal{C}_2$ such that $\Theta(x) = \theta(x)$ for all $x \in \mathcal{A}_1^0$ (notation as in Proposition 6);*
ii) *$\Theta(\mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{C}_1) = \mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2) \cap \mathcal{C}_2$; and*
iii) *if $\phi$ is a partial $R_1$-isomorphism such that $F(\phi) \in \mathcal{C}_1$, then there exists a unitary $u \in \mathcal{D}_2$ such that $\Theta(F(\phi)) = F(\tau \circ \phi \circ \tau^{-1})u$.*

**Proof.** By Corollary 4, $\theta(\mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{A}_1) = \mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2) \cap \mathcal{A}_2$. It follows that $\theta(\mathcal{A}_1^0) = \mathcal{A}_2^0$. By Proposition 6, the pair $(\mathcal{C}_i, \mathcal{D}_i)$ is a $C^*$-diagonal and $C^*(\mathcal{A}_i^0) = \mathcal{C}_i$, for $i = 1, 2$. An application of [8, Theorem 2.16] establishes (i).

Since $\Theta$ is a *-isomorphism and $\Theta(\mathcal{D}_1) = \mathcal{D}_2$, (ii) holds.

For (iii), let $\phi$ be a partial $R_1$-isomorphism such that $v = F(\phi) \in \mathcal{C}_1$. Then there exists a partial $R_2$-isomorphism $\psi$ and a unitary $u \in \mathcal{D}_2$ such that $\Theta(v) = F(\psi)u$. We aim to show that $\psi = \phi_\tau$, where $\phi_\tau = \tau \circ \phi \circ \tau^{-1}$. We have that

$$\chi_{\mathrm{dom}(\psi)} = \Theta(v)^*\Theta(v) = \Theta(v^*v) = \chi_{\mathrm{dom}(\phi)} \circ \tau^{-1} = \chi_{\tau(\mathrm{dom}(\phi))} = \chi_{\mathrm{dom}(\phi_\tau)},$$

and so $\mathrm{dom}(\psi) = \mathrm{dom}(\phi_\tau)$. For $d \in \mathcal{D}_1$ we have that

$$\Theta(v^*dv) = v^*dv \circ \tau^{-1} = \left[(d \circ \phi)\chi_{\mathrm{dom}(\phi)}\right] \circ \tau^{-1} = (d \circ \tau^{-1} \circ \phi_\tau)\chi_{\mathrm{dom}(\phi_\tau)}$$

and

$$\Theta(v)^*\Theta(d)\Theta(v) = \Theta(v)^*(d \circ \tau^{-1})\Theta(v) = (d \circ \tau^{-1} \circ \psi)\chi_{\mathrm{dom}(\psi)}.$$

Thus,

$$d(\tau^{-1}(\psi(x))) = d(\tau^{-1}(\phi_\tau(x)))$$

for all $d \in \mathcal{D}_1$ and all $x \in \mathrm{dom}(\psi) = \mathrm{dom}(\phi_\tau)$. Therefore, $\psi = \phi_\tau$. $\square$

As the next step in our proof, we show that the *-isomorphism $\Theta : \mathcal{C}_1 \to \mathcal{C}_2$ of Corollary 3 is spatial by constructing an implementing unitary $U : L^2(R_1) \to L^2(R_2)$. For this, the following definition and technical "disjointification" lemma will be useful.

**Definition 8.** Let $v, w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. We say that $v$ and $w$ are *disjoint* if $\Gamma(\phi) \cap \Gamma(\psi) = \emptyset$, where $\phi$ and $\psi$ are the partial $R$-isomorphisms associated with $v$ and $w$, respectively.

**Remark 9.** Notice that disjoint elements of $\mathcal{GN}(\mathcal{M}, \mathcal{D})$ are orthogonal in $L^2(R, \nu)$. Also, by Corollary 7, if $v, w \in \mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{C}_1$ are disjoint, then so are $\Theta(v), \Theta(w) \in \mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2) \cap \mathcal{C}_2$; indeed, if $\Gamma(\phi) \cap \Gamma(\psi) = \emptyset$, then

$$\Gamma(\tau \circ \phi \circ \tau^{-1}) \cap \Gamma(\tau \circ \psi \circ \tau^{-1}) = (\tau \times \tau)(\Gamma(\phi) \cap \Gamma(\psi)) = \emptyset.$$

**Lemma 10.** *Let $v_1, v_2, ..., v_n \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Then there exist pairwise disjoint $w_1, w_2, ..., w_N \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ ($N \leq 2^n - 1$) such that the following conditions hold:*

i) *For all $1 \leq j \leq N$, $w_j = v_{i(j)} p_j$ for some $1 \leq i(j) \leq n$ and some projection $p_j \in \mathcal{D}$.*

ii) *For all $a_1, a_2, ..., a_n \in \mathcal{D}$ there exist $b_1, b_2, ..., b_N \in \mathcal{D}$ such that $\sum_{i=1}^n v_i a_i = \sum_{j=1}^N w_j b_j$.*

**Proof.** Suppose $v_i = F(\phi_i) u_i$, where $\phi_i$ is a partial $R$-isomorphism and $u_i \in \mathcal{D}$ is unitary. Each $\Gamma(\phi_i)$ is a Borel subset of $R$ [1, Proposition 3.3.1]. An exercise in set theory shows that there exist pairwise disjoint Borel sets $G_1, G_2, ..., G_N$ ($N \leq 2^n - 1$) such that

1) For all $1 \leq j \leq N$, $G_j \subseteq \Gamma(\phi_{i(j)})$ for some $1 \leq i(j) \leq n$.

2) For all $1 \leq i \leq n$, $\Gamma(\phi_i) = \bigsqcup_{G_j \subseteq \Gamma(\phi_i)} G_j$.

For each $1 \leq j \leq N$, set $B_j = \pi_2(G_j)$, a Borel subset of $\mathrm{dom}(\phi_{i(j)})$ [2, p. 291]. Define $\psi_j$ to be the restriction of $\phi_{i(j)}$ to $B_j$. Then $\psi_j$ is a partial $R$-isomorphism such that $\Gamma(\psi_j) = G_j$. Define $p_j = \chi_{B_j}$, a projection in $\mathcal{D}$, and $w_j = v_{i(j)} p_j$, an element of $\mathcal{GN}(\mathcal{M}, \mathcal{D})$. Since

$$w_j = F(\phi_{i(j)}) u_{i(j)} \chi_{B_j} = F(\phi_{i(j)}) \chi_{B_j} u_{i(j)} = F(\psi_j) u_{i(j)},$$

we see that $w_1, w_2, ..., w_N$ are pairwise disjoint. Suppose $a_1, a_2, ..., a_n \in \mathcal{D}$. Then

$$\begin{aligned} \sum_{i=1}^n v_i a_i &= \sum_{i=1}^n F(\phi_i) u_i a_i = \sum_{i=1}^n \left( \sum_{\Gamma(\psi_j) \subseteq \Gamma(\phi_i)} F(\psi_j) \right) u_i a_i = \sum_{j=1}^N F(\psi_j) \left( \sum_{\Gamma(\psi_j) \subseteq \Gamma(\phi_i)} u_i a_i \right) \\ &= \sum_{j=1}^N F(\psi_j) u_{i(j)} \left( u_{i(j)}^* \sum_{\Gamma(\psi_j) \subseteq \Gamma(\phi_i)} u_i a_i \right) = \sum_{j=1}^N w_j b_j. \end{aligned}$$

□

**Remark 11.** If $v_1, v_2, ..., v_n \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{C}$ for some $C^*$-algebra $\mathcal{C}$ containing $\mathcal{D}$, then $w_1, w_2, ..., w_N \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{C}$ also, because of (i).

For $i = 1, 2$, we denote by $f_i \in L^2(R_i, \nu_i)$ the characteristic function of the diagonal of $R_i$. By [3, Proposition 2.5], $f_i$ is cyclic and separating for $\mathcal{M}_i$. Recalling that $\mu_1 \circ \tau^{-1}$ and $\mu_2$ are equivalent measures, define $g = \left[\frac{d(\mu_1 \circ \tau^{-1})}{d\mu_2}\right] \in L^1(X_2, \mu_2)$. We will simultaneously regard $g^{1/2}$ as an element of $L^2(X_2, \mu_2)$ and as an element of $L^2(R_2, \nu_2)$ supported on the diagonal of $R_2$. We claim that $g^{1/2}$ is cyclic for $\mathcal{M}_2$. Indeed, there exist $d_n \in L^\infty(X_2, \mu_1 \circ \tau^{-1})$, $n \in \mathbb{N}$, such that $d_n \to g^{-1/2}$ in $L^2(X_2, \mu_1 \circ \tau^{-1})$. A simple calculation shows that $d_n g^{1/2} \to 1$ in $L^2(X_2, \mu_2)$, equivalently that $d_n g^{1/2} \to f_2$ in $L^2(R_2, \nu_2)$. It follows that

$$f_2 \in \overline{\mathcal{D}_2 g^{1/2}} \text{ (closure in } L^2(R_2, \nu_2)),$$

which implies the claim. Combining the previous discussion with Proposition 6 (iv), we see that $\mathcal{C}_1 f_1$ is dense in $L^2(R_1, \nu_1)$ and $\mathcal{C}_2 g^{1/2}$ is dense in $L^2(R_2, \nu_2)$.

We define a surjective linear mapping

$$U_0 : \mathcal{C}_1 f_1 \to \mathcal{C}_2 g^{1/2} : a f_1 \mapsto \Theta(a) g^{1/2}.$$

We claim that $U_0$ is isometric, and therefore extends uniquely to a unitary $U : L^2(R_1) \to L^2(R_2)$. It suffices to show that

$$\|\Theta(v)\Theta(d)g^{1/2}\|_{L^2(R_2)} = \|vdf_1\|_{L^2(R_1)} \tag{5}$$

for all $v \in \mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{C}_1$ and all $d \in \mathcal{D}_1$. Indeed, if (5) holds, then by Remark 9 and the Pythagorean Theorem, $\|\Theta(a)g^{1/2}\|_{L^2(R_2)} = \|af_1\|_{L^2(R_1)}$ for all $a = \sum_{j=1}^N v_j d_j$ with $v_1, v_2, ..., v_N \in \mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{C}_1$ pairwise disjoint and $d_1, d_2, ..., d_N \in \mathcal{D}_1$. But then by Lemma 10 and Remark 11, $\|\Theta(a)g^{1/2}\|_{L^2(R_2)} = \|af_1\|_{L^2(R_1)}$ for all $a \in \operatorname{span}(\mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1) \cap \mathcal{C}_1)$, and so by Proposition 6 (ii), $\|\Theta(a)g^{1/2}\|_{L^2(R_2)} = \|af_1\|_{L^2(R_1)}$ for all $a \in \mathcal{C}_1$. It remains to show (5), but this follows by the calculation

$$\begin{aligned}
\|\Theta(v)\Theta(d)g^{1/2}\|^2_{L^2(R_2)} &= \int_{R_2} \chi_{\Gamma(\phi_\tau)}(x,y)|\Theta(d)(y)|^2 g(y) d\nu_2(x,y) \\
&= \int_{X_2} \chi_{\mathrm{dom}(\phi_\tau)}(y)|\Theta(d)(y)|^2 g(y) d\mu_2(y) \\
&= \int_{X_2} \chi_{\tau(\mathrm{dom}(\phi))}(y)|d(\tau^{-1}(y))|^2 d(\mu_1 \circ \tau^{-1})(y) \\
&= \int_{X_1} \chi_{\mathrm{dom}(\phi))}(y)|d(y)|^2 d\mu_1(y) \\
&= \int_{R_1} \chi_{\Gamma(\phi)}(x,y)|d(y)|^2 d\nu_1(x,y) = \|vdf_1\|^2_{L^2(R_1)}.
\end{aligned}$$

We claim that $\Theta(a) = UaU^*$ for all $a \in \mathcal{C}_1$ (as intended). Indeed,

$$U^*\Theta(a)Ubf_1 = U^*\Theta(a)\Theta(b)g^{1/2} = U^*\Theta(ab)g^{1/2} = abf_1$$

for all $a, b \in \mathcal{C}_1$.

To conclude our proof of Theorem 1, we define $\overline{\theta}(x) = UxU^*$ for all $x \in \mathcal{M}_1$. Since $\overline{\theta}|_{\mathcal{C}_1} = \Theta$, $\overline{\theta}$ is a von Neumann algebra isomorphism of $\mathcal{M}_1$ and $\mathcal{M}_2$, by Proposition 6 (iv). Since $\overline{\theta}|_{\mathcal{A}_1^0} = \theta|_{\mathcal{A}_1^0}$, $\overline{\theta}|_{\mathcal{A}} = \theta$, by Proposition 6 (iii) and Corollary 5.

## References


1. William Arveson, *An invitation to $C^*$-algebras*, Springer-Verlag, New York, 1976, Graduate Texts in Mathematics, No. 39. MR 0512360 (58 #23621), Zbl 0344.46123
2. Jacob Feldman and Calvin C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 289–324. MR 58 #28261a, Zbl 0369.22009
3. ______, *Ergodic equivalence relations, cohomology, and von Neumann algebras. II*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 325–359. MR 58 #28261b, Zbl 0369.22010
4. Alexander Kumjian, *On $C^*$-diagonals*, Canad. J. Math. **38** (1986), no. 4, 969–1008. MR 88a:46060, Zbl 0627.46071
5. Richard Mercer, *Bimodules over Cartan subalgebras*, Proceedings of the Seventh Great Plains Operator Theory Seminar (Lawrence, KS, 1987), vol. 20, 1990, pp. 487–502. MR 1065846 (91f:46081), Zbl 0722.46021
6. ______, *Isometric isomorphisms of Cartan bimodule algebras*, J. Funct. Anal. **101** (1991), no. 1, 10–24. MR 1132304 (92k:46102), Zbl 0772.46030
7. P. S. Muhly, K. Saito and B. Solel, *Coordinates for triangular operator algebras*, Ann. Math. **127** (1988), 245–278. MR 0932297 (89h:46088), Zbl 0649.47036
8. David R. Pitts, *Norming algebras and automatic complete boundedness of isomorphisms of operator algebras*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1757–1768. MR 2373606, Zbl 1137.47061
9. ______, *working title: Structure for regular inclusions*, (2011), preprint.
10. Florin Pop, Allan M. Sinclair, and Roger R. Smith, *Norming $C^*$-algebras by $C^*$-subalgebras*, J. Funct. Anal. **175** (2000), no. 1, 168–196. MR 2001h:46105, Zbl 1003.46031
11. Allan M. Sinclair and Roger R. Smith, *Hochschild cohomology for von Neumann algebras with Cartan subalgebras*, Amer. J. Math. **120** (1998), no. 5, 1043–1057. MR 1646053 (99j:46087), Zbl 0913.46050

Dept. of Mathematics, Vassar College, Poughkeepsie, NY, 12604
jacameron@vassar.edu

Dept. of Mathematics, University of Nebraska-Lincoln, Lincoln, NE, 68588-0130
dpitts2@math.unl.edu

Dept. of Mathematics, U. S. Naval Academy, Annapolis, MD, 21402
zarikian@usna.edu