Cascadic Multigrid for Finite Volume Methods for Elliptic Problems

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Abstract

In this paper, some effective cascadic multigrid methods are proposed for solving the large scale symmetric or nonsymmetric algebraic systems arising from the finite volume methods for second order elliptic problems. It is shown that these algorithms are optimal in both accuracy and computational complexity. Numerical experiments are reported to support our theory.

1 Introduction

The finite volume methods or covolume methods have become powerful tools for numerically solving PDEs. They can also be termed as box methods [1], generalized finite difference methods [15]. These methods have a simplicity for implementation comparable to the finite difference methods; on the other hand, they have a flexibility similar to that of finite element methods for handling complicated geometries and boundary conditions. Another important advantage of these methods is that the numerical solutions usually have certain conservation property, which are very desirable in many applications, especially in CFD. For a comprehensive presentation and more references of existing results in this direction, we refer to the monographs [15],[13], for details.

The algebraic systems resulting from the finite volume methods are sparse and ill-conditioned. So we should construct some effective methods like multigrid methods or domain decomposition methods for solving such kind of large scale systems. Although the convergence behavior of multigrid algorithms for standard finite element methods is by now well understood, much less is known for the finite volume element method. Recently, a V-cycle multigrid for the finite volume element method was proposed in [10] by Chou and Kwak. They show that the multigrid is optimal, which means that the convergence rate of this method is independent of the mesh size and mesh level. The aim of this paper is to present some cascadic multigrid algorithms for the discretization systems of the finite volume methods. Compared with usual multigrid, the main advantage of the cascadic multigrid method is its simplicity[2][17]. It requires no coarse grid corrections at all and may be viewed as a "one-way" multigrid method. In recent years, there have been several theoretical analysis and the applications of these methods, cf. [17][19]for nonconforming element methods and plate bending problems, [18] for parabolic problems, [14][20] for nonlinear problems, [5] for Stokes problems, [6] for mortar element methods.

In this paper, we shall first propose a cascadic multigrid algorithm for the symmetric systems resulting from finite volume method approximation of some special second order elliptic equations. In this case, the quadratic forms in different mesh levels are noninherited. We shall show that the cascadic multigrid algorithm holds optimal accuracy and computational complexity. Second, it is known that the algebraic equations arising from the finite volume methods are usually nonsymmetric, which brings us many difficulties for designing an optimal cascadic multigird algorithms. But note that the nonsymmetric equations are a small perturbation of the usual finite element discretization equations. Based on this observation, we shall construct an efficient cascadic multigrid algorithm for this large scale nonsymmetric system. In this algorithm, we shall first solve a small nonsymmetric problem on the coarsest grids which is associated with low frequencies of the discretization system, and then solve symmetric positive definite (SPD) finite element problems on the fine levels. Under this construction, we shall also prove that the cascadic multigrid is optimal in both the accuracy and computational complexity.

The rest of our paper is organized as follows: In Section 2, we give some notations used in this paper and formulate the finite volume element schemes. In Section 3, the cascadic multigrid methods for the symmetric and nonsymmetric systems are analyzed respectively. In the last section, numerical experiments are reported to support our theory.

2 A model problem and finite volume methods

We consider the following self-adjoint elliptic problem

$$\begin{cases} -\nabla\cdot(A\nabla u)+qu &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a plane polygonal domain; $f\in L^2(\Omega)$, $q\in L^\infty(\Omega)$ and $q\geq 0$ almost everywhere in Ω are two given real-valued functions; $A=(a_{ij})_{2\times 2}\in (W^{1,\infty}(\Omega))^4$ is a given real symmetric matrix-valued function. We assume that A satisfies the following ellipticity condition: there exists a constant $\alpha_1>0$ such that

$$\alpha_1\xi^T\xi\leq \xi^T A(\mathbf{x})\xi, \quad \forall\ \xi\in R^2 \text{ and } \mathbf{x}=(x,y)\in\bar{\Omega}. \tag{2.2}$$

In what follows we shall adopt the standard definitions of Sobolev spaces and their norms and semi-norms as presented in [11].

The variational formulation associated with (2.1) is to find $u \in V = H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in V, \tag{2.3}$$

where

$$\begin{aligned} a(u, v) &= \int_\Omega (A\nabla u \cdot \nabla v + quv)dx, \\ (f, v) &= \int_\Omega fvdx. \end{aligned}$$

Under the above assumptions, it is known that (2.3) holds a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Define the energy norm as:

$$\|v\|_a = a(v, v)^{\frac{1}{2}}, \quad \forall v \in H_0^1(\Omega).$$

It is easy to check that this norm is equivalent to the usual norm $\|\cdot\|_1$ over the space $H_0^1(\Omega)$.

In order to present the cascadic multigrid algorithm for finite volume methods, we first construct a sequence of nested triangulations of Ω as follows. Suppose that a coarse triangulation $\mathcal{T}_0$ of Ω is given, we define the finer triangulation $\mathcal{T}_l$ for $l \geq 1$ by subdividing a triangle in $\mathcal{T}_{l-1}$ into four subtriangles by connecting the midpoints of the edges. Assume that the coarse triangulation $\mathcal{T}_0$ is regular and quasi-uniform, then every $\mathcal{T}_l$, $(l >)1$ is regular and quasi-uniform too. Let h_l denote the maximum mesh size of $\mathcal{T}_l$, then $h_l = \frac{h_{l-1}}{2}$. Let L be an integer greater than or equal to zero, and for $l = 0, 1, \cdots, L$, construct the usual piecewise linear conforming finite element space V_l on $\mathcal{T}_l$ as

$$V_l = \{v \in C^0(\bar{\Omega}) : \ v|_K \text{ is linear}, \ \forall \ K \in \mathcal{T}_l, v = 0 \text{ on } \partial\Omega\}$$

Since the triangulations are nested, it follows that

$$V_0 \subset V_1 \subset \cdots \subset V_L.$$

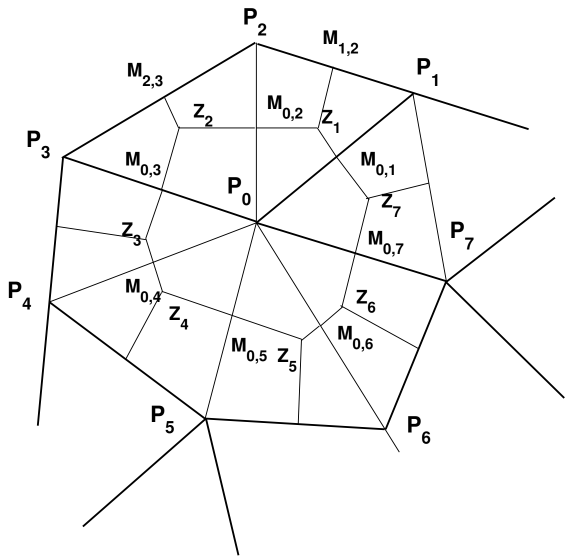
Then the standard finite element approximation of (2.3) is to find $\bar{u}_l \in V_l$ such that

$$a(\bar{u}_l, v) = (f, v), \ \ \forall \ v \in \ V_l. \tag{2.4}$$

Next, we describe the finite volume methods. First, associated with the primal partition $\mathcal{T}_l$, we define its dual partition $\mathcal{T}_l^*$ as follows: choose any interior point or median, Z_K of $K \in \mathcal{T}_l$. Let P_0 be an interior node and P_i be its adjacent nodes, $M_i = M_{0i}$ be the midpoint of $\overline{P_0P_i}$. There are two well-known duals, i.e., the so called Voronoi meshes (Z_K chosen as the circumcenters) and the barycentric dual (Donald dual) in which Z_K are barycenters of K. In Figure 2.1, we choose Z_K as the circumcenter of K, then connect successively the points $M_1, Z_1, M_2, Z_2, \cdots, M_7, Z_7, M_1$ to obtain the dual polygonal element $K_{P_0}^*$. The dual element $K_{P_2}^*$ based at a typical boundary node P_2 is defined by restricting the dual element to the interior of Ω. Then

$$\mathcal{T}_l^* = \{K_{P_i}^* : \; P_i \text{ is the vertex of } T_l\},$$

which constitutes a dual partition of the domain Ω.



Associated with the partition $\mathcal{T}_l^*$ we define the test function space W_l as the piecewise constants over every dual element and which vanish on $\partial\Omega$. Let χ_P be the characteristic function of the dual element K_P^*, all the characteristic functions on $\mathcal{T}_l^*$ form a basis of the space W_l. Then the finite volume methods of (2.3) is to find $u_l \in V_l$ such that

$$b_l^*(u_l, w_l) = (f, w_l) \qquad \forall \; w_l \in V_l, \tag{2.5}$$

where

$$b_l^*(v_l, w_l) = \sum_{P \in \Omega_l^o} w_l(P) b_l^*(v_l, \chi_P),$$

and

$$b_l^*(v_l, \chi_P) = -\int_{\partial K_P^*} (A\nabla v_l) \cdot \mathbf{n} ds + \int_{K_P^*} q v_l dx,$$

where $\mathbf{n}$ is the unit outward normal to ∂K_P^*. It should be noted that the above formulation is a way of stating that we have an integral conservation form on dual domains using the divergence theorem.

Define a one-to-one operator r_l: $V_l \rightarrow \; W_l$ such that for $v_l \in V_l$

$$r_l v_l = \sum_{p \in \Omega_l^0} v_l(P) \chi_P,$$

where Ω_l^0 is the set of interior nodes of $\mathcal{T}_l$, and it is easy to check that the operator has the following approximation property

$$\|v_l - r_l v_l\|_{0,q} \leq C h_l |v_l|_{1,q}. \qquad q > 1 \tag{2.6}$$

Employing the above defined operator r_l, set

$$a_l^*(u_l, v_l) = b_l^*(u_l, r_l v_l), \quad \forall\ u_l,\ v_l \in V_l,$$

then the finite volume scheme (2.5) can be rewritten as: Find $u_l \in V_l$ such that

$$a_l^*(u_l, v_l) = (f, r_l v_l), \quad \forall v_l \in V_l. \tag{2.7}$$

For the bilinear form $a_l^*(\cdot,\cdot)$, we have that there exist $\beta_0, \beta_1 > 0$ such that [7][12]

$$\beta_0 \|v\|_a \leq a_l^*(v, v) \leq \beta_1 \|v\|_a. \tag{2.8}$$

On the other hand, it holds that[12][9]

Lemma 2.1 *There exists a constant C independent of the mesh size h_l and the level l such that for any $u_l,\ v_l\ \in V_l$,*

$$|a(u_l, v_l) - a_l^*(u_l, v_l)| \leq C h_l \|u_l\|_a \|v_l\|_a. \tag{2.9}$$

From this lemma, we can see that the bilinear form $a_l^*(\cdot,\cdot)$ is a perturbation of the form $a(\cdot,\cdot)$. As a corollary of this result, we have the coerciveness and boundedness of the form $a_l^*(\cdot,\cdot)$. Moreover, it is proved in [12][9][8] that the following error estimates are true.

Theorem 2.1 *Let $u_l,\ u$ be the solutions of (2.7) and (2.3) respectively, then*

$$\|u - u_l\|_a \leq C h_l \|f\|_0. \tag{2.10}$$

Moreover if $f \in\ W^{1,p}(\Omega)$ $p > 1$, it holds

$$\|u - u_l\|_0 \leq C h_l^2 (\|f\|_0 + \|f\|_{1,p}), \tag{2.11}$$

where the constant C is independent of the mesh size h_l and the level l.

Define the operator $A_l^* : V_l \to V_l$ with respect to the bilinear form $a_l^*(\cdot,\cdot)$ as:

$$(A_l^* u, v) = a_l^*(u, v) \quad \forall\ u, v \in V_l. \tag{2.12}$$

Then (2.7) can be rewritten in operator form as

$$A_l^* u_l = f_l, \tag{2.13}$$

where $f_l \in V_l$, $(f_l, v) = (f, r_l v)\ \ \forall v \in V_l$.

3 The Cascadic multigrid algorithms

We know that the algebraic system derived from the finite volume scheme is nonsymmetric in general even if the underlying PDE is symmetric, but there are two exceptions below:

- Case 1. A is a constant symmetric matrix,
- Case 2. $A = c(x)I$, and the point Z_K is chosen to be the circumcenter of K.

In Case 1, the bilinear form $a_l^*(\cdot,\cdot)$ is equivalent to $a(\cdot,\cdot)$, the discussion of its cascadic multigrid scheme reduces to the finite element case. So we only need to discuss the Case 2. The general nonsymmetric case will be discussed in Section 3.2

3.1 The symmetric case

In this case the bilinear form $a_l^*(\cdot,\cdot)$ is symmetric, the cascadic multigrid algorithm for this problem can be written as follows:
Algorithm I
(1) Let $u_0^0 = u_0^* \hat{=} u_0$ be the exact solution of (2.7).
(2) For $l = 1, 2, \cdots, L$, let

$$u_l^0 = I_l u_{l-1}^*.$$

Do iterations for (2.7):

$$u_l^{m_l} = K_l^{m_l} u_l^0.$$

(3) Set $u_l^* \hat{=} u_l^{m_l}$,
where in the above algorithm, $I_l : V_{l-1} \to V_l$ is the natural injection operator, K_l denotes the smoothing operator on the level l, such as the Richardson, Jacobi, Gauss-seidel, or CG iteration. m_l is the number of iteration steps on the level l.

Similar as in the finite element case, we call a cascadic multigrid for the finite volume methods optimal in the energy norm on the level L, if we obtain both the accuracy

$$\|u_L - u_L^*\|_a \approx \|u - u_L\|_a,$$

which means that the iterative error is comparable to the approximation, and the multigrid complexity

$$amount\ of\ work\ = O(n_L), \quad n_L = \dim V_L.$$

Moreover, define the norm

$$\|v\|_l^2 \hat{=} a_l^*(v,v), \quad \forall v \in V_l.$$

By (2.8), we know that this norm is equivalent to the usual energy norm $\|\cdot\|_a$.

It is known that for the smoothing operator mentioned above, there exists a linear operator $T_l : V_l \to V_l$ such that

$$u_l - K_l^{m_l} u_l^0 = T_l^{m_l}(u_l - u_l^0), \tag{3.1}$$

and for the symmetric operator A_l^* it holds that

$$\|T_l^{m_l} v\|_l \leq C \frac{h_l^{-1}}{m_l^{\gamma}} \|v\|_0, \quad \forall v \in V_l, \tag{3.2}$$

$$\|T_l^{m_l} v\|_l \leq \|v\|_l, \quad \forall v \in V_l, \tag{3.3}$$

where γ is a positive number depending on the given iteration. Using fully same arguments as in [2],[3],[4], we can show that (3.2), (3.3) are valid with $\gamma = 1$ for the CG iteration and $\gamma = \frac{1}{2}$ for the other three standard iterations.

Note that in the finite volume methods the test function spaces are not nested because a dual element $K_l^* \in \mathcal{T}_{h_l}^*$ is not a subset of a dual element $K_{l-1}^* \in \mathcal{T}_{h_{l-1}}^*$. Then the associated quadratic form $a_l^*(\cdot,\cdot)$ is noninherited in nature, i.e,

$$a_l^*(I_l v, I_l v) \neq a_{l-1}^*(v,v), \quad v \in V_{l-1}. \tag{3.4}$$

In the following, we will give an example to show that the bilinear form $a_l^*(\cdot,\cdot)$ is indeed noninherited and give an explicit difference between these two quadratic forms.

Example: Consider the problem (2.1) on the domain $\Omega = [0,1] \times [0,1]$ with $q = 0$ and $A = a(x,y)I$, where $a(x,y) = x^2 + y^2 + 1$.

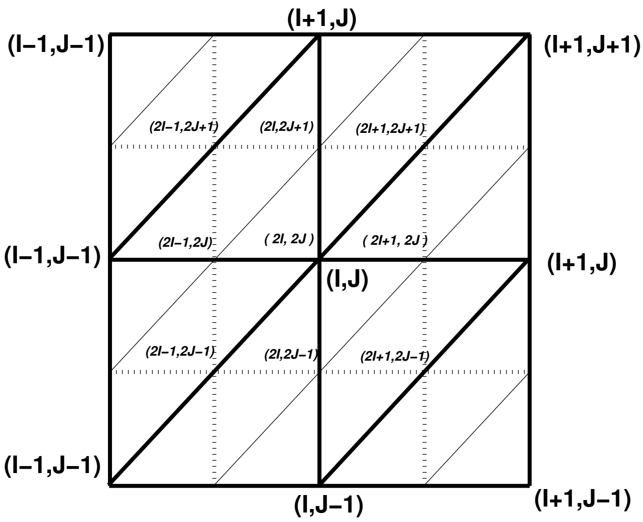


Figure 3.1

First decompose the domain Ω into N^2 equal-size squares with the edge length $h = 1/N$, then each square is divided into two right triangles in the same direction. We denote this partition as $\mathcal{T}_{l-1}$. Based on this partition, as in Figure 3.1 we construct the dual partition by connecting the medians of all $K \in \mathcal{T}_{l-1}$. Refine the mesh once we get $\mathcal{T}_l$ and its dual partition $\mathcal{T}_l^*$. Note that v_{l-1} is piecewise linear on $\mathcal{T}_{l-1}$, and

$$
\begin{aligned}
& a_{l-1}^*(v_{l-1}, v_{l-1}) \\
&= \sum_{K_{I,J}\in \mathcal{T}_{l-1}} \sum_{K_i^*\in \mathcal{T}_{h_{l-1}}^*} v_{l-1}(P_i) \int_{\partial K_i^*\cap K_{I,J}} a(x)\nabla v_{l-1}\cdot \mathbf{n} ds \\
&= \sum_{K_{2I,J}} \{V_{I,J+1}\ (\int_{\overline{m_0m_1}} a(x)\nabla u_{l-1}\cdot \mathbf{n} ds + \int_{\overline{m_4m_0}} a(x)\nabla u_{l-1}\cdot \mathbf{n} ds) \\
&\quad + V_{I,J}\int_{\overline{m_4m_0}} a(x)\nabla v_{l-1}\cdot \mathbf{n} ds + V_{I+1,J+1}\int_{\overline{m_0m_1}} a(x)\nabla v_{l-1}\cdot \mathbf{n} ds\} \\
&\quad + \sum_{K_{2I+1,J}} \{V_{I+1,J}\ (\int_{\overline{m_3m_0}} a(x,y)\nabla v_{l-1}\cdot \mathbf{n} ds + \int_{\overline{m_0m_2}} a(x,y)\nabla v_{l-1}\cdot \mathbf{n} ds) \\
&\quad + V_{I,J}\int_{\overline{m_3m_0}} a(x,y)\nabla_{l-1}\cdot \mathbf{n} ds + V_{I+1,J+1}\int_{\overline{m_0m_2}} a(x,y)\nabla_{l-1}\cdot \mathbf{n} ds\} \\
&= \sum_{I=0}^{N-1}\sum_{J=1}^{N-1} \{\int_{x_{J-\frac{1}{2}}}^{x_{J+\frac{1}{2}}} a(x, y_{I+\frac{1}{2}})dx \frac{(V_{I+1,J}-V_{I,J})^2}{h}\} \qquad (3.5) \\
&\quad + \sum_{I=1}^{N-1}\sum_{J=0}^{N-1} \{\int_{y_{I-\frac{1}{2}}}^{y_{I+\frac{1}{2}}} a(x, y_{J+\frac{1}{2}})dy \frac{(V_{I,J+1}-V_{I,J})^2}{h}\} \\
&= \sum_{I=0}^{N-1}\sum_{J=1}^{N-1} \{(I^2+J^2+I+\frac{1}{3}) * h^2 + 1\}(V_{I+1,J}-V_{I,J})^2 \\
&\quad + \sum_{I=1}^{N-1}\sum_{J=0}^{N-1} \{(I^2+J^2+J+\frac{1}{3}) * h^2 + 1\}(V_{I,J+1}-V_{I,J})^2,
\end{aligned}
$$

where $x_J = J\cdot h,\ y_I = I\cdot h$ and $V_{I,J} = v_{l-1}(x_J, y_I)$.

On the other hand, in a similar way

$$
\begin{aligned}
a_l^*(I_l v_{l-1}, I_l v_{l-1}) \;=\; & \sum_{i=0}^{2N-1}\sum_{j=0}^{2N-1} \{\int_{\hat{x}_{j-\frac{1}{2}}}^{\hat{x}_{j+\frac{1}{2}}} a(\hat{x}, \hat{y}_{i+\frac{1}{2}})d\hat{x} \frac{(\hat{V}_{i+1,j}-\hat{V}_{i,j})^2}{\frac{h}{2}}\} \\
+ & \sum_{i=0}^{2N-1}\sum_{j=0}^{2N-1} \{\int_{\hat{y}_{j-\frac{1}{2}}}^{\hat{y}_{j+\frac{1}{2}}} a(\hat{x}_{j+\frac{1}{2}}, \hat{y})d\hat{y} \frac{(\hat{V}_{i,j+1}-\hat{V}_{i,j})^2}{\frac{h}{2}}\},
\end{aligned}
\qquad (3.6)
$$

where $\hat{x}_j = j\cdot\dfrac{h}{2},\ \hat{y}_i = i\cdot\dfrac{h}{2}$ and $\hat{V}_{i,j} = I_l v_{l-1}(\hat{x}_j, \hat{y}_i)$.
Since v_{l-1} is piecewise linear on $\mathcal{T}_{h_{l-1}}$, we have that for $I, J = 0, 1, \cdots, N$,

$$
\hat{V}_{2I,2J} = V_{I,J}, \qquad \hat{V}_{2I+1,2J+1} = \frac{V_{I,J}+V_{I+1,J+1}}{2},
$$

$$
\hat{V}_{2I+1,2J} = \frac{V_{I+1,J}+V_{I,J}}{2}, \qquad \hat{V}_{2I,2J+1} = \frac{V_{I,J+1}+V_{I,J}}{2}.
$$

Substituting these equalities into (3.6) to get

$$
\begin{aligned}
& a_l^*(I_l v_{l-1}, I_l v_{l-1}) \\
= & \sum_{I=0}^{N-1}\sum_{J=0}^{N-1}\{\int_{\hat{x}_{2J-\frac{3}{2}}}^{\hat{x}_{2J+\frac{1}{2}}} a(\hat{x},\hat{y}_{2I+\frac{1}{2}})d\hat{x} + \int_{\hat{x}_{2J-\frac{1}{2}}}^{\hat{x}_{2J+\frac{3}{2}}} a(\hat{x},\hat{y}_{2I+\frac{3}{2}})d\hat{x}\}\ \frac{(V_{I+1,J}-V_{I,J})^2}{2h} \\
+ & \sum_{I=0}^{N-1}\sum_{J=0}^{N-1}\{\int_{\hat{y}_{2I-\frac{3}{2}}}^{\hat{y}_{2I+\frac{1}{2}}} a(\hat{x}_{2J+\frac{1}{2}},\hat{y})d\hat{y} + \int_{\hat{y}_{2I-\frac{1}{2}}}^{\hat{y}_{2I+\frac{3}{2}}} a(\hat{x}_{2J+\frac{3}{2}},\hat{y})d\hat{y}\}\ \frac{(V_{I,J+1}-V_{I,J})^2}{2h} \\
= & \sum_{I=0}^{N-1}\sum_{J=0}^{N-1} ((I^2+J^2+I+\frac{11}{24})h^2+1)(V_{I+1,J}-V_{I,J})^2 \\
+ & \sum_{I=0}^{N-1}\sum_{J=0}^{N-1} ((I^2+J^2+J+\frac{11}{24})h^2+1)(V_{I,J+1}-V_{I,J})^2.
\end{aligned}
\tag{3.7}
$$

Comparing (3.7) with (3.5), we have

$$
\begin{aligned}
& a_l^*(I_l v_{l-1}, I_l v_{l-1}) - a_{l-1}^*(v_{l-1}, v_{l-1}) \\
= & \frac{1}{8}h^2\ (\sum_{I=0}^{N-1}\sum_{J=0}^{N-1}(V_{I+1,J}-V_{I,J})^2 + \sum_{I=0}^{N-1}\sum_{J=0}^{N-1}(V_{I,J+1}-V_{I,J})^2) \\
\geq & \frac{1}{8}h^2\cdot\frac{1}{5}a_{l-1}^*(v_{l-1},v_{l-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
& a_l^*(I_l v_{l-1}, I_l v_{l-1}) - a_{l-1}^*(v_{l-1}, v_{l-1}) \\
& \leq \frac{1}{8}h^2 a_{l-1}^*(v_{l-1},v_{l-1}).
\end{aligned}
$$

Finally, based on the above two inequalities, we get

$$
a_l^*(I_l v_{l-1}, I_l v_{l-1}) \geq (1+\frac{1}{40}h^2)a_{l-1}^*(v_{l-1},v_{l-1}), \tag{3.8}
$$

and

$$
a_l^*(I_l v_{l-1}, I_l v_{l-1}) \leq (1+\frac{1}{8}h^2)a_{l-1}^*(v_{l-1},v_{l-1}). \tag{3.9}
$$

From (3.8) and (3.9), we know that $a_l^*(\cdot,\cdot)$ are noninherited for the special model problem, but are just a small perturbation in different mesh levels. In general, the difference between these two terms can be estimated as follows:

Lemma 3.1 *There exists a constant C independent of the mesh size h_l and the level l, such that*

$$
|a_l^*(I_l v_{l-1}, I_l v_{l-1}) - a_{l-1}^*(v_{l-1}, v_{l-1})| \leq C h_l \|v_{l-1}\|_{l-1}^2. \tag{3.10}
$$

Proof : Using the fact

$$
a(I_l v_{l-1}, I_l v_{l-1}) = a(v_{l-1}, v_{l-1}),
$$

we have

$$
\begin{aligned}
&|a_l^*(I_l v_{l-1}, I_l v_{l-1}) - a_{l-1}^*(v_{l-1}, v_{l-1})| \\
&\leq |a_l^*(I_l v_{l-1}, I_l v_{l-1}) - a(I_l v_{l-1}, I_l v_{l-1})| \\
&\quad + |a_{l-1}^*(v_{l-1}, v_{l-1}) - a(v_{l-1}, v_{l-1})|.
\end{aligned}
$$

Then an application of Lemma 2.1 completes the proof of this lemma.

Let $m_l,\ 0 \leq l \leq L$, be the smallest integer satisfying

$$
m_l \geq \beta^{L-l} m_L \tag{3.11}
$$

for some fixed $\beta > 1$, where m_L be the number of the iterations on the finest level L.

Theorem 3.1 *Let u_l be the solution of (2.7) and u_l^* be the iteration solution of* **Algorithm I***, then*

$$
\|u_L - u_L^*\|_L \leq C_0 \sum_{l=1}^{L} \frac{h_l}{m_l^\gamma} (\|f\|_0 + \|f\|_{1,p}) \quad p > 1. \tag{3.12}
$$

Proof: It follows from Lemma 2.1, Lemma 3.1, Theorem 2.1 and (3.2) that

$$
\begin{aligned}
\|u_l - u_l^*\|_l &= \|T_l^{m_l}(u_l - u_{l-1}^*)\|_l \leq \|T_l^{m_l}(u_l - u_{l-1})\|_l + \|T_l^{m_l}(u_{l-1} - u_{l-1}^*)\|_l \\
&\leq C \frac{h_l^{-1}}{m_l^\gamma} \|u_l - u_{l-1}\|_0 + \|u_{l-1} - u_{l-1}^*\|_l \\
&\leq C \frac{h_l}{m_l^\gamma} (\|f\|_0 + \|f\|_{1,p}) + (1 + C h_l^{\frac{1}{2}}) \|u_{l-1} - u_{l-1}^*\|_{l-1}.
\end{aligned} \tag{3.13}
$$

Recurrently, we get

$$
\|u_L - u_L^*\|_L \leq C \sum_{l=0}^{L-1} \prod_{i=0}^{l-1} (1 + C h_{L-i}^{\frac{1}{2}}) \frac{h_{L-l}}{m_{L-l}^\gamma} (\|f\|_0 + \|f\|_{1,p}) \tag{3.14}
$$

Note that $h_l = 2^{-l} h_0$, we obtain

$$
\prod_{i=0}^{L-1} (1 + C h_{L-i}^{\frac{1}{2}}) \leq \exp(C \sum_{i=0}^{L-1} h_{L-i}^{\frac{1}{2}}) \leq \exp(C \frac{h_1^{\frac{1}{2}}}{1 - \sqrt{\frac{1}{2}}}) \leq C_0. \tag{3.15}
$$

Inserting (3.15) into (3.14) proves (3.12).

A similar argument of Lemmas 1.3 and 1.4 in [2] leads to

Lemma 3.2 *If m_l, the number of iterations on level l is given by (3.11), then the accuracy of the cascadic algorithm I is*

$$\|u_L - u_L^*\|_L \leq \begin{cases} C\dfrac{1}{1-(\frac{2}{\beta^\gamma})}\dfrac{h_L}{m_L^\gamma}(\|f\|_0 + \|f\|_{1,p}) \quad for \quad \beta > 2^{\frac{1}{\gamma}},\ p > 1, \\ CL\dfrac{h_L}{m_L^\gamma}(\|f\|_0 + \|f\|_{1,p}) \quad for \quad \beta = 2^{\frac{1}{\gamma}},\ p > 1. \end{cases}$$

Lemma 3.3 *The computational cost of the cascadic algorithm I is proportional to*

$$\sum_{l=0}^{L} m_l n_l \leq \begin{cases} C\dfrac{1}{1-\frac{\beta}{2^d}} m_L n_L \quad for \quad \beta < 2^d, \\ CLm_L n_L \quad for \quad \beta = 2^d, \end{cases}$$

where d is the dimension of the domain Ω.

Based on Lemmas 3.2 and 3.3, we have the following

Theorem 3.2 *It holds that*
(1). If $\gamma = \frac{1}{2}$, $d = 3$, then the cascadic multigrid I is optimal.
(2). If $\gamma = 1, d = 2, 3$, then the cascadic multigrid I is optimal.
(3). If $\gamma = \frac{1}{2}, d = 2$, and the number of iterations on the level L is

$$m_L = [m_* L^2],$$

then the error is

$$\|u_L - u_L^*\|_L \leq C\frac{h_L}{m_*^{\frac{1}{2}}}(\|f\|_0 + \|f\|_{1,p}),\ p > 1,$$

and the complexity of computation is

$$\sum_{l=0}^{L} m_l n_l \leq cm_* n_L (1 + logn_L)^3.$$

It means that the cascadic multigrid is nearly optimal in this case.

3.2 The nonsymmetric case

It is known that the bilinear form associated with finite volume methods for the self-adjoint problem is nonsymmetric in general. To our knowledge, there is no any cascadic multigrid algorithm for nonsymmetric systems, in this section, based on the special property of the finite volume method, i.e., the finite volume quadratic form is a small perturbation of the finite element quadratic form, we shall propose an effective cascadic multigrid algorithm for the nonsymmetric system.

Algorithm II

(1) let $u_0^0 = u_0^* \hat{=} u_0$ be the exact solution of (2.7).

(2) for $l = 1, 2, \cdots, L$, let $\hat{u}_l$ be the exact solution of the following problem

$$a(\hat{u}_l, v) = (f, r_l v) - N_l(u_{l-1}^*, v) \quad \forall v \in V_l, \tag{3.16}$$

where $N_l(u, v) = a_l^*(u, v) - a(u, v),\ \forall u, v \in V_l$.

Let $u_l^0 = I_l u_{l-1}^*$, for (3.16), do iterations

$$u_l^{m_l} = G_l^{m_l} u_l^0 \tag{3.17}$$

(3) Set $u_l^* \hat{=} u_l^{m_l}$,

where $G_l : V_l \to V_l$ is the iterative operator on the level l. Note that this iterative operator is based on the finite element equation (3.16).

Similar as in the above subsection, it is known that for the smoothing operators such as Richardson, Jacobi, Gauss-seidel, or CG iteration, there exists a linear operator $S_l : V_l \to V_l$ such that

$$u_l - G_l^{m_l} u_l^0 = S_l^{m_l}(u_l - u_l^0), \tag{3.18}$$

and for the symmetric operator A_l it holds that

$$\begin{aligned} \|S_l^{m_l} v\|_a &\leq C_2 \frac{h_l^{-1}}{m_l^{\gamma}} \|v\|_0, \quad \forall v \in V_l, && (3.19)\\ \|S_l^{m_l} v\|_a &\leq \|v\|_a, \quad \forall v \in V_l, && (3.20) \end{aligned}$$

where γ is a positive number depending on the given iteration, and A_l is defined by

$$(A_l u, v) = a(u, v), \quad \forall u, v \in V_l.$$

For the convenience of the following analysis, we denote the constants in (2.10),(2.11) as C_1, and the constant in (2.9) as C_3 respectively.

In order to get the main result of this section, we first give two lemmas.

Lemma 3.4 *Let $u_l, \hat{u}_l$ be the solutions of (2.7) and(3.16) respectively, then*

$$\|u_l - \hat{u}_l\|_a \leq 3C_1 C_3 h_l^2 (\|f\|_0 + \|f\|_{1,p}) + C_3 h_l (\|u_{l-1} - \hat{u}_{l-1}\|_a + \|\hat{u}_{l-1} - u_{l-1}^*\|_a). \tag{3.21}$$

Proof: Since

$$\begin{aligned} a(u_l - \hat{u}_l, v) &= a(u_l, v) - (f, r_l v) + a_l^*(u_{l-1}^*, v) - a(u_{l-1}^*, v) \\ &= a(u_l - u_{l-1}^*, v) - a_l^*(u_l - u_{l-1}^*, v) \\ &\leq C_3 h_l \|u_l - u_{l-1}^*\|_a \|v\|_a, \qquad \forall v \in V_l, \end{aligned}$$

we get

$$\begin{aligned} \|u_l - \hat{u}_l\|_a &\leq C_3 h_l \|u_l - u_{l-1}^*\|_a \\ &\leq C_3 h_l (\|u_l - u_{l-1}\|_a + \|u_{l-1} - \hat{u}_{l-1}\|_a + \|\hat{u}_{l-1} - u_{l-1}^*\|_a), \end{aligned}$$

which, together with Theorem 2.1, completes the proof of (3.21).

Lemma 3.5 *For the solutions $\hat{u}_l$ and u_l^* in* Algorithm II, *we have*

$$\begin{aligned}\|\hat{u}_l - u_l^*\|_a \;\leq\; & 3C_1C_2\frac{h_l}{m_l^r}(\|f\|_0 + \|f\|_{1,p}) + \|\hat{u}_{l-1} - u_{l-1}^*\|_a \\ & + C_2\frac{h_l^{-1}}{m_l^r}(\|u_l - \hat{u}_l\|_a + \|u_{l-1} - \hat{u}_{l-1}\|_a)\end{aligned} \tag{3.22}$$

Proof: Since

$$\begin{aligned} & \|\hat{u}_l - u_l^*\|_a = \|S_l^{m_l}(\hat{u}_l - u_{l-1}^*)\|_a \\ \leq\; & \|S_l^{m_l}(\hat{u}_l - \hat{u}_{l-1})\|_a + \|S_l^{m_l}(\hat{u}_{l-1} - u_{l-1}^*)\|_a \\ \leq\; & C_2\frac{h_l^{-1}}{m_l^r}(\|u_l - u_{l-1}\|_0 + \|u_l - \hat{u}_l\|_0 + \|u_{l-1} - \hat{u}_{l-1}\|_0) \\ & + \|\hat{u}_{l-1} - u_{l-1}^*\|_a \end{aligned}$$

As in Lemma 3.4, an application of Theorem 2.1 gives (3.22).

Based on these two lemmas, we can get the optimal error estimation between the finite volume solution u_l and the cascadic multigrid solution u_l^*.

Theorem 3.3 *Assume that the coarse partition is small enough such that*

$$4C_3h_0 < 1 \tag{3.23}$$

and the number of the iteration step m_L on the last level L satisfies

$$m_L^\gamma \geq 3C_2 \cdot (1 + 7C_3) \cdot \frac{4}{\beta^\gamma - 2} \qquad if \quad \beta^\gamma > 2, \tag{3.24}$$

$$m_L^\gamma \geq 3C_2 \cdot (1 + 7C_3) \cdot 2L \qquad if \quad \beta^\gamma = 2, \tag{3.25}$$

then

$$\|u_L - u_L^*\|_a \leq C_* \cdot (h_L^2 + \sum_{l=1}^{L}\frac{h_l}{m_l^\gamma})\ (\|f\|_0 + \|f\|_{1,p}), \quad p > 1, \tag{3.26}$$

where $C_* = \max\ \{\ 4C_1C_3,\ 3C_1C_2(1 + 7C_3)\ \}$.

Proof: We first prove the following two inequalities inductively,

$$\|u_l - \hat{u}_l\|_a \;\leq\; 4C_1C_3h_l^2 \cdot (\|f\|_0 + \|f\|_{1,p}), \tag{3.27}$$

$$\|\hat{u}_l - u_l^*\|_a \;\leq\; 3C_1C_2(1 + 7C_3) \cdot \sum_{i=1}^{l}\frac{h_l}{m_l^\gamma}(\|f\|_0 + \|f\|_{1,p}) \tag{3.28}$$

(1) For $l = 0$, by the definition of Algorithm II,

$$\|u_0 - \hat{u}_0\|_a = \|\hat{u}_0 - u_0^*\|_a = 0,$$

so (3.27), (3.28) hold.
(2) For $l = 1$, by (3.21), (3.22)

$$\|u_1 - \hat{u}_1\|_a \leq 3C_1C_3h_1^2(\|f\|_0 + \|f\|_{1,p}),$$

and

$$\begin{aligned}\|\hat{u}_1 - u_1^*\|_a &\leq (3C_1C_2\frac{h_1}{m_1^\gamma} + C_2\frac{h_1^{-1}}{m_1^\gamma} \cdot 3C_1C_3h_1^2)\ (\|f\|_0 + \|f\|_{1,p}) \\ &\leq 3C_1C_2(1+C_3)\frac{h_1}{m_1^\gamma}(\|f\|_0 + \|f\|_{1,p}),\end{aligned}$$

which are (3.27), (3.28) with $l = 1$.

(3) Assume that (3.27),(3.28) hold for $l-1, l$, then by (3.21), we have

$$\begin{aligned}\|u_{l+1} - \hat{u}_{l+1}\|_a &\leq \{\ 3C_1C_3h_{l+1}^2 + C_3h_{l+1} \cdot 4C_1C_3h_l^2 \\ &\quad +C_3h_{l+1} \cdot 3C_1C_2(1+7C_3)\sum_{i=1}^{l}\frac{h_i}{m_i^\gamma}\ \}(\|f\|_0 + \|f\|_{1,p}) \\ &\leq \{3C_1C_3h_{l+1}^2 + \frac{1}{2}C_1C_3h_{l+1}^2 \cdot 16C_3h_l \\ &+ \quad C_1C_3h_{l+1}h_L \cdot 3C_2(1+7C_3)\sum_{i=1}^{l}\frac{2^{L-i}}{m_i^r}\ \}(\|f\|_0 + \|f\|_{1,p}),\end{aligned} \tag{3.29}$$

Since $l < L$,

$$\sum_{i=1}^{l}\frac{2^{L-i}}{m_i^\gamma} = \frac{1}{m_L^\gamma}\sum_{i=1}^{l}\frac{2^{L-i}}{\beta^{\gamma(L-i)}} \leq \begin{cases} \dfrac{1}{m_L^\gamma}\dfrac{2}{\beta^\gamma - 2} & \beta^\gamma > 2, \\ \dfrac{L}{m_L^\gamma} & \beta^\gamma = 2, \end{cases} \tag{3.30}$$

and $l \geq 2$, by (3.23), we know

$$16C_3h_l \leq 1.$$

Combining above equalities with (3.24), (3.25), we have that (3.27) holds for $l+1$. On the other hand, by (3.22)

$$\begin{aligned}\|\hat{u}_{l+1} - u_{l+1}^*\|_a &\leq \{\ 3C_1C_2\frac{h_{l+1}}{m_{l+1}^\gamma} + C_2\frac{h_{l+1}^{-1}}{m_{l+1}^\gamma} \cdot 4C_1C_3(h_{l+1}^2 + h_l^2) \\ &\quad +3C_1C_2(1+7C_3)\sum_{i=1}^{l}\frac{h_i}{m_i^\gamma}\ \}\ (\|f\|_0 + \|f\|_{1,p}) \\ &\leq 3C_1C_3(1+7C_3)\sum_{i=1}^{l+1}\frac{h_i}{m_i^\gamma}\ (\|f\|_0 + \|f\|_{1,p}).\end{aligned} \tag{3.31}$$

Then (3.28) holds for $l+1$. By induction we know that (3.27) and (3.28) hold for any $l = 0, 1, \cdots, L$.

Finally,

$$\begin{aligned}\|u_L - u_L^*\|_a &\leq \|u_L - \hat{u}_L\|_a + \|\hat{u}_L - u_L^*\|_a \\ &\leq \{4C_1C_3h_L^2 + 3C_1C_2(1+7C_3)\sum_{i=1}^{L}\frac{h_i}{m_L^\gamma}\}\ (\|f\|_0 + \|f\|_{1,p}) \\ &\leq C_*(h_L^2 + \sum_{i=1}^{L}\frac{h_i}{m_i^\gamma})(\|f\|_0 + \|f\|_{1,p}),\end{aligned} \tag{3.32}$$

which completes the proof of this theorem.

Based on Theorem 3.3, and using a similar argument of Lemmas 1.3 and 1.4 in [2] leads to

Lemma 3.6 *If m_l, the number of iterations on level l is given by (3.11), and the h_0 and m_L satisfy the assumptions of Theorem 3.3. Then the accuracy of the cascadic multigrid algorithm II is*

$$\|u_L - u_L^*\|_a \leq \begin{cases} C_*(h_L + \dfrac{1}{1-(\frac{2}{\beta^\gamma})}\dfrac{1}{m_L^\gamma})h_L(\|f\|_0 + \|f\|_{1,p}) & for \quad \beta > 2^{\frac{1}{\gamma}},\ p > 1, \\ C_*(h_L + L\dfrac{1}{m_L^\gamma})h_L(\|f\|_0 + \|f\|_{1,p}) & for \quad \beta = 2^{\frac{1}{\gamma}},\ p > 1. \end{cases}$$

Lemma 3.7 *The computational cost of the cascadic multigrid algorithm II is proportional to*

$$\sum_{l=0}^{L} m_l n_l \leq \begin{cases} C\dfrac{1}{1-\frac{\beta}{2^d}} m_L n_L & for \quad \beta < 2^d, \\ CLm_L n_L & for \quad \beta = 2^d, \end{cases}$$

where d is the dimension of the domain Ω.

Based on Lemmas 3.6 and 3.7, we have the following

Theorem 3.4 *Let h_0 and m_L satisfy the assumptions of Theorem 3.3. Then*
(1). If $\gamma = \frac{1}{2}$, $d = 3$, then the cascadic multigrid algorithm II is optimal.
(2). If $\gamma = 1, d = 2, 3$, then the cascadic multigrid algorithm II is optimal.
(3). If $\gamma = \frac{1}{2}, d = 2$, and the number of iterations on the level L is

$$m_L \geq L^2,$$

then the error in the energy norm is

$$\|u_L - u_L^*\|_a \leq Ch_L(\|f\|_0 + \|f\|_{1,p}), \quad p > 1,$$

and the complexity of computation is

$$\sum_{l=0}^{L} m_l n_l \leq Cn_L(1 + log n_L)^3.$$

4 Numerical Experiments

In this section we will give two examples to test the **Algorithm I** and **Algorithm II** respectively.

Example 1. We use the **Algorithm I** to solve the following problem:

$$
\begin{cases} -\nabla \cdot (a(x,y)\nabla u) &= f, \quad \text{in } \Omega = (0,1) \times (0,1), \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{4.1}
$$

where $a(x,y) = \exp(x+y)$ and let f be chosen as $2\pi^2 \exp(x+y)\sin(\pi x)\sin(\pi y) - \pi \exp(x+y)(\sin(\pi x)\cos(\pi y) + \cos(\pi x)\sin(\pi y))$ such that the exact solution of the problem is $u(x,y) = \sin(\pi x)\sin(\pi y)$.

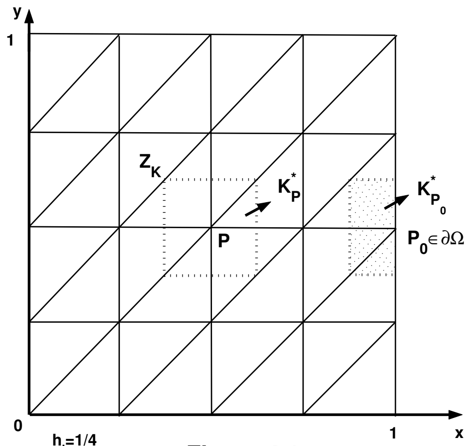


Figure 4.1

Partitioning the domain $[0,1] \times [0,1]$ into the uniform triangular mesh as in Figure 4.1, choosing Z_K as the circumcenter of the triangular, we can get the dual partition correspondingly. Using the Gauss-Seidel and CG iterations as the smoothing operators respectively, we list the energy norm of the error between the cascadic multigrid solution and the exact solution on the last level L in Tables 1-2, We can see from these tables that for both of the smoothers, if the mesh is refined once, the energy error is decreasing by half independent of the coarse mesh. It means that the convergence rate of the **Algorithm I** is one independent of the refinement level.

Algorithm I with G − S smoother

Mesh	# level (L)	$\|u_L^* - u\|_1$	CPU time
512×512	3	6.827557e-03	204(s)
	4	6.866367e-03	45(s)
	5	6.853683e-03	48(s)
	6	6.844400e-03	70(s)
1024×1024	3	3.415217e-03	2908(s)
	4	3.421522e-03	343(s)
	5	3.424024e-03	222(s)
	6	3.418322e-03	332(s)
2048×2048	4	1.716457e-03	3311(s)
	5	1.714659e-03	1040(s)
	6	1.710306e-03	1329(s)
	7	1.708597e-03	2044(s)

$m_L = 2L^2, \beta = 5.0$

Table 1.

Algorithm I with CG smoother

Mesh	# level (L)	$\|u_L^* - u\|_1$	CPU time
512×512	3	6.826660e-03	31(s)
	4	6.848449e-03	28(s)
	5	6.848575e-03	29(s)
	6	6.848575e-03	29(s)
1024×1024	3	3.410303e-03	158(s)
	4	3.418454e-03	117(s)
	5	3.418599e-03	116(s)
	6	3.418578e-03	118(s)
2048×2048	4	1.706217e-03	511(s)
	5	1.708165e-03	479(s)
	6	1.708200e-03	488(s)
	7	1.708201e-03	498(s)

$m_L = 10,\ \beta = 3.0$

Table 2.

Example 2. The problem will be computed is

$$\begin{cases} -\nabla \cdot (A\nabla u) &= f, \quad \text{in } \Omega = (0,1) \times (0,1), \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where

$$A = \begin{pmatrix} 2x^2 + y^2 + 1 & x^2 + y^2 \\ x^2 + y^2 & x^2 + 2y^2 + 1 \end{pmatrix}$$

and

$$\begin{aligned} f \;=\; & -2\pi(2x + y)\cos(\pi x)\sin(\pi y) + \pi^2(3x^2 + 3y^2 + 2)\sin(\pi x)\sin(\pi y) \\ & -2\pi(x + 2y)\sin(\pi x)\cos(\pi y) - 2\pi^2(x^2 + y^2)\cos(\pi x)\cos(\pi y). \end{aligned}$$

Then the exact solution can be expressed as $u = \sin(\pi x)\sin(\pi y)$.

Using the same partition as in Example 1, since the coefficient A is a matrix function with full entries, the system deriving from the finite volume discretization of this problem is nonsymmetric. We use the **Algorithm II** to solve the discrete systems.

Algoritm II with $\mathrm{G-S}$ smoother

Mesh	# level (L)	$\|u_L^* - u\|_1$	CPU time
512×512	3	6.824011e-03	203(s)
	4	6.838624e-03	55(s)
	5	6.831931e-03	55(s)
	6	6.827161e-03	69(s)
1024×1024	3	3.412430e-03	2526(s)
	4	3.415616e-03	350(s)
	5	3.417416e-03	234(s)
	6	3.413925e-03	286(s)
2048×2048	4	1.711151e-03	3144(s)
	5	1.709525e-03	1078(s)
	6	1.707189e-03	1176(s)
	7	1.706368e-03	1513(s)

$m_L = 2L^2,\ \beta = 4.0$

Table 3.

Algorithm II with CG smoother

Mesh	# level (L)	$\|u_L^* - u\|_1$	CPU time
512×512	3	6.820292e-03	187(s)
	4	6.823242e-03	50(s)
	5	6.823173e-03	42(s)
	6	6.823173e-03	42(s)
1024×1024	3	3.411521e-03	2754(s)
	4	3.411351e-03	347(s)
	5	3.410722e-03	190(s)
	6	3.410723e-03	183(s)
2048×2048	4	1.709745e-03	2917(s)
	5	1.704839e-03	843(s)
	6	1.704851e-03	705(s)
	7	1.704851e-03	705(s)

$m_L = 10,\ \beta = 3.0$

Table 4.

The error of energy norm between the cascadic multigrid solution u_L^* and the exact solution u on the last level L is given in Table 3-4 respectively for the Gauss-seidel and CG smoother. Similar results as in Example 1 can be seen from these tables. The above numerical experiments show that the convergence rate of the energy error is indeed of order one as proved by the theoretical analysis for the two cascadic multigrid algorithms we propose for the finite volume method.

Finally, in table 5, we list the error of the usual full multigrid method developed in [10]. It can be seen that the cascadic multigrid and usual full multigrid hold same computational accuracy. But it seems that convergence speed of the usual multigrid method is a little bit faster than the cascadic multigrid. Due to no coarse grid corrections, it is obvious that the computational code of the cascadic multigrid is simper than the usual multigrid methods.

Full Multigrid with $\mathrm{G-S}$ smoother

Mesh	# level (L)	$\|u_L^* - u\|_1$	CPU time
512×512	3	6.816698e-03	217(s)
	4	6.816763e-03	51(s)
	5	6.816832e-03	37(s)
	6	6.816882e-03	36(s)
1024×1024	3	3.410086e-03	2390(s)
	4	3.408419e-03	324(s)
	5	3.408338e-03	159(s)
	6	3.408346e-03	147(s)
2048×2048	4	1.707871e-03	2859(s)
	5	1.704383e-03	749(s)
	6	1.704163e-03	599(s)
	7	1.704153e-03	595(s)

Table 5.

References

[1] R.E. Bank and D.J. Rose, Some error estimates for the box method, SIAM J. Numer. Anal., **24**(1987), 777-787.

[2] F. Bornemann and P. Deuflhard, The cascadic multigrid method for elliptic problems, Numer. Math. **75**(1996), 135-152.

[3] F. Bornemann and P. Deuflhard, The cascadic multigrid method, The Eighth International Conference on Domain Decomposition Methods for Partial Differential Equations, R.Glowinski, J.Periaux, Z.Shi, O.Widlund, eds., Wiley and Sons, 1997.

[4] F. Bornemann and R. Krause, Classical and Cascadic multigrid- methodogical comparison, Proceedings of the 9th International Conference on Domain Decomposition, P. Bjorstad, M. Espedal, D. Keyes, eds., John Wiley and Sons, NY, 1998.

[5] D. Braess and W. Dahmen, A cascade multigrid algorithm for the Stokes equations, Numer. Math., **82**(1999), 179-192.

[6] D. Braess, P. Deuflhard, and K. Lipnikov, A subspace cascadic multigrid method for mortar elements, Computing **69** (2002), 205–225.

[7] Z. Cai, On the finite volume element method, Numer. Math., **58**(1991), 713-735.

[8] Z. Cai and S. McCormick, On the accuracy of the finite volume element method for diffusion equations on composite grids, SIAM J. Numer. Anal., **27**(1990), 636-655.

[9] S.H. Chou and Q. Li, Error estimates in L^2, H^1 and L^∞ in covolume methods for elliptic and parabolic problems: A unified approach, Math. Comp., **69**(2000), 103-120.

[10] S.H. Chou and D.Y. Kwak, Multigrid algorithms for a vertex-centered covolume method for elliptic problems, Numer. Math., **90**(2002), 459-486.

[11] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, (1978).

[12] R. E. Ewing, T. Lin and Y. Lin, On the accuracy of the finite volume element method based on piecewise linear polynomials, SIAM J. Numer. Anal. **39**(2002), 1865-1888.

[13] R. Eymard, T. Gallouet and R. Herbin, Finite Volume Methods, Handbook of Numerical Analysis, Vol. VII, 713-1020, North Holland, 2001.

[14] Y. Huang, Z.C. Shi, T. Tang, and W. Xue, A multilevel successive iteration method for nonlinear elliptic problems, to appear in Math. Comp., 2004.

[15] R.H. Li, Z. Chen and W. Wu, Generalized Difference Methods for Differential Equations, Marcel Dekker, New York, 2000.

[16] V. Shaidurov, Some estimates of the rate of convergence for the cascadic conjugate gradient method, Comp. Math. Applic., **31, No 4/5** (1996), 161-171.

[17] Z. Shi and X. Xu, Cascadic multigrid method for elliptic problems, East-West J. Numer. Math., **7**(1999), 199-209.

[18] Z. Shi and X. Xu, Cascadic multigrid methods for parabolic problems, J. Comput. Math., **18**(2000), 551-560.

[19] R. Stevenson, Nonconforming finite elements and the cascadic multigrid method, Numer. Math. **91**(2002), 351-387.

[20] G. Timmermann, A cascadic multigrid algorithm for semilinear elliptic problems, Numer. Math., **86**(2000), 717-731.