Multigrid Algorithm for the Coupling System of Natural Boundary Element Method and Finite Element Method for Unbounded Domain Problem *

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Abstract

In this paper, some V-cycle multigrid algorithms are presented for the coupling system arising from discretization the Dirichlet exterior problem by coupling the natural boundary element method and finite element method. The convergence of these multigrid algorithms is obtained even with only one smoothing on all levels and the rate of convergence is uniformly bounded independent of the number of levels and the mesh sizes of all levels which indicates that these multigrid algorithms are optimal. Some numerical results are also reported.

1 Introduction

In many fields of scientific and engineering computing, it is necessary to solve boundary values problems of partial differential equations over unbounded domains. The standard techniques such as the finite element method, which is effective for most problems in bounded domain, will meet some difficulties and the corresponding computing cost will be very high for unbounded domain problems. So for problems of this kind, it is a good choice to use the coupling method of boundary element method and finite element method because it enables us to combine the advantages of boundary element method for treating domains extended to infinity with those of finite element method in treating the complicated bounded domain problems. Research toward this direction is of great importance both in theory and practical computation.

Generally speaking, the procedure of this kind of coupling is described as follows. The unbounded domain is divided into two subregions, a bounded inner one and an unbounded outer one, by introducing a artificial common boundary. Then, the problem is reduced to an equivalent one in the bounded region. There are many approaches to accomplish this reduction (refer to [3, 4, 5, 7, 8, 9, 11, 12, 14, 16, 17, 19, 21]). Natural boundary reduction method is one of them.

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Natural boundary reduction method and its coupling with finite element method, which is also known as the exact artificial boundary condition method, are suggested and developed first by K. Feng in 1980, D. Yu in 1982 and H. Han in 1985. And a very similar method, the so-called DtN method, has also been devised by J. B. Keller and D. Givoli in 1989. In this reduction, the problem over unbounded exterior domain is reduced into an bounded problem with a hyper-singular integral equation on the artificial boundary by using a Green function to get the exact artificial boundary condition with hyper-singular integral. It is fully compatible with the variational principle over the domain, and the boundary element are also fully compatible with the domain elements. This coupling is natural and direct. Moreover, the coupled bilinear form preserves automatically the symmetry and coerciveness of the original bilinear form, which result that not only the analysis of the discrete problem is simplified, but also the error estimates and the numerical stability are restored (see [8] [19]). In this paper, we follow this approach.

With a discretization scheme, construction of efficient algorithms for solving the resulting discrete system is also of great importance. So, our aim is to construct efficient algorithms for the resulting discrete system of the coupling of natural boundary element method and finite element method .

It is well known that multigrid algorithms are among the most efficient methods for solving discretization equations arising from various finite element approximations of boundary value problem on bounded domain (for multigrid method, refer to [1, 10] for detail). And during the last three decades, there has been intensive research toward such methods. The purpose of this paper is to construct multigrid algorithms for discretization equations arising from the coupling of the natural boundary element method and finite element method for the Dirichlet exterior problem and to investigate their convergence.

In the following sections, some V-cycle multigrid algorithms of this kind are constructed and we obtain the convergence of these multigrid algorithms even with only one smoothing on all levels. The rate of convergence is shown to be uniformly bounded independent of the number of levels and the mesh sizes of all levels which indicates that these multigrid algorithms are optimal.

The remainder of the paper is organized as follows: In section 2, we present our model problem and introduce the natural boundary reduction method. Multigrid algorithm is described and analyzed in section 3. And some numerical results are reported in section 4.

2 Model problem and natural boundary reduction

We adopt the notations for Sobolev space, their norms and semi-norms as presented in [2, 6]. Let Ω be a Lipschitz bounded domain in $\mathbb{R}^2$, $\Omega^c = \mathbb{R}^2 \setminus (\Omega \cup \partial\Omega)$, $f \in L^2(\Omega^c)$ be a given compactly supported function. We consider the following model problem

$$\begin{cases} -\triangle u &= f, \quad in \;\; \Omega^c, \\ \quad u &= 0, \quad on \;\; \partial\Omega, \end{cases} \tag{2.1}$$

subject to the asymptotic conditions

$$u(x,y) = \alpha + O(1/r) , \qquad |\nabla u(x,y)| = O(1/r^2) ,$$

as $r = \sqrt{x^2+y^2} \to \infty$ where α is a constant.

Define

$$H^1_\triangle(\Omega^c) = \{v| \quad \frac{v}{\sqrt{r^2+1}\ln(r^2+2)} , \frac{\partial v}{\partial x} , \frac{\partial v}{\partial y} \in L^2(\Omega^c) , \ v|_{\partial\Omega} = 0\}$$

and

$$a(w,v) = \int\int_{\Omega^c} \nabla w \cdot \nabla v dxdy , \qquad \forall w, v \in H^1_\triangle(\Omega^c) .$$

Then the corresponding variational form of (2.1) can be written as:
Find $u \in H^1_\triangle(\Omega^c)$ such that

$$a(u,v) = (f,v) , \qquad \forall v \in H^1_\triangle(\Omega^c) . \tag{2.2}$$

According to the hypothesis on f, we choose a circle disc Ω_0 containing $\bar{\Omega}$ and *supp* f. Let $\Omega_1 = \Omega^c \cap \Omega_0$, $\Omega_2 = \Omega_0^c = \mathbb{R}^2 \setminus (\Omega_0 \cup \partial\Omega_0)$ and $\Gamma = \partial\Omega_0$. Then we have

$$a(u,v) = a_1(u,v) + a_2(u,v) , \tag{2.3}$$

where $a_i(u,v) = \int\int_{\Omega_i} \nabla u \cdot \nabla v dxdy$, $i = 1, 2$.

Next, we introduce the natural boundary reduction method and derive a coupled variational form equivalent to (2.3).

From Green's formula on Ω_2, we have

$$a_2(u,v) = \int_\Gamma \frac{\partial}{\partial n} u(z) \cdot v(z) dz + \int\int_{\Omega_2} f v dxdy . \tag{2.4}$$

Let $V(z,z')$ be the Green's function for the Laplace operator on the domain Ω_2, which satisfies

$$\begin{cases} -\triangle V(z,z') &= \delta(z-z'), \quad \forall z, z' \in \Omega_2, \\ V(z,z')|_{z\in\Gamma} &= 0, \qquad\qquad \forall z' \in \Omega_2, \end{cases}$$

subject to the same asymptotic conditions as u. By taking $w = V(z,z'), v = u$ in Green's second formula

$$\int\int_{\Omega_2} (w\triangle v - v\triangle w) dz' = \int_\Gamma (w\frac{\partial v}{\partial n} - v\frac{\partial w}{\partial n}) dz' ,$$

we get (refer to [19])

$$u(z) = \int\int_{\Omega_2} f(z')V(z,z')dz' - \int_\Gamma \frac{\partial}{\partial n'} V(z,z')u(z')dz' , \qquad \forall z \in \Omega_2 ,$$

where n and n' denote the exterior normal vectors on Γ (viewed as the boundary of Ω_2) at the respective points z and z'. Thus we obtain

$$\frac{\partial u}{\partial n}(z) = \int\int_{\Omega_2} f(z')\frac{\partial}{\partial n} V(z,z')dz' - \int_\Gamma \frac{\partial^2}{\partial n \partial n'} V(z,z')u(z')dz' , \qquad \forall z \in \Gamma . \tag{2.5}$$

Let

$$Ku(z) = -\int_\Gamma \frac{\partial^2}{\partial n \partial n'} V(z,z')u(z')dz' \ , \qquad z \in \Gamma \ . \tag{2.6}$$

Then, it follows from (2.4), (2.5), (2.6) and the fact that *supp* $f \subset \Omega_0$ that

$$a_2(u,v) = \int_\Gamma Ku(z) \cdot v(z)dz \ . \tag{2.7}$$

Define $H^1_*(\Omega_1) = \{v| \ \ v \in H^1(\Omega_1), \ v|_{\partial\Omega} = 0\}$ and

$$b(u,v) = a_1(u,v) + < Ku, v >_\Gamma \ , \tag{2.8}$$

where $< \cdot, \cdot >_\Gamma$ denotes the L^2 inner product on Γ. With (2.3) and (2.7), we can rewrite the variation form (2.2) as:
Find $u \in H^1_*(\Omega_1)$ such that

$$b(u,v) = \int\int_{\Omega_1} fv dx dy \ , \qquad \forall v \in H^1_*(\Omega_1) \ . \tag{2.9}$$

Remark 2.1 *The operator $K : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is shown to be just the Dirichlet-Neumann operator (Steklov-Poincaré operator) for Ω_2 in [18]. So, it is symmetric and semi-positive definite with respect to the inner product $< \cdot, \cdot >_\Gamma$ (see [19]), which indicates that $b(u,v)$ is symmetric, bounded and coercive in $H^1_*(\Omega_1)$. Thus, it follows from the well known Lax-Milgram Theorem that the variational problem (2.9) has unique solution $u \in H^1_*(\Omega_1)$.*

Remark 2.2 *As Γ is a circle, the Green's function $V(z,z')$ can be expressed explicitly. For example, in the case that the center of the circle Γ is the origin and its radius is R,*

$$V(z,z') = \frac{1}{4\pi} \ln \frac{R^4 + r^2 r'^2 - 2R^2 r r' \cos(\theta - \theta')}{R^2(r^2 + r'^2 - 2rr' \cos(\theta - \theta'))} \ , \qquad z = (r,\theta) \ , \quad z' = (r', \theta') \in \Omega_2 \ .$$

Moreover, we have (refer to [19])

$$\frac{\partial^2}{\partial n \partial n'} V(z,z') = \frac{1}{4\pi \sin^2((\theta - \theta')/2)} \ , \qquad z = (r,\theta) \ , \quad z' = (r', \theta') \in \Gamma \ .$$

It is worth pointing out that these explicit expressions ensure the practical use of the natural boundary reduction method in practical computation. And these expressions also imply another advantage of the natural boundary reduction method compared with many other approaches: we need not to solve any boundary integral equation associated with the unbounded subdomain and instead only calculation of certain singular integrations is needed.

Remark 2.3 *In order to show how to calculate the singular integrations involved in the bilinear form, we divide the artificial boundary Γ into m circular arcs with the same length. Let $\{\phi_i\}_{i=1}^m$ be the set of the nodal basis functions on Γ. Noticing that, in polar* *coordinates (r,θ), the nodal basis functions associated with Γ are piecewise linear with respect to the variable θ, we can obtain (refer to [19])*

$$\begin{aligned} < K\phi_i, \phi_j >_\Gamma &= -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\phi_i(\theta)\phi_j(\theta')}{\sin^2((\theta-\theta')/2)}d\theta d\theta' \\ &= \frac{4m^2}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^3}\sin^4\frac{k\pi}{m}\cos\frac{2k(i-j)\pi}{m}\ , \qquad i,j=1,\cdots,m\ . \end{aligned}$$

From this expression, we can easily find that the stiffness matrix of K is symmetric and circulant, which also allows for more efficiency and implies only small memory storage for such stiffness matrix. Moreover, since the series converges quickly, suitable short finite sum can be used to simplify the calculation.

3 Multigrid algorithm

In this section, we introduce the multigrid algorithm and analyze its convergence.

First, we introduce some multi-level triangulations and notations. Because of the appearance of curved triangles, if we use the usual approach to refine mesh by obtaining the $k+1$ level triangulation $\mathcal{T}_{k+1}$ by dividing the triangle in the k level into four by connecting the midpoint of each edge and construct the corresponding finite element space, then, unfortunately, the resulting spaces are not nested. And obviously non-nested spaces will cause some additional difficulty and trouble for the analysis of the convergence of the multigrid algorithm. In order to avoid this additional difficulty caused by non-nested spaces, we do not use the usual approach to refine mesh, but introduce another approach by employing the initial triangulation as a parametrization of Ω_1 and obtaining the refinement step from subdividing the reference triangle which leads to a sequence of nested spaces. (refer to [15])

More precisely, let Γ be parameterized by a 1-periodic function $\psi : [0,1] \to \Gamma$ such that

$$\beta(z) := |\psi'(z)| > 0$$

for all $z \in [0,1]$. And let $0 = z_0^{(1)} < z_1^{(1)} < \cdots < z_{N_1}^{(1)} = 1$, $N_1 \in \mathbb{N}$, be a uniform partition of $[0,1]$ with $z_i^{(1)} - z_{i-1}^{(1)} = h_1 := 1/N_1$, $i = 1,2,\cdots,N_1$. We denote by Ω_{h1} the polygonal domain whose vertices on Γ are $\psi(z_0^{(1)}), \psi(z_1^{(1)}), \cdots, \psi(z_{N_1}^{(1)})$. Let $\mathcal{T}_1$ be a regular triangulation of $\bar{\Omega}_{h1}$ by triangles of diameter satisfying $diam\ \tau_i \le h_1 \sup_{z\in[0,1]}\beta(z)$ for all $\tau_i \in \mathcal{T}_1$. Then there exists affine mapping G_i such that $G_i(\hat{\tau}) = \tau_i$ for each $\tau_i \in \mathcal{T}_1$ where $\hat{\tau} = \triangle((0,0),(1,0),(0,1))$ is the reference triangle. Next, we replace each triangle $\tau_i \in \mathcal{T}_1$ with two vertices on Γ by the corresponding curved triangle. Without loss of generality we may suppose that the vertices $p_0,\ p_1,\ p_2$ of a curved triangle τ_i satisfy $p_1 = \psi(z_j^{(1)}),\ p_2 = \psi(z_{j+1}^{(1)})$, respectively. Then, a C^∞-mapping $\tilde{G}_i$ with $\tilde{G}_i(\hat{\tau}) = \tau_i$ is given by (refer to [20])

$$\tilde{G}_i = G_i + U_i$$

with

$$U_i(t) = \frac{t_1}{1-t_2}[\psi((1-t_2)z_j^{(1)} + t_2 z_{j+1}^{(1)}) - (1-t_2)\psi(z_j^{(1)}) - t_2\psi(z_{j+1}^{(1)})]\ .$$

We denote this initial triangulation with non-curved and curved triangles by $\tilde{\mathcal{T}}_1$. Subdividing in the usual way the reference triangle $\hat{\tau}$ in $4,\ 16,\ 64,\ \cdots$ triangles yields a sequence of meshes

$$\tilde{\mathcal{T}}_1 \subset \tilde{\mathcal{T}}_2 \subset \tilde{\mathcal{T}}_3 \subset \cdots$$

with step width $diam\ \tau_i \leq h_j \sup_{t\in[0,1]} \beta(t)$ for all $\tau_i \in \tilde{\mathcal{T}}_j$, where $h_j = 2^{-j} h_1$.

The finite element spaces on these meshes are considered to be piecewise linear and continuous. Let the degrees of freedom on the mesh $\tilde{\mathcal{T}}_j$ be $\tilde{N}_j$ and denote the corresponding nodal basis function by $\phi_{j,k},\ k = 1, 2, \cdots, \tilde{N}_j$. With the notation $W_j = span\ \{\phi_{j,k},\ k = 1, 2, \cdots, \tilde{N}_j\}$ for $j = 1, 2, \cdots, J$, we obtain

$$W_1 \subset W_2 \subset \cdots \subset W_J \subset H^1_*(\Omega_1)\ .$$

Then the corresponding J-level discrete variational problem of (2.9) is:
Find $u_J \in W_J$ such that

$$b(u_J, v) = \int\int_{\Omega_1} f v dx dy\ , \qquad \forall v \in W_J\ . \tag{3.1}$$

It is not difficult to obtain that (3.1) has a unique solution (refer to [19]). And the corresponding error estimates in H^1, L^2 and L^∞ norm can be found in [19].

In what follows, we denote c or C with or without subscript a generic positive constant, which can take different values in different occurrences but always be independent of the mesh size and the number of levels.

Define operators $A_k : W_k \mapsto W_k,\ \hat{A}_k : W_k \mapsto W_k,\ S_k : W_J \mapsto W_k,\ \hat{S}_k : W_J \mapsto W_k$ and $T_k : W_J \mapsto W_k,\ k = 1, 2, \cdots, J$, by

$$(A_k w, v) = b(w, v)\ , \qquad \forall w, v \in W_k\ , \tag{3.2}$$

$$(\hat{A}_k w, v) = a_1(w, v)\ , \qquad \forall w, v \in W_k\ , \tag{3.3}$$

$$b(S_k w, v) = b(w, v)\ , \qquad \forall w \in W_J,\ v \in W_k\ , \tag{3.4}$$

$$a_1(\hat{S}_k w, v) = a_1(w, v)\ , \qquad \forall w \in W_J,\ v \in W_k\ , \tag{3.5}$$

$$(T_k w, v) = (w, v)\ , \qquad \forall w \in W_J,\ v \in W_k\ . \tag{3.6}$$

From above definitions, we can easily obtain

$$T_k A_J = A_k S_k\ , \tag{3.7}$$

and

$$T_k \hat{A}_J = \hat{A}_k \hat{S}_k\ . \tag{3.8}$$

Let Q_k be a certain smoother, then the V-cycle multigrid algorithm can be described as follows:

Algorithm 3.1

Set $B_1 = A_1^{-1}$. For $k > 1$ define $B_k : W_k \mapsto W_k$ in terms of B_{k-1} as follows:
Let $g \in W_k$,
1. Set $x_0 = 0$.

2. Define x_i for $i = 1, 2, \cdots, m(k)$ by

$$x_i = x_{i-1} + Q_k^t(g - A_k x_{i-1}) \ .$$

3. Set $y_{m(k)} = x_{m(k)} + q_k$, where q_k is defined by

$$q_k = B_{k-1} T_{k-1}(g - A_k x_{m(k)}) \ .$$

4. Define y_i for $i = m(k)+1, m(k)+2, \cdots, 2m(k)$ by

$$y_i = y_{i-1} + Q_k(g - A_k y_{i-1}) \ .$$

5. Set $B_k g = y_{2m(k)}$.

where Q_k^t denotes the adjoint of Q_k with respect to inner product $(\cdot,\cdot)$ and we take $m(k) = 1$ for all k which is suffices in our analysis. The case $m(k) > 1$ and the cases with only pre-smoothing or post-smoothing can be analyzed similarly.

Let $P_k = I - Q_k A_k, k = 1, 2, \cdots, J$, $D_k = Q_k A_k S_k$ for $k > 1$ and $D_1 = S_1$, then it is easy to check that the error operator associated with the discretization equation

$$A_k u = f \tag{3.9}$$

is given by

$$\tilde{E}_k = I - B_k A_k S_k = E_k E_k^* \tag{3.10}$$

where the superscript $*$ denotes the adjoint with respect to the inner product $b(\cdot,\cdot)$ and

$$E_k = (I - D_k)(I - D_{k-1}) \cdots (I - D_1) \ . \tag{3.11}$$

In order to analyze the convergence of the multigrid algorithm, we make some assumptions, which will be verified later. Let $\tilde{D}_k = A_k S_k / \lambda_k = T_k A_J / \lambda_k$ for $k > 1$ and $\tilde{D}_1 = S_1$, where λ_k denotes the largest eigenvalue of A_k.

(A1) There exists a constant $C_b > 0$ independent of k such that

$$b(v, v) \le C_b \sum_{k=0}^{J} b(\tilde{D}_k v, v) \ , \qquad \forall v \in W_J \ . \tag{3.12}$$

(A2) There exist $0 < \zeta < 1$ and $\widetilde{C} > 0$ independent of k such that

$$b(\tilde{D}_k w, w) \le (\widetilde{C} \zeta^{k-j})^2 b(w, w) \ , \qquad \forall w \in W_j, j \le k \ . \tag{3.13}$$

For the smoother Q_k, we assume the following two condition are satisfied.

(A3) There exits a constant $C_Q \ge 1$ independent of k such that

$$\frac{(v, v)}{\lambda_k} \le C_Q(\bar{Q}_k v, v) \ , \qquad \forall v \in W_k \ , \tag{3.14}$$

where $\bar{Q}_k = (I - P_k^* P_k) A_k^{-1}$.

(A4) There exists a positive constant $\sigma < 2$ independent of k satisfying

$$b(D_k v, D_k v) \le \sigma b(D_k v, v) \ , \qquad \forall v \in W_J \ . \tag{3.15}$$

With these assumptions, we can obtain the convergence theorem of multigrid algorithm given below by following the frame work of [1]. For the self-containedness of this paper, we still provide a proof here.

Theorem 3.1 *If (A1), (A2), (A3) and (A4) are satisfied, then there exists a positive constant $\delta < 1$ independent of h and J such that*

$$0 \le b((I - B_J A_J)v, v) \le \delta b(v, v) \ , \qquad \forall u \in W_J \ . \tag{3.16}$$

Proof. From (3.10), it is obvious that the lower inequality holds since

$$b((I - B_J A_J)v, v) = b(E_J^* v, E_J^* v) := |||E_J^* v|||^2 \ge 0 \ .$$

And by the fact that $|||E_J^* v||| = |||E_J v|||$, we only need to estimate $|||E_J v|||$ for the upper inequality.

From (3.11), we get

$$E_k = (I - D_k) E_{k-1} \ , \tag{3.17}$$

from which follows

$$b(E_k v, E_k v) = b(E_{k-1} v, E_{k-1} v) - 2b(E_{k-1} v, D_k E_{k-1} v) + b(D_k E_{k-1} v, D_k E_{k-1} v) \ ,$$

i.e.

$$b(E_{k-1} v, E_{k-1} v) - b(E_k v, E_k v) = b((2I - D_k) E_{k-1} v, D_k E_{k-1} v) \ . \tag{3.18}$$

Let $E_0 = I$. Then it follows from (3.18) that

$$b(v, v) - b(E_J v, E_J v) = \sum_{i=1}^{J} b((2I - D_k) E_{k-1} v, D_k E_{k-1} v) \ . \tag{3.19}$$

Define $\bar{D}_k = \bar{Q}_k A_k S_k = (I - P_k^* P_k) S_k$ for $k > 1$ and $\bar{D}_k = S_1$. From $P_k = I - Q_k A_k$ and the definition of P_k^*, it is easy to check that $P_k^* = I - Q_k^t A_k$. Combining this with (3.2), (3.4) and the definition of D_k, we have

$$\begin{aligned}
b(\bar{D}_k E_{k-1} v, E_{k-1} v) &= b((I - (I - Q_k^t A_k)(I - Q_k A_k)) S_k E_{k-1} v, E_{k-1} v) \\
&= b((Q_k^t + Q_k) A_k S_k E_{k-1} v, E_{k-1} v) \\
&\quad - b(Q_k^t A_k Q_k A_k S_k E_{k-1} v, E_{k-1} v) \\
&= b((Q_k^t + Q_k) A_k S_k E_{k-1} v, S_k E_{k-1} v) \\
&\quad - b(Q_k^t A_k Q_k A_k S_k E_{k-1} v, S_k E_{k-1} v) \\
&= ((Q_k^t + Q_k) A_k S_k E_{k-1} v, A_k S_k E_{k-1} v) \\
&\quad - (Q_k^t A_k Q_k A_k S_k E_{k-1} v, A_k S_k E_{k-1} v) \\
&= 2(A_k S_k E_{k-1} v, Q_k A_k S_k E_{k-1} v) \\
&\quad - (A_k Q_k A_k S_k E_{k-1} v, Q_k A_k S_k E_{k-1} v) \\
&= b((2I - D_k) E_{k-1} v, D_k E_{k-1} v) \ .
\end{aligned} \tag{3.20}$$

Thus, (3.19) and (3.20) imply

$$b(v, v) - b(E_J v, E_J v) = \sum_{i=1}^{J} b(\bar{D}_k E_{k-1} v, E_{k-1} v) \ . \tag{3.21}$$

From (A1), (3.2), (3.4), triangle inequality, the fact $\tilde{D}_1 = \bar{D}_1 = S_1$ and $E_0 = I$, we have

$$\begin{aligned} b(v,v) &\le C_b \textstyle\sum_{i=1}^J b(\tilde{D}_k v, v) \\ &= C_b[b(\tilde{D}_1 v, v) + \textstyle\sum_{i=2}^J b(A_k S_k v, v)/\lambda_k] \\ &= C_b[b(\tilde{D}_1 v, v) + \textstyle\sum_{i=2}^J \|A_k S_k v\|_0^2/\lambda_k] \\ &\le C_b[b(\tilde{D}_1 v, v) + 2\textstyle\sum_{i=2}^J (\|A_k S_k E_{k-1} v\|_0^2 + \|A_k S_k (I-E_{k-1})v\|_0^2)/\lambda_k] \\ &= C_b[b(\bar{D}_1 E_0 v, E_0 v) + 2\textstyle\sum_{i=2}^J \|A_k S_k E_{k-1} v\|_0^2/\lambda_k \\ &\quad +2\textstyle\sum_{i=2}^J b(A_k S_k (I-E_{k-1})v, (I-E_{k-1})v)/\lambda_k] \\ &\le 2C_b[b(\bar{D}_1 E_0 v, E_0 v) + \textstyle\sum_{i=2}^J \|A_k S_k E_{k-1} v\|_0^2/\lambda_k \\ &\quad + \textstyle\sum_{i=2}^J b(\tilde{D}_k (I-E_{k-1})v, (I-E_{k-1})v)]\ . \end{aligned} \tag{3.22}$$

For $k = 2, 3, \cdots, J$, (A3), (3.2) and (3.4) imply that

$$\begin{aligned} \textstyle\sum_{i=2}^J \|A_k S_k E_{k-1} v\|_0^2/\lambda_k &\le C_Q \textstyle\sum_{i=2}^J (\bar{Q}_k A_k S_k E_{k-1} v, A_k S_k E_{k-1} v) \\ &= C_Q \textstyle\sum_{i=2}^J b(\bar{Q}_k A_k S_k E_{k-1} v, E_{k-1} v) \\ &= C_Q \textstyle\sum_{i=2}^J b(\bar{D}_k E_{k-1} v, E_{k-1} v)\ . \end{aligned} \tag{3.23}$$

Let $\tilde{v} = (I - E_{k-1})v$. Noting (3.17), we get

$$E_{i-1} - E_i = D_i E_{i-1}\ ,$$

from which follows

$$I - E_k = \sum_{i=1}^k D_i E_{i-1}\ . \tag{3.24}$$

Let $w_i = D_i E_{i-1} v$. By (3.24), (3.2), (3.4), Cauchy-Schwarz inequality, (A2) and (A4), it follows

$$\begin{aligned} \textstyle\sum_{k=2}^J b(\tilde{D}_k \tilde{v}, \tilde{v}) &\le \textstyle\sum_{k=2}^J \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} b(\tilde{D}_k D_i E_{i-1} v, D_j E_{j-1} v) \\ &= \textstyle\sum_{k=2}^J \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (A_k S_k D_i E_{i-1} v, A_k S_k D_j E_{j-1} v)/\lambda_k \\ &\le \textstyle\sum_{k=2}^J \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \|A_k S_k D_i E_{i-1} v\|_0 \|A_k S_k D_j E_{j-1} v\|_0/\lambda_k \\ &= \textstyle\sum_{k=2}^J \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} b(\tilde{D}_k w_i, w_i)^{1/2} b(\tilde{D}_k w_j, w_j)^{1/2} \\ &\le \tilde{C}^2 \textstyle\sum_{k=2}^J \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \zeta^{2k-i-j} b(w_i, w_i)^{1/2} b(w_j, w_j)^{1/2} \\ &\le \tilde{C}^2 \textstyle\sum_{k=2}^J \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \zeta^{2k-i-j} [b(w_i, w_i) + b(w_j, w_j)]/2 \\ &= \tilde{C}^2 \textstyle\sum_{k=2}^J \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \zeta^{k-j} \zeta^{k-i} b(w_i, w_i) \\ &\le \tfrac{\zeta \tilde{C}^2}{1-\zeta} \textstyle\sum_{k=2}^J \sum_{i=1}^{k-1} \zeta^{k-i} b(w_i, w_i) \\ &= \tfrac{\zeta \tilde{C}^2}{1-\zeta} \textstyle\sum_{i=1}^{J-1} \sum_{k=i+1}^{J} \zeta^{k-i} b(w_i, w_i) \\ &\le \tfrac{\zeta^2 \tilde{C}^2}{(1-\zeta)^2} \textstyle\sum_{i=1}^{J-1} b(D_i E_{i-1} v, D_i E_{i-1} v) \\ &\le \tfrac{\sigma \zeta^2 \tilde{C}^2}{(1-\zeta)^2} \textstyle\sum_{i=1}^{J-1} b(D_i E_{i-1} v, E_{i-1} v)\ . \end{aligned} \tag{3.25}$$

On the other hand, from (3.20) and (A4), we have

$$\begin{aligned} b(\bar{D}_k E_{i-1} v, E_{i-1} v) &= b((2I - D_k) E_{i-1} v, D_k E_{i-1} v) \\ &\ge (2-\sigma) b(D_k E_{i-1} v, E_{i-1} v)\ . \end{aligned} \tag{3.26}$$

Noting $b(\bar{D}_J E_{J-1}v, E_{J-1}v) = (\bar{Q}_J A_J E_{J-1}v, A_J E_{J-1}v) \geq 0$, with (3.25) and (3.26), we obtain

$$\begin{array}{ll} \sum_{k=2}^{J} b(\tilde{D}_k \tilde{v}, \tilde{v}) & \leq \frac{\sigma\zeta^2\tilde{C}^2}{(1-\zeta)^2} \sum_{i=1}^{J-1} b(D_i E_{i-1}v, E_{i-1}v) \\ & \leq \frac{\sigma}{2-\sigma}(\frac{\zeta}{1-\zeta})^2 \tilde{C}^2 \sum_{k=1}^{J-1} b(\bar{D}_k E_{k-1}v, E_{k-1}v) \\ & \leq \frac{\sigma}{2-\sigma}(\frac{\zeta}{1-\zeta})^2 \tilde{C}^2 \sum_{k=1}^{J} b(\bar{D}_k E_{k-1}v, E_{k-1}v) \ . \end{array} \tag{3.27}$$

Thus, from (3.22), (3.23), (3.27) and (3.21), it follows

$$\begin{array}{ll} b(v,v) & \leq C_m \sum_{k=1}^{J} b(\bar{D}_k E_{k-1}v, E_{k-1}v) \\ & = C_m[b(v,v) - b(E_J v, E_J v)] \end{array}$$

where $C_m = 2C_b[C_Q + \frac{\sigma}{2-\sigma}(\frac{\zeta}{1-\zeta})^2\tilde{C}^2]$. Let $\delta = 1 - 1/C_m < 1$, then

$$b(E_J v, E_J v) \leq (1 - 1/C_m) b(v,v) = \delta b(v,v) \ .$$

This completes the proof. $\square$

With Theorem 3.1, it is obvious that we only need to verify (A1), (A2), (A3) and (A4) to achieve the convergence of the multigrid algorithm. To this end, we still need some notations. Let $\hat{D}_k = \hat{A}_k \hat{S}_k / \hat{\lambda}_k = T_k \hat{A}_J / \hat{\lambda}_k$ for $k > 1$ and $\hat{D}_1 = \hat{S}_1$, where $\hat{\lambda}_k$ denotes the largest eigenvalue of $\hat{A}_k$. It is well known that (A1) and (A2) hold for $a_1(\cdot,\cdot)$ which are denoted as (A1D) and (A2D) respectively here (refer to [1]).

(A1D) There exists a constant $C_a > 0$ independent of k such that

$$a_1(v,v) \leq C_a \sum_{k=0}^{J} a_1(\hat{D}_k v, v) \ , \qquad \forall v \in W_J \ , \tag{3.28}$$

(A2D) There exist $0 < \zeta_a < 1$ and $\widetilde{C}_a > 0$ independent of k such that

$$a_1(\hat{D}_k w, w) \leq (\widetilde{C}_a \zeta_a^{k-j})^2 a_1(w,w) \ , \qquad \forall w \in W_j, j \leq k \ . \tag{3.29}$$

Next, we show that (A.1) and (A.2) also hold for $b(\cdot,\cdot)$. First, some lemmas are introduced.

Let $\hat{w}$ denote the discrete harmonic extension of $w|_\Gamma$, which is defined by

$$\begin{cases} a_1(\hat{w}, \hat{v}) = 0 \ , & \forall \hat{v} \in W_J^0 \ , \\ \hat{w}|_{\partial\Omega} = 0 \ , & \hat{w}|_\Gamma = w|_\Gamma \ , \end{cases}$$

where $W_J^0 = \{v | \ \ v \in W_J \ , v|_\Gamma = 0\}$. Then, it follows (refer to [13])

Lemma 3.1 *For any $w \in W_J$, we have*

$$< Kw, w >_\Gamma \leq |\hat{w}|_{1,\Omega_1} \ . \tag{3.30}$$

With this lemma, we can obtain the following lemma.

Lemma 3.2 *For any $w \in W_J$, we have*

$$a_1(w,w) \leq b(w,w) \leq C_0 a_1(w,w) \ . \tag{3.31}$$

Proof. Since the lower inequality is obvious, we only need to prove the upper inequality. To this end, let $\tilde{w} \in W_J$ be the discrete harmonic extension of $w|_\Gamma$, which is defined by

$$\begin{cases} a_1(\tilde{w}, \psi) = 0 \ , & \forall \psi \in W_J^0 \ , \\ \tilde{w}|_{\partial\Omega} = 0 \ , & \tilde{w}|_\Gamma = w|_\Gamma \ . \end{cases}$$

Set $w_1 = w - \tilde{w} \in W_J^0$. Then we get

$$a_1(w, w) = |w_1|^2_{1,\Omega_1} + |\tilde{w}|^2_{1,\Omega_1} \ , \tag{3.32}$$

and

$$b(w, w) = |w_1|^2_{1,\Omega_1} + |\tilde{w}|^2_{1,\Omega_1} + < Kw, w >_\Gamma \ . \tag{3.33}$$

With (3.32), (3.33) and Lemma 3.1, we obtain

$$\begin{aligned} b(w, w) &= |w_1|^2_{1,\Omega_1} + |\tilde{w}|^2_{1,\Omega_1} + < Kw, w >_\Gamma \\ &\le |w_1|^2_{1,\Omega_1} + 2|\tilde{w}|^2_{1,\Omega_1} \\ &\le C_0 a_1(w, w) \ . \end{aligned}$$

This completes the proof. □

The following two lemmas can be found in [1].

Lemma 3.3 *Suppose $\bar{A}$ and $\tilde{A}$ are two symmetric positive definite operators on W_J. Then for all $w \in W_J$,*

$$C_1(\bar{A}w, w) \le (\tilde{A}w, w) \le C_2(\bar{A}w, w) \tag{3.34}$$

if and only if

$$C_1(\tilde{A}^{-1}w, w) \le (\bar{A}^{-1}w, w) \le C_2(\tilde{A}^{-1}w, w) \tag{3.35}$$

where C_1 and C_2 are the same constants in both inequalities.

Lemma 3.4 *Assume that two symmetric positive definite operators $\bar{A}$ and $\tilde{A}$ on W_J with corresponding bilinear forms $\bar{a}(\cdot, \cdot)$ and $\tilde{a}(\cdot, \cdot)$ such that*

$$C_1 \tilde{a}(w, w) \le \bar{a}(w, w) \le C_2 \tilde{a}(w, w) \qquad \forall w \in W_J \ . \tag{3.36}$$

Then (A1) holds for $\tilde{A}$ if and only if (A1) holds for $\bar{A}$.

With the help of Lemma 3.2, lemma 3.4 and (A1D), we obtain

Theorem 3.2 *(A.1) holds for $b(\cdot, \cdot)$.*

Next, we show that assumption (A.2) also holds for $b(\cdot, \cdot)$.

Theorem 3.3 *(A.2) holds for $b(\cdot, \cdot)$.*

Proof. For $k = 1$, there is nothing to prove. For $k > 1$, with (3.3), (3.5), (3.8), (3.2), (3.4) and (3.7), we have

$$a_1(\hat{D}_k w, w) = \hat{\lambda}_k^{-1} \|\hat{A}_k \hat{S}_k w\|_0^2 = \hat{\lambda}_k^{-1} \|T_k \hat{A}_J w\|_0^2 \tag{3.37}$$

and

$$b(\tilde{D}_k w, w) = \lambda_k^{-1} \|A_k S_k w\|_0^2 = \lambda_k^{-1} \|T_k A_J w\|_0^2 \ . \tag{3.38}$$

Then , from (3.37) and (A2D), we have

$$\begin{aligned} \hat{\lambda}_k^{-1} \|T_k \hat{A}_J w\|_0^2 &= a_1(\hat{D}_k w, w) \\ &\le (\widetilde{C}_a \zeta_a^{k-j})^2 a_1(w, w) \\ &= (\widetilde{C}_a \zeta_a^{k-j})^2 (\hat{A}_J^{-1} \hat{A}_J w, \hat{A}_J w) \ . \end{aligned} \tag{3.39}$$

Set $v = \hat{A}_J w$. Then, it follows

$$\hat{\lambda}_k^{-1} \|T_k v\|_0^2 \le (\widetilde{C}_a \zeta_a^{k-j})^2 (\hat{A}_J^{-1} v, v) \ .$$

From the above inequality, Lemma 3.2 and Lemma 3.3, we obtain

$$\begin{aligned} \lambda_k^{-1} \|T_k v\|_0^2 &\le \hat{\lambda}_k^{-1} \|T_k v\|_0^2 \\ &\le (\widetilde{C}_a \zeta_a^{k-j})^2 (\hat{A}_J^{-1} v, v) \\ &\le C_2 (\widetilde{C}_a \zeta_a^{k-j})^2 (A_J^{-1} v, v) \ . \end{aligned}$$

Let $\widetilde{C} = C_2^{1/2} \widetilde{C}_a$, $\zeta = \zeta_a$ and $v = A_J w$ in the above inequality, we obtain from (3.38) that

$$\begin{aligned} b(\tilde{D}_k w, w) &= \lambda_k^{-1} \|T_k A_J w\|_0^2 \\ &\le (\widetilde{C} \zeta^{k-j})^2 b(w, w) \end{aligned}$$

This completes the proof. □

Next, we introduce some smoothers such that (A3) and (A4) are satisfied. Due to the appearance of the item $< Kw, w >_\Gamma$ in the bilinear form, it makes the nodal basis function on the artificial boundary do not have local support and results that smoothers constructed in [1] can not be used directly because they may not satisfy (A3) or (A4) any more in this case. So, for the smoothers, we should choose and check carefully to overcome this difficulty.

Before presenting smoothers satisfying (A3) and (A4), we make some insight of smoothers of the form $Q_k = \frac{\mu}{\lambda_k} I$, where μ is a parameter. In the following, we will discuss the condition under which (A3) and (A4) are satisfied by this kind of smoothers.

First, we check the assumption (A3). Noting that for this kind of smoother, we have $P_k = I - \frac{\mu}{\lambda_k} A_k$ and

$$\begin{aligned} b((I - P_k^* P_k) v, v) &= b(v, v) - b(P_k v, P_k v) \\ &= b(v, v) - [b(v, v) - \tfrac{2\mu}{\lambda_k} b(A_k v, v) + \tfrac{\mu^2}{\lambda_k^2} b(A_k v, A_k v)] \\ &= \tfrac{2\mu}{\lambda_k} \|A_k v\|_0^2 - \tfrac{\mu^2}{\lambda_k^2} b(A_k v, A_k v) \ . \end{aligned} \tag{3.40}$$

From the fact that $b(A_kv, A_kv) \le \lambda_k \|A_kv\|_0^2$, it follows

$$\frac{\mu^2}{\lambda_k^2} b(A_kv, A_kv) \le \frac{\mu^2}{\lambda_k^2} \lambda_k \|A_kv\|_0^2 = \frac{\mu^2}{\lambda_k} \|A_kv\|_0^2 \ . \tag{3.41}$$

Thus, with (3.40) and (3.41), we obtain

$$b((I - P_k^* P_k)v, v) \ge \frac{\mu(2-\mu)}{\lambda_k} \|A_kv\|_0^2 \ . \tag{3.42}$$

For $0 < \mu < 2$, we chose $C_Q = 1/(\mu(2-\mu)) \ge 1$. Then, it follows from (3.42) that

$$\begin{aligned} C_Q(\bar{Q}A_kv, A_kv) \ &= C_Q((I - P_k^* P_k) A_k^{-1} A_kv, A_kv) \\ &= C_Q b((I - P_k^* P_k)v, v) \\ &\ge C_Q \tfrac{\mu(2-\mu)}{\lambda_k} \|A_kv\|_0^2 \\ &= \|A_kv\|_0^2 / \lambda_k \ . \end{aligned}$$

Setting $w = A_kv$, we have (A3) holds for this kind of smoother.

To check (A4), we notice that for this kind of smoother, it follows from (3.2) and (3.4) that

$$b(D_kv, D_kv) = \frac{\mu^2}{\lambda_k^2} b(A_kS_kv, A_kS_kv) \tag{3.43}$$

and

$$\begin{aligned} b(D_kv, v) \ &= \mu b(A_kS_kv, v)/\lambda_k \\ &= \mu b(A_kS_kv, S_kv)/\lambda_k \\ &= \mu \|A_kS_kv\|_0^2 / \lambda_k \ . \end{aligned} \tag{3.44}$$

Since $b(A_kS_kv, A_kS_kv) \le \lambda_k \|A_kS_kv\|_0^2$, we obtain from (3.43) and (3.44) that

$$\begin{aligned} b(D_kv, D_kv) \ &= \tfrac{\mu^2}{\lambda_k^2} b(A_kS_kv, A_kS_kv) \\ &\le \tfrac{\mu^2}{\lambda_k^2} \lambda_k \|A_kS_kv\|_0^2 \\ &= \mu b(D_kv, v) \ . \end{aligned}$$

Taking $\sigma = \mu$, for $0 < \mu < 2$, we get that (A4) holds.

Thus, we obtain

Theorem 3.4 *For $0 < \mu < 2$, smoothers of the form $Q_k = \frac{\mu}{\lambda_k} I$ satisfy (A3) and (A4).*

Remark 3.1 *Since the largest eigenvalue of the matrix is involved in the construct of this kind of smoothers, which is not easy to obtain in practical computation, it makes some difficulty in using this kind of smoothers directly in practical computation. But this theorem is still important and useful in both construction smoothers for practical computation and providing us with a better understanding of the role the smoother plays in the convergence of the multigrid algorithm (see the analysis below).*

Let us consider smoothers of the form

$$Q_k = \frac{1}{\eta} I \ . \tag{3.45}$$

In order to get smoother of this kind such that (A3) and (A4) are satisfied, η should satisfy some condition. Next, we will give some conditions of this kind based on Theorem 3.4.

In this paper, two cases are considered.

The first one is the case $\eta \geq \lambda_k$. In this case, it is obvious that there exists a positive constant $\mu \leq 1$ such that $\frac{1}{\eta} = \frac{\mu}{\lambda_k}$. From Theorem 3.4, it follows the desired smoothers. This case is of great importance for the practical computation and many practical smoothers can be obtained from this case. As mentioned above, it is not easy to get the largest eigenvalue of the matrix, but a upper bound of the largest eigenvalue of the matrix can be easily obtained in many different ways and by using many different methods. All these upper bounds can be used to construct smoothers of the form (3.45) for the purpose of the practical computation.

The second case is $\frac{\lambda_k}{2} < \eta \leq \lambda_k$. In this case, we can see that there exists a constant $1 \leq \mu < 2$ such that $\frac{1}{\eta} = \frac{\mu}{\lambda_k}$. Also, for this case, we still can obtain desired smoothers. This case indicates that the multigrid algorithm is still convergent for any upper bounds η of $\lambda_k/2$ even if $\eta < \lambda_k$. It is a really interesting result, which may provide us a better insight and understanding of the role the smoother plays in the convergence of the multigrid algorithm. On the other hand, it also implies that if you can obtain some upper bounds of $\lambda_k/2$ in any way or by using any methods, all these bounds can also be used to construct smoothers of form (3.45) for practical use and the convergence of the multigrid algorithm is still ensured in this case. So, it is quite interesting and useful both in the theory and in the practical computation.

To sum up, we obtain the following theorem

Theorem 3.5 *Let $\frac{\lambda_k}{2} < \eta$, then smoothers of the form (3.45) satisfy (A3) and (A4).*

With Theorem 3.2, Theorem 3.3, Theorem 3.5 and Theorem 3.1, we complete the construction and analysis of multigrid algorithm.

4 Numerical results

Let us consider the following model problem for our numerical experiment

$$\begin{cases} -\triangle u &= f, \quad in \ \ \Omega^c, \\ \quad\ u &= 0, \quad on \ \ \partial\Omega, \end{cases} \tag{4.1}$$

subject to the asymptotic conditions

$$u(x,y) = \alpha + O(1/r) \ , \qquad |\nabla u(x,y)| = O(1/r^2) \ , \qquad r = \sqrt{x^2+y^2} \to \infty \ ,$$

where

$$f = \begin{cases} \frac{4}{(x^2+y^2)^2} \ , & 1 < x^2+y^2 < \frac{9}{4} \ , \\ 0 \ , & \frac{9}{4} \leq x^2+y^2 \ , \end{cases}$$

Table 1: Numerical experiments for $J = 3$

m	$\mathcal{N}$	$\|u - u_h\|_D$	ITn
128	2048	2.9060e-3	10
256	8192	7.4122e-4	10
512	32768	1.8700e-4	9
1024	131072	4.6633e-5	8

Table 2: Numerical experiments for $J = 4$

m	$\mathcal{N}$	$\|u - u_h\|_D$	ITn
128	2048	2.9354e-3	12
256	8192	7.5595e-4	11
512	32768	1.9390e-4	10
1024	131072	4.9087e-5	9
2048	524288	1.2272e-5	8

Ω is unit circle disc and $\alpha = 1$.

We make the coupling at the circle Γ with radius 2. The exact solution of the model problem and the computational solution of the finest level are denoted as u and u_h respectively. The discrete norm $\|\cdot\|_D$ is defined as

$$\|w\|_D = h_J(\sum_i w(x_i)^2)^{\frac{1}{2}}$$

where the sum is taken over all nodes x_i of the finest level finite element space U_J. It is well known that this discrete norm is equivalent to the standard L^2 norm.

In what follows, the number of circular arcs Γ is divided into on the finest level and the number of unknowns of the finest level are denoted as m and $\mathcal{N}$ respectively. ITn stands for the number of iterations needed to achieve the corresponding error $\|u-u_h\|_D$. In all our numerical experiments, multigrid algorithm with only pre-smoothing is used and on all levels only one smoothing is done.

The results for the cases $J = 3$, $J = 4$ and $J = 5$ are presented in Table 1, Table 2 and Table 3 respectively.

From these tables, we find that, for all the cases, the number of iterative steps is independent of the mesh size even if the number of unknowns is very large, which match with our theory well.

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Table 3: Numerical experiments for $J = 5$

m	$\mathcal{N}$	$\|u - u_h\|_D$	ITn
256	8192	8.9339e-4	11
512	32768	2.1598e-4	11
1024	131072	5.8905e-5	10
2048	524288	1.3499e-5	10
4096	2097152	3.0680e-6	10

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