

M -tensors and The Positive Definiteness of a Multivariate Form

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Abstract. We study M -tensors and various properties of M -tensors are given. Specially, we show that the smallest real eigenvalue of M -tensor is positive corresponding to a nonnegative eigenvector. We propose an algorithm to find the smallest positive eigenvalue and then apply the property to study the positive definiteness of a multivariate form.

Key words. M -tensors, eigenvalue, algorithm, positive definiteness.

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1. Introduction

M -matrices [8, 12, 26] are known to have many applications in modeling dynamic systems in economics, ecology, and engineering [23]. Various properties of M -matrices were used in establishing stability results for dynamic systems in general [23, 24]. Especially, it follows from the famous Perron-Frobenius theorem for nonnegative matrices that for any M -matrix A , all real eigenvalues of A are positive [12, 26]. The theory of M -matrices has many applications in many fields such as computational mathematics, mathematical physics, mathematical economics, wireless communications, etc. [8, 26]. Since avoidance conditions were linked to a stability type of result via Lyapunov-type functions, the theory of M -matrices was also used to certify avoidance conditions [25]. On the other hand, testing positive definiteness of a multivariate form is an important problem in the stability study

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of nonlinear autonomous systems via Lyapunov's direct method in automatic control [16]. Researchers in automatic control studied the conditions of such positive definiteness intensively [1, 2, 3, 11]. For $n \geq 3$ and $m \geq 4$, this problem is a hard problem in mathematics. There are only a few methods to solve the problem [2, 3, 16]. In practice, when $n > 3$ and $m \geq 4$, these methods are computationally expensive. Note that an eigenvalue method was proposed in [16] to solve the problem. In [14], a method for computing the largest eigenvalue of an irreducible nonnegative tensor was applied to test the positive definiteness of a class of multivariate forms. Motivated by these applications and methods, we study higher-order M -tensors in this paper. We show that all real eigenvalues of an M -tensor are positive. Hence, testing the positive definiteness of an even-order multivariate form is equivalent to testing a tensor is an M -tensor.

A tensor can be regarded as a higher-order generalization of a matrix, which takes the form

$$\mathcal{A} = (A_{i_1 \dots i_m}), \quad A_{i_1 \dots i_m} \in R, \quad 1 \leq i_1, \dots, i_m \leq n.$$

Such a multi-array \mathcal{A} is said to be an m -order n -dimensional square real tensor with n^m entries $A_{i_1 \dots i_m}$. In this regard, a vector is a first-order tensor and a matrix is a second-order tensor. Tensors of order more than two are called higher-order tensors. M -tensor we defined in this paper is ultimately related to the nonnegative tensor. It is a higher order generalization of the so-called M -matrix.

Nonnegative tensors, arising from multilinear pagerank [13], spectral hypergraph theory [4], and higher-order Markov chains [15], etc., form a singularly important class of tensors and have attracted more and more attention since they share some intrinsic properties with those of the nonnegative matrices. One of those properties is the Perron-Frobenius theorem on eigenvalues. In [5], Chang, Pearson, and Zhang generalized the Perron-Frobenius theorem from nonnegative matrices to irreducible nonnegative tensors. Later, Yang and Yang [28] generalized the weak Perron-Frobenius theorem to general nonnegative tensors, and proved that the spectral radius is an eigenvalue of a nonnegative tensor. Numerical methods for finding the spectral radius of nonnegative tensors are subsequently proposed. Ng, Qi, and Zhou [15] provided an iterative method to find the largest eigenvalue of an irreducible nonnegative tensor. The Ng-Qi-Zhou method is efficient but it is not always convergent for irreducible nonnegative tensors. Zhang and Qi [29] established an explicit linear convergence rate of the Ng-Qi-Zhou method for essentially positive tensors [19]. Liu, Zhou and Ibrahim [14] modified the Ng-Qi-Zhou method such that the modified algorithm is always convergent for finding the largest eigenvalue of an irreducible nonnegative tensor. The linear convergence rate of the algorithm was established in [30] for weakly positive tensors. Chang, Pearson and Zhang [7] introduced primitive tensors and established convergence of the Ng-Qi-Zhou method for primitive tensors. Friedland, Gaubert and Han [9] pointed out that the Perron-Frobenius theorem for nonnegative tensors has a very close link with

the Perron-Frobenius theorem for homogeneous monotone maps, initiated by Nussbaum [17, 18] and further studied by Gaubert and Gunawardena [10]. Friendland, Gaubert and Han [9] gave a weaker definition for irreducible nonnegative tensors, and established the Perron-Frobenius theorem in this context.

By using the eigenvalue theory of nonnegative tensors, we give some properties of M -tensors. We prove that the smallest eigenvalue of an M -tensor is positive with a nonnegative eigenvector. Let \mathcal{A} be a tensor with nonpositive off-diagonal entries. Two necessary and sufficient conditions for \mathcal{A} as an M -tensor are given. Specially, we prove that \mathcal{A} is an M -tensor if and only if its all real eigenvalues are positive. We also give a sufficient condition for a tensor to be an M -tensor, which is easily verified. Finally, we propose an algorithm for computing the smallest real eigenvalue. The proposed algorithm is always convergent for any M -tensor. Furthermore, we link M -tensors with a class of multivariate forms and then apply the proposed method to study the positive definiteness of a multivariate form. It should be pointed out that the class of multivariate forms studied in [14] is a special case of our model. We do not need the assumption that the diagonal entries are positive.

This paper is organized as follows. In Section 2, we will recall some preliminary results. We will introduce M -tensors and characterize some basic properties of M -tensors in Section 3. In Section 4, we will propose an iterative algorithm for finding the smallest real eigenvalue of an M -tensor, and some numerical results are reported. In Section 5, we will present an algorithm to for testing positive definiteness of a class of multivariate forms. We conclude the paper with some remarks in Section 6.

2. Preliminaries

We start this section with some fundamental notions and properties on tensors. An m -order n -dimensional tensor \mathcal{A} is called nonnegative (or, respectively, positive) if $A_{i_1 \dots i_m} \geq 0$ (or, respectively, $A_{i_1 \dots i_m} > 0$). The tensor \mathcal{A} is called symmetric if its entries $A_{i_1 \dots i_m}$ are invariant under any permutation of their indices $\{i_1 \dots i_m\}$ [20]. The m -order n -dimensional unit tensor, denoted by \mathcal{I} , is the tensor whose entries are $\delta_{i_1 \dots i_m}$ with $\delta_{i_1 \dots i_m} = 1$ if and only if $i_1 = \dots = i_m$ and otherwise zero. A tensor \mathcal{A} is called reducible, if there exists a nonempty proper index subset $I \subset \{1, 2, \dots, n\}$ such that

$$A_{i_1 \dots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \dots, i_m \notin I.$$

Otherwise, we say \mathcal{A} is irreducible.

Analogous with that of matrices, the theory of eigenvalues and eigenvectors is one of the fundamental and essential components in tensor analysis. Wide range of practical applications can be found in [13, 21, 22]. Compared with that of matrices, eigenvalue problems for higher-order tensors are nonlinear due to their multilinear structure.

Definition 2.1 Let C be the complex field. A pair $(\lambda, x) \in C \times (C^n \setminus \{0\})$ is called an eigenvalue-eigenvector pair of \mathcal{A} , if they satisfy:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \quad (1)$$

where n -dimensional column vectors $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are defined as

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2, \dots, i_m=1}^n A_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leq i \leq n} \quad \text{and} \quad x^{[m-1]} := (x_i^{m-1})_{1 \leq i \leq n},$$

respectively.

This definition was introduced in [6, 13, 20]. We call $\rho(\mathcal{A})$ the spectral radius of tensor \mathcal{A} if

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\},$$

where $|\lambda|$ denotes the modulus of λ . An immediate consequence on the spectral radius follows directly from Corollary 3 in [20].

Lemma 2.1 Let \mathcal{A} be an m -order n -dimensional tensor. Suppose that $\mathcal{B} = a(\mathcal{A} + b\mathcal{I})$, where a and b are two real numbers. Then μ is an eigenvalue of \mathcal{B} if and only if $\mu = a(\lambda + b)$ and λ is an eigenvalue of \mathcal{A} . In this case, they have the same eigenvectors.

Clearly, from this lemma, we have $\rho(\mathcal{B}) \leq |a|(\rho(\mathcal{A}) + |b|)$.

Nice properties such as the Perron-Frobenius theorem for eigenvalues of nonnegative square tensors have been established in [5].

Theorem 2.1 If \mathcal{A} is an irreducible nonnegative tensor of order m and dimension n , then there exist $\lambda_0 > 0$ and $x_0 > 0, x_0 \in R^n$ such that

$$\mathcal{A}x_0^{m-1} = \lambda_0 x_0^{[m-1]}.$$

Moreover, if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \lambda_0$. If λ is an eigenvalue of \mathcal{A} , then $|\lambda| \leq \lambda_0$.

Clearly, by this theorem, λ_0 is the largest eigenvalue of \mathcal{A} .

Yang and Yang [28] asserted that the spectral radius of a nonnegative tensor is an eigenvalue. In the following we state the results of [28, Theorem 2.3 and Lemma 5.8] for reference.

Theorem 2.2 *Assume that \mathcal{A} is a nonnegative tensor of order m dimension n , then $\rho(\mathcal{A})$ is an eigenvalue of \mathcal{A} with a nonzero nonnegative eigenvector. Moreover, for any $x > 0, x \in R^n$ we have*

$$\min_{1 \leq i \leq n} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} \leq \rho(\mathcal{A}) \leq \max_{1 \leq i \leq n} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}.$$

The following continuity of the spectral radius was given in the proof of [28, Theorem 2.3].

Lemma 2.2 *Let \mathcal{A} be a nonnegative tensor of order m and dimension n , and $\varepsilon > 0$ be a sufficiently small number. If $\mathcal{A}_\varepsilon = \mathcal{A} + \mathcal{E}$ where \mathcal{E} denotes the tensor with every entry being ε , then*

$$\lim_{\varepsilon \rightarrow 0} \rho(\mathcal{A}_\varepsilon) = \rho(\mathcal{A}).$$

Now we state an iterative algorithm for calculating the largest eigenvalue of a nonnegative tensor \mathcal{A} , which is proposed in [14, 15] based on Theorems 2.1 and 2.2.

Algorithm 1

Step 0. *Choose $x^{(1)} > 0, x^{(1)} \in R^n$ and $\rho > 0$. Let $\mathcal{B} = \mathcal{A} + \rho\mathcal{I}$, and set $k := 1$.*

Step 1. *Compute*

$$\begin{aligned} y^{(k)} &= \mathcal{B}(x^{(k)})^{m-1}, \\ \underline{\lambda}_k &= \min_{x_i^{(k)} > 0} \frac{(y^{(k)})_i}{(x_i^{(k)})^{m-1}}, \\ \bar{\lambda}_k &= \max_{x_i^{(k)} > 0} \frac{(y^{(k)})_i}{(x_i^{(k)})^{m-1}}. \end{aligned}$$

Step 2. *If $\bar{\lambda}_k = \underline{\lambda}_k$, then let $\lambda = \bar{\lambda}_k$ and stop. Otherwise, compute*

$$x^{(k+1)} = \frac{(y^{(k)})^{\left[\frac{1}{m-1}\right]}}{\left\| (y^{(k)})^{\left[\frac{1}{m-1}\right]} \right\|},$$

replace k by $k + 1$ and go to Step 1.

Step 3. *Output $\lambda - \rho$, the largest eigenvalue of \mathcal{A} .*

It is proved in [14] that Algorithm 1 is always convergent for irreducible nonnegative tensors; see the following theorem.

Theorem 2.3 *Suppose \mathcal{A} be an irreducible nonnegative tensor. Let $\mathcal{B} = \mathcal{A} + \rho\mathcal{I}$, where $\rho > 0$. Assume that λ is the largest eigenvalue of \mathcal{B} . Then, Algorithm 1 produces a value of λ in a finite number of steps, or generates two sequences $\{\underline{\lambda}_k\}$ and $\{\bar{\lambda}_k\}$ which converge to λ . Furthermore, $\lambda - \rho$ is the largest eigenvalue of \mathcal{A} .*

Note that for any nonnegative tensor \mathcal{A} , $\mathcal{A} + \mathcal{E}$ is an irreducible nonnegative tensor. Therefore, if we set $\mathcal{B} = \mathcal{A} + \rho\mathcal{I} + \mathcal{E}$ in Algorithm 1, then by Lemma 2.2, we can prove that the modified algorithm is also convergent in the similar way.

3. M -tensors

We now extend the notion of M -matrices to higher-order tensors and introduce the definition of an M -tensor.

Definition 3.1 *Let \mathcal{A} be an m -order and n -dimensional tensor. \mathcal{A} is called an M -tensor if there exist a nonnegative tensor \mathcal{B} and a real number $c > \rho(\mathcal{B})$, where $\rho(\mathcal{B})$ is the spectral radius of \mathcal{B} , such that*

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

Clearly, when $m = 2$, if \mathcal{A} is an M -tensor then \mathcal{A} is an M -matrix.

Note that the off-diagonal entries of an M -tensor are nonpositive. Denote \mathbb{Z} the set of m -order and n -dimensional real tensors whose off-diagonal entries are nonpositive.

Note that eigenvalues defined in Definition 2.1 exist for an M -tensor [20]. We will show that every eigenvalue of an M -tensor has a positive real part, and hence all of its real eigenvalues are positive.

Theorem 3.1 *Let \mathcal{A} be an M -tensor and denote by $\tau(\mathcal{A})$ the minimal value of the real part of all eigenvalues of \mathcal{A} . Then $\tau(\mathcal{A}) > 0$ is an eigenvalue of \mathcal{A} with a nonnegative eigenvector. Moreover, there exist a nonnegative tensor \mathcal{B} and a real number $c > \rho(\mathcal{B})$ such that $\tau(\mathcal{A}) = c - \rho(\mathcal{B})$. If \mathcal{A} is irreducible, then $\tau(\mathcal{A})$ is the unique eigenvalue with a positive eigenvector.*

Proof. Since \mathcal{A} is an M -tensor, by Definition 3.1, there exist a nonnegative tensor \mathcal{B} and a real number $c > \rho(\mathcal{B})$ such that

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

Let λ be an eigenvalue of \mathcal{A} and $\text{Re}\lambda$ be the real part of λ . By Lemma 2.1, $c - \lambda$ is an eigenvalue of \mathcal{B} . Since $\rho(\mathcal{B})$ is the spectral radius of \mathcal{B} ,

$$\rho(\mathcal{B}) \geq |c - \lambda| \geq c - \text{Re}\lambda > \rho(\mathcal{B}) - \text{Re}\lambda,$$

which implies that $\operatorname{Re}\lambda > 0$, and hence

$$\rho(\mathcal{B}) \geq \max_{\lambda \in \lambda(\mathcal{A})} \{c - \operatorname{Re}\lambda\} = c - \min_{\lambda \in \lambda(\mathcal{A})} \{\operatorname{Re}\lambda\} = c - \tau(\mathcal{A}). \quad (2)$$

On the other hand, Theorem 2.2 shows that $\rho(\mathcal{B})$ is an eigenvalue of \mathcal{B} . By Lemma 2.1, $c - \rho(\mathcal{B})$ is a real eigenvalue of \mathcal{A} , and hence

$$c - \rho(\mathcal{B}) \geq \tau(\mathcal{A}),$$

which, together with (2), implies

$$\tau(\mathcal{A}) = c - \rho(\mathcal{B}).$$

Therefore, $\tau(\mathcal{A})$ is an eigenvalue of \mathcal{A} . Since $\rho(\mathcal{B})$ has a nonnegative eigenvector x^* , x^* is also an eigenvector of $\tau(\mathcal{A})$. Moreover, if \mathcal{A} is irreducible, then \mathcal{B} is also irreducible. Hence $\rho(\mathcal{B})$ is the unique eigenvalue of \mathcal{B} with a positive eigenvector. Thus, we complete the proof. ■

We state a few of the alternative necessary and sufficient conditions of a tensor in \mathbb{Z} to be an M -tensor.

Theorem 3.2 *A tensor in \mathbb{Z} is an M -tensor if and only if any of its eigenvalues has a positive real part.*

Proof. Let $\mathcal{A} \in \mathbb{Z}$, and suppose that every eigenvalue of \mathcal{A} has a positive real part. Let $a = \max_{1 \leq i \leq n} \{A_{i\dots i}\}$. Then $\mathcal{B} = a\mathcal{I} - \mathcal{A}$ is nonnegative. By Lemma 2.1 and Theorem 2.2, $a - \rho(\mathcal{B})$ is a real eigenvalue of \mathcal{A} . It follows that $a - \rho(\mathcal{B}) > 0$. Thus $\mathcal{A} = a\mathcal{I} - \mathcal{B}$ is an M -tensor.

Now, let $\mathcal{A} = c\mathcal{I} - \mathcal{B}$ be an M -tensor. Then, \mathcal{B} is nonnegative and $c > \rho(\mathcal{B})$. Let λ be an eigenvalue of \mathcal{A} . Let $\operatorname{Re}\lambda$ be the real part of λ . Suppose $\operatorname{Re}\lambda \leq 0$. Since λ is an eigenvalue of \mathcal{A} , there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

which yields

$$\mathcal{B}x^{m-1} = (c - \lambda)x^{[m-1]}.$$

It follows from Definition 1.1 that $c - \lambda$ is an eigenvalue of \mathcal{B} . Since $c > \rho(\mathcal{B})$ and $\operatorname{Re}\lambda \leq 0$, we have

$$|c - \lambda| \geq c - \operatorname{Re}\lambda \geq c > \rho(\mathcal{B}),$$

which contradicts the maximality of $\rho(\mathcal{B})$. We complete the proof. ■

Another equivalent condition for an M -tensor is presented as follows, which is more easily certified than the one in Theorem 3.2.

Theorem 3.3 *A tensor $\mathcal{A} \in \mathbb{Z}$ is an M -tensor if and only if all of its real eigenvalues are positive.*

Proof. The necessity of the condition follows directly from Theorem 3.2. Suppose now that all real eigenvalues of $\mathcal{A} \in \mathbb{Z}$ are positive. Let $\mathcal{B} = a\mathcal{I} - \mathcal{A}$ where $a = \max_{1 \leq i \leq n} \{A_{i\dots i}\}$. Then \mathcal{B} is nonnegative and hence it follows from Theorem 2.2 that $\rho(\mathcal{B})$ is an eigenvalue of \mathcal{B} . By Lemma 2.1, $a - \rho(\mathcal{B})$ is a real eigenvalue of \mathcal{A} , it must be positive. That is, $a > \rho(\mathcal{B})$, and therefore \mathcal{A} is an M -tensor. ■

Clearly, we easily obtain the following results from the above theorems.

Corollary 3.1 *If $\mathcal{A} \in \mathbb{Z}$ is an M -tensor, then $\max_{1 \leq i \leq n} \{A_{i\dots i}\} > 0$.*

Proof. Let $a = \max_{1 \leq i \leq n} \{A_{i\dots i}\}$ and $\mathcal{B} = a\mathcal{I} - \mathcal{A}$. Then $\mathcal{B} \geq 0$ and hence $\rho(\mathcal{B})$ is an eigenvalue of \mathcal{B} . So, $a - \rho(\mathcal{B})$ is a real eigenvalue of \mathcal{A} . Since \mathcal{A} is an M -tensor, by Theorem 3.3, $a - \rho(\mathcal{B}) > 0$, which implies $a > \rho(\mathcal{B}) \geq 0$. ■

Note that, If the tensor $\mathcal{A} \in \mathbb{Z}$ then \mathcal{A} can be deposed into the form $\mathcal{A} = \lambda\mathcal{I} - \mathcal{B}$ with $\lambda \geq \max_{1 \leq i \leq n} \{A_{i\dots i}\}$ and $\mathcal{B} \geq 0$. Thus, we have the following sufficient and necessary condition for a tensor being an M -tensor, which is a generalization of Theorem 7 in [27].

Corollary 3.2 *Let $\mathcal{A} \in \mathbb{Z}$. Decompose the tensor \mathcal{A} into the form $\mathcal{A} = \lambda\mathcal{I} - \mathcal{B}$, where $\lambda \geq \max_{1 \leq i \leq n} \{A_{i\dots i}\}$. Then \mathcal{A} is an M -tensor if and only if $\lambda > \rho(\mathcal{B})$.*

Proof. Clearly, $\mathcal{B} \geq 0$. By Theorem 2.2, $\rho(\mathcal{B})$ is an eigenvalue of \mathcal{B} . By Lemma 2.1, $\lambda - \rho(\mathcal{B})$ is a real eigenvalue of \mathcal{A} . If \mathcal{A} be an M -tensor then we have, by Theorem 3.3, $\lambda - \rho(\mathcal{B}) > 0$, i.e., $\lambda > \rho(\mathcal{B})$. If $\lambda > \rho(\mathcal{B})$ then, by Definition 3.1, \mathcal{A} is an M -tensor. ■

Note that this result provides us an easy method for determining whether a tensor \mathcal{A} is an M -tensor. We only need to compute the spectral radius of the tensor $\lambda\mathcal{I} - \mathcal{A}$, where $\lambda \geq \max_{1 \leq i \leq n} \{A_{i\dots i}\}$.

Finally, we give a sufficient condition for a tensor to be an M -tensor, which is easily verified. First, we state a definition which is a generalization from matrices to tensors [8, 26].

Definition 3.2 *Let \mathcal{A} be an m -order and n -dimensional tensor. \mathcal{A} is diagonally dominant*

if

$$\sum_{i_2, \dots, i_m=1, \delta_{i_2 \dots i_m}=0}^n |A_{ii_2 \dots i_m}| \leq |A_{ii \dots i}|, \quad i = 1, 2, \dots, n, \quad (3)$$

where the symbol $\delta_{i_2 \dots i_m} = 0$ is defined as the entry of the unit tensor \mathcal{I} . \mathcal{A} is strictly diagonally dominant if the strict inequality holds in (3) for all i . \mathcal{A} is irreducibly diagonally dominant if \mathcal{A} is irreducible, diagonally dominant, and the strict inequality in (3) holds for at least one i .

Theorem 3.4 *If a tensor $\mathcal{A} \in \mathbb{Z}$ is strictly or irreducibly diagonally dominant with all nonnegative diagonal entries, then \mathcal{A} is an M -tensor.*

Proof. Let λ be an eigenvalue of \mathcal{A} with a nonzero eigenvector x . Denote

$$|x_i| = \max_{1 \leq j \leq n} |x_j|.$$

Then

$$\sum_{i_2, \dots, i_m=1}^n A_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_i^{m-1}, \quad (4)$$

which implies that

$$|\lambda - A_{ii \dots i}| \leq \sum_{i_2, \dots, i_m=1; \delta_{i_2 \dots i_m}=0}^n |A_{ii_2 \dots i_m}|.$$

Hence, the diagonal dominance of \mathcal{A} implies that

$$|\operatorname{Re} \lambda - A_{ii \dots i}| \leq |\lambda - A_{ii \dots i}| \leq |A_{ii \dots i}|. \quad (5)$$

Since $A_{j \dots j} \geq 0$ for $j = 1, 2, \dots, n$, (5) yields

$$\operatorname{Re} \lambda - A_{ii \dots i} \geq -A_{ii \dots i}. \quad (6)$$

Suppose that \mathcal{A} is strictly diagonally dominant. Then the strict inequality holds in (3) for all j , so the strict inequality holds in (6). This yields $\operatorname{Re} \lambda > 0$, i.e., any eigenvalues of \mathcal{A} has a positive real part. By Theorem 3.2, $\mathcal{A} \in \mathbb{Z}$ is an M -tensor.

Suppose now that \mathcal{A} is irreducibly diagonally dominant. Define

$$J = \{l : |x_l| = \max_{1 \leq i \leq n} |x_i|, |x_l| > |x_i| \text{ for some } i\}.$$

If $J = \emptyset$, then (4) and the diagonal dominance of \mathcal{A} imply that for $i = 1, 2, \dots, n$,

$$|\lambda - A_{ii \dots i}| \leq \sum_{i_2, \dots, i_m=1; \delta_{i_2 \dots i_m}=0}^n |A_{ii_2 \dots i_m}| \leq |A_{ii \dots i}|.$$

Let

$$|A_{kk\dots k}| > \sum_{i_2, \dots, i_m=1; \delta_{ki_2\dots i_m}=0}^n |A_{ki_2\dots i_m}|$$

for some k . We have

$$|\operatorname{Re}\lambda - A_{k\dots k}| \leq |\lambda - A_{k\dots k}| < |A_{k\dots k}| = A_{k\dots k}$$

which implies that $\operatorname{Re}\lambda > 0$.

If $J \neq \emptyset$, then the irreducibility of \mathcal{A} implies that there exist $l \in J$ and $i_2, \dots, i_m \notin J$ such that $A_{li_2\dots i_m} \neq 0$. Hence (4) yields

$$|\lambda - A_{ll\dots l}| \leq \sum_{i_2, \dots, i_m=1; \delta_{li_2\dots i_m}=0}^n |A_{li_2\dots i_m}| \frac{|x_{i_2}|}{|x_l|} \cdots \frac{|x_{i_m}|}{|x_l|} < \sum_{i_2, \dots, i_m=1; \delta_{li_2\dots i_m}=0}^n |A_{li_2\dots i_m}| \leq |A_{ll\dots l}|,$$

which implies that $\operatorname{Re}\lambda > 0$. By Theorem 3.2, $\mathcal{A} \in \mathbb{Z}$ is an M -tensor. \blacksquare

4. An Algorithm

Theorem 3.3 shows that a tensor $\mathcal{A} \in \mathbb{Z}$ is an M -tensor if and only if the smallest real eigenvalue of \mathcal{A} is positive. In this section we propose an algorithm to determine whether or not a tensor with nonpositive off-diagonal entries is an M -tensor.

Lemma 4.1 *Let \mathcal{A} be an m -order and n -dimensional tensor. Define*

$$L_{\mathcal{A}} = \min_{1 \leq i \leq n} \{A_{ii\dots i} - C_i\}, \quad U_{\mathcal{A}} = \max_{1 \leq i \leq n} \{A_{ii\dots i} + C_i\}, \quad (7)$$

where

$$C_i = \sum_{i_2, \dots, i_m=1, \delta_{ii_2\dots i_m}=0}^n |A_{ii_2\dots i_m}|, \quad i = 1, 2, \dots, n.$$

Then $L_{\mathcal{A}}$ and $U_{\mathcal{A}}$ are the lower and upper bounds of real eigenvalues of \mathcal{A} , respectively.

Proof. Let λ be a real eigenvalue of \mathcal{A} with an eigenvector $x \neq 0$. That is,

$$\sum_{i_2, \dots, i_m=1}^n A_{ii_2\dots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_i^{m-1}, \quad i = 1, 2, \dots, n. \quad (8)$$

Let $|x_k| = \max_{1 \leq i \leq n} |x_i|$. Then (8) implies that

$$|\lambda - A_{kk\dots k}| \leq \sum_{i_2, \dots, i_m=1, \delta_{ki_2\dots i_m}=0}^n |A_{ki_2\dots i_m}| \frac{|x_{i_2}|}{|x_k|} \cdots \frac{|x_{i_m}|}{|x_k|} \leq C_k,$$

which yields $A_{kk\dots k} - C_k \leq \lambda \leq A_{kk\dots k} + C_k$. This shows $L_{\mathcal{A}} \leq \lambda \leq U_{\mathcal{A}}$. ■

For a tensor $\mathcal{A} \in \mathbb{Z}$, we define a tensor \mathcal{C} as

$$\mathcal{C} = U_{\mathcal{A}}\mathcal{I} - \mathcal{A}, \quad (9)$$

where $U_{\mathcal{A}}$ is defined in (7). Clearly, the tensor \mathcal{C} is a nonnegative tensor. By Lemma 2.1 and Theorem 2.1, $U_{\mathcal{A}} - \rho(\mathcal{C})$ is the smallest eigenvalue of \mathcal{A} . By Theorem 3.3 and Definition 3.1, if $U_{\mathcal{A}} - \rho(\mathcal{C})$ is positive then \mathcal{A} is an M -tensor. Based on this observation, in the following, we establish an algorithm for computing the smallest eigenvalue of a tensor $\mathcal{A} \in \mathbb{Z}$. If the smallest eigenvalue is positive then \mathcal{A} is an M -tensor. Otherwise, \mathcal{A} is not an M -tensor.

Based on the above discussion, we propose the following algorithm to determine whether or not a tensor \mathcal{A} with nonpositive off-diagonal entries is an M -tensor.

Algorithm 2

Step 0. Compute the upper bound of real eigenvalues, $U_{\mathcal{A}}$ by the formula (7) and let $\mathcal{C} = U_{\mathcal{A}}\mathcal{I} - \mathcal{A}$.

Step 1. By using Algorithm 1, compute λ , the largest eigenvalue of \mathcal{C} .

Step 2. Let $\mu = U_{\mathcal{A}} - \lambda$. If $\mu > 0$ then \mathcal{A} is an M -tensor. Otherwise, \mathcal{A} is not an M -tensor.

In order to show the viability of Algorithm 2, we used Matlab Version 7.7(R2008b) to test it on some tensors with nonpositive off-diagonal entries which are randomly generated by the following procedure.

Procedure 1.

1. Give (m, n, A_d) , where n and m are the dimension and the order of the randomly generated tensor, respectively, and $A_d > 0$.
2. Randomly generate an m -order n -dimensional tensor \mathcal{D} such that all elements of \mathcal{D} are in the interval $(0, 1)$.
3. Let $\mathcal{A} = (A_{i_1\dots i_m})$, where $A_{i\dots i} = A_d + D_{i\dots i}$, $i = 1, 2, \dots, n$, otherwise, $A_{i_1\dots i_m} = -D_{i_1\dots i_m}$, $1 \leq i_1, \dots, i_m \leq n$.

Our numerical results are reported in Table 1 In this table, \mathbf{n} and \mathbf{m} specify the dimension and the order of the randomly generated tensor, respectively. A_d is a parameter in Procedure

1. Given (m, n, A_d) , we generate 100 tensors and determine whether or not they are M -tensors by Algorithm 2. In the **Yes** column we show the number of tensors which are M -tensors. In the **No** column, we give the number of tensors which are not M -tensors. CPU(s) denotes the average computer time in seconds used for Algorithm 2. The results reported in Table 1 show that Algorithm 2 performs well for these tensors.

5. An Application: The positive definiteness of a multivariate form

In this section we apply the proposed algorithm for testing the positive definiteness of a class of multivariate forms.

An m th degree homogeneous polynomial form of n variables $f(x)$, where $x \in R^n$, can be denoted as

$$f(x) := \sum_{i_1, i_2, \dots, i_m=1}^n A_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}. \quad (10)$$

When m is even, $f(x)$ is called *positive definite* if

$$f(x) > 0, \quad \forall x \in R^n, x \neq 0. \quad (11)$$

Testing positive definiteness of a multivariate form is an important problem in the stability study of nonlinear autonomous systems [2, 3, 16]. It is proved in [22] that $f(x)$ is positive definite if and only if its corresponding tensor $\mathcal{A} = (A_{i_1 \dots i_m})$ is symmetric and all of real eigenvalues are positive. That is, testing positive definiteness of a class of even-order multivariate forms is equivalent to determine whether or not the even-order symmetric tensor is an M -tensor. It should be pointed out that the positive definiteness of a special class of multivariate forms was studied in [14]. It can be regarded as our special case. Some numerical results can be referred [14]. We omit them here.

6. Conclusions

We have defined and studied M -tensors. An M -tensor is the generation of an M -matrix. Many important characterizations of M -matrices has been extended to M -tensors. We have proposed two sufficient and necessary conditions, a necessary condition, and a sufficient condition of an M -tensor. We have shown that the smallest eigenvalue of an M -tensor is positive, and then proposed an algorithm to determine whether or not a tensor with nonpositive off-diagonal entries is an M -tensor. Finally, we link an M -tensor with a multivariate form and apply the algorithm to judge the positive definiteness of the multivariate form. Numerical results are reported.

m	n	A_d	Yes	No	CPU(s)
3	10	5	0	100	0.0025
3	10	10	0	100	0.0030
3	10	100	100	0	0.0028
3	10	1000	100	0	0.0028
3	20	5	0	100	0.0094
3	20	10	0	100	0.0102
3	20	100	0	100	0.0088
3	20	1000	100	0	0.0084
3	30	5	0	100	0.0181
3	30	10	0	100	0.0187
3	30	100	0	100	0.0186
3	30	1000	100	0	0.0195
3	40	5	0	100	0.0352
3	40	10	0	100	0.0358
3	40	100	0	100	0.0362
3	40	1000	100	0	0.0355
3	50	5	0	100	0.0619
3	50	10	0	100	0.0619
3	50	100	0	100	0.0598
3	50	1000	0	100	0.0692
4	10	5	0	100	0.0592
4	10	10	0	100	0.0603
4	10	100	0	100	0.0611
4	10	1000	0	100	0.0620
4	20	5	0	100	0.2945
4	20	10	0	100	0.3097
4	20	100	0	100	0.3187
4	20	1000	0	100	0.3116
4	30	5	0	100	1.3233
4	30	10	0	100	1.3125
4	30	100	0	100	1.3170
4	30	1000	0	100	1.3453
4	40	5	0	100	6.5375
4	40	10	0	100	6.5358
4	40	100	0	100	6.4925
4	40	1000	0	100	6.5520
4	50	5	0	100	15.2086
4	50	10	0	100	15.1844
4	50	100	0	100	15.2102
4	50	1000	0	100	15.2039

Table 1: Output of Algorithm 2 for randomly generated tensors.

There are some questions are still in study. For example, whether the condition “there exists $x \in R^n, x \geq 0$ such that $\mathcal{A}x^{m-1} > 0$ ” is a necessary and sufficient condition for a tensor $\mathcal{A} \in \mathbb{Z}$ to be an M -tensor?

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