Matrix integrals and several integrable differential-difference systems

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Abstract

In this paper, the relations between three special forms of matrix integrals and their associated integrable differential-difference systems are considered. It turns out that these matrix integrals with special $\beta = 2$ and $1, 4$ satisfy the differential-difference KP equation, the two-dimensional Toda lattice, the semi-discrete Toda equation and their Pfaffianized systems, respectively.

KEYWORDS: Matrix integrals, Pfaffian, Casorati determinant, two dimensional Toda lattice, differential-difference KP equation, semi-discrete Toda lattice

1 Introduction

It is known that integrals of the form

$$\int \left[\prod_{k=1}^{N} \mu(x_k)\right] \left[\prod_{1\le i<j\le N} |x_i - x_j|^{\beta}\right] dx_{m+1} \cdots dx_N \tag{1}$$

have played a crucial role in various branches of mathematics and physics (see, e.g. [1]-[4] and references therein). One of interesting aspects in the research is to link integrals of the form (1) with their associated differential equations. Over the past decade, many remarkable and excellent results have been achieved towards clarifying close relations between integrals of the form (1) with $\beta = 1, 2, 4$ and nonlinear integrable systems (see, e.g. [5]-[10]). An important step involved to induce these integrable equations is to insert time-parameters into the integrals. In this paper, we shall consider the following special cases of the matrix integral (1):

$$Z_N^{(\beta)}(t_1,t_2,n)=\frac{1}{N!}\int_{-1}^{\infty}\cdots\int_{-1}^{\infty}\prod_{i=1}^{N}(1+x_i)^n\prod_{1\le i<j\le N}|x_i-x_j|^\beta\exp[\sum_{i=1}^{N}\eta(x_i,t)]dx_1\cdots dx_N, \quad (2)$$

$$Z_N^{(\beta)}(t_1,s_1,n)=\frac{1}{N!}\int_{0}^{\infty}\cdots\int_{0}^{\infty}\prod_{i=1}^{N}x_i^n\prod_{1\le i<j\le N}|x_i-x_j|^\beta\exp[\sum_{i=1}^{N}\hat\eta(x_i,t_1,s_1)]dx_1\cdots dx_N, \quad (3)$$

$$Z_N^{(\beta)}(t_1,n,k)=\frac{1}{N!}\int_{0}^{\infty}\cdots\int_{0}^{\infty}\prod_{i=1}^{N}x_i^n(1+x_i)^k\prod_{1\le i<j\le N}|x_i-x_j|^\beta\exp[\sum_{i=1}^{N}\bar\eta(x_i,t_1)]dx_1\cdots dx_N, \quad (4)$$

where $\eta(x,t)=\sum_{m=1}^{2}x^mt_m+\eta_0(x)$, $\hat\eta(x,t_1,s_1)=xt_1-x^{-1}s_1+\hat\eta_0(x)$, $\bar\eta(x,t_1)=xt_1+\bar\eta_0(x)$ and $\eta_0(x),\hat\eta_0(x),\bar\eta_0(x)$ are three suitable functions of x satisfying that

$$\lim_{x\to-1}(1+x)^le^{\eta(x,t)}=0,\quad \lim_{x\to\infty}x^le^{\eta(x,t)}=0,$$
$$\lim_{x\to0}x^le^{\hat\eta(x,t_1,s_1)}=0,\quad \lim_{x\to\infty}x^le^{\hat\eta(x,t_1,s_1)}=0,$$
$$\lim_{x\to0}x^le^{\bar\eta(x,t_1)}=0,\quad \lim_{x\to\infty}x^le^{\bar\eta(x,t_1)}=0,$$

for any integer l.

The purpose of this paper is to show that the matrix integrals (2), (3) and (4) with $\beta=2$ and $\beta=1,4$ satisfy the so-called differential-difference KP equation, the two-dimensional Toda lattice and the semi-discrete Toda lattice and their Pfaffianized coupled systems, respectively.

The paper is organized as follows. In section 2, we will show that the matrix integral (2) with $\beta=2$ and $\beta=1,4$ satisfy the so-called differential-difference KP equation and the Pfaffianized coupled form, respectively. Section 3 is devoted to clarifying the relation between the matrix integral (3) and its induced integrable systems. It is shown that the matrix integral (3) with $\beta=2$ and $\beta=1,4$ satisfy the two-dimensional Toda lattice and its Pfaffianized coupled system. The similar results will be established for the matrix integral (4) in section 4. It is found that the matrix integral (4) with $\beta=2$ and $\beta=1,4$ satisfy the semi-discrete Toda lattice and its Pfaffianized coupled system. Finally, summary and discussions are given in section 5.

2 Matrix integral (2) with $\beta=1,2,4$ and the induced integrable equations

In this section, we will show that the matrix integrals (2) with $\beta=2$ and $\beta=1,4$ satisfy the so-called differential-difference KP equation and the Pfaffianized coupled form, respectively. We first review some known results concerning the differential-difference KP equation. The so-called differential-difference KP(DΔKP) equation reads [11][12],

$$\Delta(\frac{\partial u(n)}{\partial t_2}+2\frac{\partial u(n)}{\partial t_1}-2u(n)\frac{\partial u(n)}{\partial t_1})=(2+\Delta)\frac{\partial^2u(n)}{\partial t_1^2}, \quad (5)$$

where Δ denotes the forward difference operator defined by $\Delta f(n)=f(n+1)-f(n)$ and the shift operator E defined by $Ef(n)=f(n+1)$. The operators Δ and E are connected by $\Delta=E-1$.

By the dependent variable transformation $u(n) = \ln(\tau(n+1)/\tau(n))_{t_1}$, Eq.(5) is transformed into the bilinear form:

$$(D_{t_2} + 2D_{t_1} - D_{t_1}^2)\exp(\frac{D_n}{2})\tau(n)\cdot\tau(n) = 0, \tag{6}$$

where the Hirota's bilinear differential operator $D_y^m D_t^k$ and the bilinear difference operator $\exp(\delta)D_n$ are defined by [13]-[15]

$$D_y^m D_t^k a\cdot b \equiv \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(y,t)b(y',t')|_{y'=y,t'=t}.$$

$$\exp(\delta D_n)a(n)\cdot b(n) \equiv \exp\left[\delta(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'})\right] a(n)b(n')\mid_{n'=n} = a(n+\delta)b(n-\delta).$$

Tamizhmani et al [12] proved that the N-soliton solution to equation (6) takes the form of an $N\times N$ Wronskian (Casorati) determinant

$$\begin{aligned}\tau(n) &= \begin{vmatrix} f_1(n) & f_1(n+1) & \cdots & f_1(n+N-1) \\ f_2(n) & f_2(n+1) & \cdots & f_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ f_N(n) & f_N(n+1) & \cdots & f_N(n+N-1) \end{vmatrix} \\ &= \begin{vmatrix} f_1(n) & \Delta f_1(n) & \cdots & \Delta^{N-1} f_1(n) \\ f_2(n) & \Delta f_2(n) & \cdots & \Delta^{N-1} f_2(n) \\ \vdots & \vdots & \ddots & \vdots \\ f_N(n) & \Delta f_N(n) & \cdots & \Delta^{N-1} f_N(n) \end{vmatrix}\end{aligned}$$

where $f_j(n) = f_j(t_1, t_2, n), \quad (j = 1, 2, \cdots, N)$, are the solutions of the following set of linear partial differential-difference equations

$$\frac{\partial f_j(n)}{\partial t_1} = \Delta f_j(n), \quad \frac{\partial f_j(n)}{\partial t_2} = \Delta^2 f_j(n), \quad \text{for} \quad j = 1, 2, \cdots, N. \tag{7}$$

In [16], we have derived the following Pfaffianized differential-difference KP equation:

$$(D_{t_2} + 2D_{t_1} - D_{t_1}^2)\exp(\frac{D_n}{2})\tau(n)\cdot\tau(n) = -2\exp(\frac{D_n}{2})\hat{\tau}(n)\cdot\tilde{\tau}(n), \tag{8}$$

$$\left[(D_{t_2} - 2D_{t_1} + D_{t_1}^2)\exp(\frac{D_n}{2}) + 4\sinh(\frac{D_n}{2})\right]\tau(n)\cdot\tilde{\tau}(n) = 0, \tag{9}$$

$$\left[(D_{t_2} - 2D_{t_1} - D_{t_1}^2)\exp(-\frac{D_n}{2}) + 4\sinh(\frac{D_n}{2})\right]\tau(n)\cdot\hat{\tau}(n) = 0. \tag{10}$$

It has also been shown that $\tau(n), \tilde{\tau}(n)$ and $\hat{\tau}(n)$ given by

$$\tau(n) = \mathrm{pf}(1, 2, \cdots, N)_n, \tag{11}$$

$$\tilde{\tau}(n) = \mathrm{pf}(1, 2, \cdot, N-1, N, N+1, N+2)_n, \tag{12}$$

$$\hat{\tau}(n) = \mathrm{pf}(1, 2, \cdots, N-2)_n, \tag{13}$$

with N even, is a solution to the system (8)-(10) if the elements in our Pfaffians are chosen to satisfy

$$\mathrm{pf}(i,j)_{n+1} = \mathrm{pf}(i,j)_n + \mathrm{pf}(i+1,j)_n + \mathrm{pf}(i,j+1)_n + \mathrm{pf}(i+1,j+1)_n, \tag{14}$$

$$\frac{\partial}{\partial t_1}\mathrm{pf}(i,j)_n = \mathrm{pf}(i+1,j)_n + \mathrm{pf}(i,j+1)_n, \tag{15}$$

$$\frac{\partial}{\partial t_2}\mathrm{pf}(i,j)_n = \mathrm{pf}(i+2,j)_n + \mathrm{pf}(i,j+2)_n. \tag{16}$$

Through the dependent variable transfromation,

$$u(n) = \frac{\partial}{\partial t_1}\ln\frac{\tau(n+1)}{\tau(n)}, \quad v(n) = \frac{\tilde{\tau}(n)}{\tau(n)}, \quad w(n) = \frac{\hat{\tau}(n)}{\tau(n)},$$

the Pfaffianized DΔKP system (8),(9) and (10) are transformed into the following coupled differential-difference equations,

$$\Delta[u_{t_2}(n) + 2u_{t_1}(n) - 2u(n)u_{t_1}(n) + 2(v(n)w(n+1))_{t_1}] = (2+\Delta)u_{t_1t_1}(n), \tag{17}$$

$$\begin{aligned} &v(n)\left[2U_{t_1}(n) + u_{t_1}(n) + u^2(n) - 2u(n) + \int^{t_1} u_{t_2}(n)dt_1\right] - 2u(n)v_{t_1}(n) + v_{t_1t_1}(n) + 2v_{t_1}(n) \\ &\quad - v_{t_2}(n) + 2(v(n) - v(n+1)) = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &w(n+1)\left[2U_{t_1}(n) + u_{t_1}(n) + u^2(n) + 2u(n) - \int^{t_1} u_{t_2}(n)dt_1\right] + 2u(n)w_{t_1}(n+1) - w_{t_1t_1}(n+1) \\ &\quad + 2w_{t_1}(n+1) - w_{t_2}(n+1) + 2(w(n) - w(n+1)) = 0, \end{aligned} \tag{19}$$

where $U(n) = \frac{\partial}{\partial t_1}\ln\tau(n)$ and $U(n+1) - U(n) = u(n)$.

In the following, we firstly show that the matrix integral (2) with $\beta = 2$ satisfies the differential-difference KP equation (6). In fact, the matrix integral (2) with $\beta = 2$ can be rewritten as

$$Z_N^{(\beta=2)}(t_1,t_2,n) = \det\left[\int_{-1}^{\infty}(1+x)^n x^{i+j-2} e^{\eta(x,t)}dx\right]_{i,j=1,\cdots,N}. \tag{20}$$

If we choose

$$f_i(n) = \int_{-1}^{\infty}(1+x)^n x^{i-1} e^{\eta(x,t)}dx,$$

it is easy to verify that $f_i(n)$ given above satisfies

$$\frac{\partial f_i(n)}{\partial t_1} = \Delta f_i(n), \quad \frac{\partial f_i(n)}{\partial t_2} = \Delta^2 f_i(n), \quad \text{for} \quad i = 1,2,\cdots,N. \tag{21}$$

Therefore, $Z_N^{(\beta=2)}(t_1,t_2,n)$ is a solution of (6).

We now turn to establish the relation between the matrix integral (2) with $\beta = 1,4$ and the Pfaffianized differential-difference KP system (8)-(10). In order to do so, we need the following formulae due to de Bruijn [17]:

$$\begin{aligned} &\frac{1}{N!}\int\cdots\int_{x_1\le\cdots\le x_N}\det[\phi_i(x_j)]_{i,j=1,\cdots,N}dx_1\cdots dx_N \\ &\quad = \mathrm{Pf}\left[\int\int sgn(y-x)\phi_i(x)\phi_j(y)dxdy\right]_{i,j=1,\cdots,N} \\ &\quad = \mathrm{Pf}\left[\int\int_{x<y}(\phi_i(x)\phi_j(y) - \phi_j(y)\phi_i(x))dxdy\right]_{i,j=1,\cdots,N}, N \quad \text{even}, \end{aligned} \tag{22}$$

$$\begin{aligned} &\frac{1}{N!}\int\cdots\int\det[\phi_i(x_j),\psi_i(x_j)]_{i=1,\cdots,2N,j=1,\cdots,N}dx_1\cdots dx_N \\ &\quad = Pf\left[\int(\phi_i(x)\psi_j(x) - \phi_j(x)\psi_i(x))dx\right]_{i,j=1,\cdots,2N}. \end{aligned} \tag{23}$$

Similar to [10], we can rewrite $Z_N^{(\beta=1)}(t_1,t_2,n)$ in terms of a Pfaffian [18][13] as

$$\begin{aligned}
&Z_N^{(\beta=1)}(t_1,t_2,n)\\
&=\mathrm{Pf}\left[\int_{-1}^{\infty}\int_{x}^{\infty}(x^{i-1}y^{j-1}-y^{i-1}x^{j-1})(1+x)^n(1+y)^n\exp[\eta(x,t)+\eta(y,t)]dydx\right]_{i,j=1,\cdots,N}, \quad (24)\\
&Z_N^{(\beta=4)}(t_1,t_2,n)=\mathrm{Pf}\left[\int_{-1}^{\infty}(i-j)x^{i+j-3}(1+x)^{2n}\exp[2\eta(x,t)]dx\right]_{i,j=1,\cdots,2N}. \quad (25)
\end{aligned}$$

It is easy to verify that $\mathrm{pf}(i,j)_n$ respectively defined by

$$\begin{aligned}
\mathrm{pf}(i,j)_n&=\int_{-1}^{\infty}\int_{x}^{\infty}(x^{i-1}y^{j-1}-y^{i-1}x^{j-1})(1+x)^n(1+y)^n\exp[\eta(x,t)+\eta(y,t)]dydx,\\
\mathrm{pf}(i,j)_n&=\int_{-1}^{\infty}(i-j)x^{i+j-3}(1+x)^{2n}\exp[2\eta(x,t)]dx,
\end{aligned}$$

satisfy (14)-(16). Therefore

$$\begin{aligned}
\tau^{(\beta)}(n)&=Z_N^{(\beta)}(t_1,t_2,n),\\
\tilde{\tau}^{(\beta)}(n)&=Z_{N+1+\delta_{1,\beta}}^{(\beta)}(t_1,t_2,n),\\
\hat{\tau}^{(\beta)}(n)&=Z_{N-1-\delta_{1,\beta}}^{(\beta)}(t_1,t_2,n)
\end{aligned}$$

with $\beta=1,4$, satisfy (8)-(10) where

$$\delta_{1,\beta}=\begin{cases}1, & \beta=1\\ 0, & \beta\neq 1.\end{cases}$$

3 Matrix integral (3) with $\beta=1,2,4$ and the induced integrable equations

Similar to section 2, we will establish the relation between the matrix integral (3) with $\beta=1,2,4$ and the two-dimensional Toda lattce and its Pfaffianized system. Fist of all, let us review some known results. The so-called two-dimensional Toda lattice is

$$\frac{\partial^2}{\partial t_1\partial s_1}\ln u(n)=u(n+1)+u(n-1)-2u(n). \quad (26)$$

By the dependent variable transformation $u(n)=(\ln\tau_n)_{t_1s_1}$, equation (26) can be transformed into the following bilinear form

$$D_{t_1}D_{s_1}\tau_n\cdot\tau_n=2(e^{D_n}-1)\tau_n\cdot\tau_n. \quad (27)$$

It is known that the two-dimensional Toda lattice (27) has the following solutions expressed by Casorati determinant [13]:

$$\tau_n=\begin{vmatrix}\phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1)\\ \phi_2(n) & \phi_2(n+1) & \cdots & \phi_2(n+N-1)\\ \vdots & \vdots & \ddots & \vdots\\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1)\end{vmatrix}, \quad (28)$$

where $\phi_i(m)$ satisfies the following relations

$$\frac{\partial \phi_i(m)}{\partial t_1} = \phi_i(m+1), \tag{29}$$

$$\frac{\partial \phi_i(m)}{\partial s_1} = -\phi_i(m-1). \tag{30}$$

In [19], we have considered the Pfaffianization of the two-dimensional Toda lattice (27). As a result, the following Pfaffianized two-dimensional Toda lattice is derived:

$$D_{t_1} D_{s_1} f_n \cdot f_n = 2(e^{D_n} - 1) f_n \cdot f_n - 2 g_n \hat{g}_n, \tag{31}$$

$$D_{t_1} e^{-\frac{1}{2}D_n} g_n \cdot f_n = -D_{s_1} e^{\frac{1}{2}D_n} g_n \cdot f_n, \tag{32}$$

$$D_{t_1} e^{-\frac{1}{2}D_n} f_n \cdot \hat{g}_n = -D_{s_1} e^{\frac{1}{2}D_n} f_n \cdot \hat{g}_n. \tag{33}$$

By the dependent variable transformation

$$u(n) = \ln f_n, \quad v(n) = g_n/f_n, \quad w(n) = \hat{g}_n/f_n,$$

we may transform the Pfaffianized two-dimensional Toda lattice (31)-(33) into

$$u_{t_1 s_1}(n) = e^{u(n+1)+u(n-1)-2u(n)} - 1 - \frac{1}{2} v(n) w(n), \tag{34}$$

$$v_{t_1}(n) + v_{s_1}(n+1) + v(n+1)(u_{s_1}(n+1) - u_{s_1}(n)) - v(n)(u_{t_1}(n+1) - u_{t_1}(n)) = 0, \tag{35}$$

$$w_{s_1}(n) + w_{t_1}(n+1) + w(n+1)(u_{t_1}(n+1) - u_{t_1}(n)) - w(n)(u_{s_1}(n+1) - u_{s_1}(n)) = 0. \tag{36}$$

It has also been shown that the Pfaffianized two-dimensional Toda lattice (31)-(33) has the following Pfaffian solution:

$$f_n = \mathrm{pf}(1, 2, \cdots, N)_n, \qquad g_n = \mathrm{pf}(0, 1, \cdots, N, N+1)_n, \qquad \hat{g}_n = \mathrm{pf}(2, 3, \cdots, N-2, N-1)_n,$$

where the Pfaffian entries are chosen to satisfy

$$\frac{\partial}{\partial t_1} \mathrm{pf}(i, j)_n = pf(i+1, j)_n + pf(i, j+1)_n, \tag{37}$$

$$\frac{\partial}{\partial s_1} \mathrm{pf}(i, j)_n = -\mathrm{pf}(i-1, j)_n - \mathrm{pf}(i, j-1)_n, \tag{38}$$

$$\mathrm{pf}(i, j)_{n+1} = \mathrm{pf}(i+1, j+1)_n. \tag{39}$$

In the following, we firstly show that the matrix integral (3) with $\beta = 2$ satisfies the two-dimensional Toda lattice (27). In fact, the matrix integral (3) with $\beta = 2$ can be rewritten as

$$Z_N^{(\beta=2)}(t_1, s_1, n) = \det \left[\int_0^\infty x^{n+i+j-2} e^{\hat{\eta}(x, t_1, s_1)} dx \right]_{i,j=1,\cdots,N}. \tag{40}$$

If we choose

$$\phi_i(n) = \int_0^\infty x^{n+i-1} e^{\hat{\eta}(x, t_1, s_1)} dx,$$

it is easy to verify that $\phi_i(n)$ given above satisfies

$$\frac{\partial \phi_i(n)}{\partial t_1} = \phi_i(n+1), \quad \frac{\partial \phi_i(n)}{\partial s_1} = -\phi_i(n-1), \quad \text{for} \quad i = 1, 2, \cdots, N. \tag{41}$$

Therefore, $Z_N^{(\beta=2)}(t_1, s_1, n)$ is a solution of (27).

We now turn to establish the relation between the matrix integral (3) with $\beta = 1, 4$ and the Pfaffianized two-dimensional Toda system (31)-(33). In order to do so, we need the formulae (22)-(23) due to de Bruijn [17] again. Similar to [10], we can rewrite $Z_N^{(\beta=1)}(t_1, s_1, n)$ and $Z_N^{(\beta=4)}(t_1, s_1, n)$ in terms of Pfaffians as

$$
\begin{aligned}
& Z_N^{(\beta=1)}(t_1, s_1, n) \\
&= \mathrm{Pf}\left[\int_0^\infty \int_x^\infty (x^{i-1}y^{j-1} - y^{i-1}x^{j-1}) x^n y^n \exp[\hat{\eta}(x, t_1, s_1) + \hat{\eta}(y, t_1, s_1)] dy dx\right]_{i,j=1,\cdots,N}, \quad (42)\\
& Z_N^{(\beta=4)}(t_1, s_1, n) = \mathrm{Pf}\left[\int_0^\infty (i-j) x^{i+j-3} x^{2n} \exp[2\hat{\eta}(x, t_1, s_1)] dx\right]_{i,j=1,\cdots,2N}. \quad (43)
\end{aligned}
$$

It is easy to verify that $\mathrm{pf}(i, j)_n$ defined by

$$
\begin{aligned}
\mathrm{pf}(i, j)_n &= \int_0^\infty \int_x^\infty (x^{i-1}y^{j-1} - y^{i-1}x^{j-1}) x^n y^n \exp[\hat{\eta}(x, t_1, s_1) + \hat{\eta}(y, t_1, s_1)] dy dx,\\
\mathrm{pf}(i, j)_n &= \int_0^\infty (i-j) x^{i+j-3} x^{2n} \exp[2\hat{\eta}(x, t_1, s_1)] dx,
\end{aligned}
$$

respectively, satisfy (41). Therefore

$$
\begin{aligned}
f_n^{(\beta)} &= Z_N^{(\beta)}(t_1, s_1, n),\\
g_n^{(\beta)} &= Z_{N+1+\delta_{1,\beta}}^{(\beta)}(t_1, s_1, n-1),\\
\hat{g}_n^{(\beta)} &= Z_{N-1-\delta_{1,\beta}}^{(\beta)}(t_1, s_1, n+1)
\end{aligned}
$$

with $\beta = 1, 4$, satisfy (31)-(33) where

$$
\delta_{1,\beta} = \begin{cases} 1, & \beta = 1\\ 0, & \beta \neq 1. \end{cases}
$$

4 Matrix integral (4) with $\beta = 1, 2, 4$ and the induced integrable equations

In this section, we will show that the matrix integral (4) with $\beta = 1, 2, 4$ satisfy the semi-discrete two-dimensional Toda equation and its Pfaffianized coupled system. Before going into the details, let us recall some known results on these equations. The semi-discrete 2-dimensional Toda equation is written as [20]

$$
\frac{d}{dt_1} \log \frac{\nu_n^{(k+1)}}{\nu_n^{(k)}} = \nu_{n+1}^{(k+1)} + \nu_{n-1}^{(k)} - \nu_n^{(k)} - \nu_n^{(k+1)}. \quad (44)
$$

Through the dependent variable transformation

$$
\nu_n^{(k)} = \frac{\tau_{n+1}^{(k+1)} \tau_{n-1}^{(k)}}{\tau_n^{(k+1)} \tau_n^{(k)}},
$$

equation (1) is transformed into the following bilinear equation

$$
D_{t_1} \tau_n^{(k+1)} \cdot \tau_n^{(k)} - \tau_{n+1}^{(k+1)} \tau_{n-1}^{(k)} + \tau_n^{(k+1)} \tau_n^{(k)} = 0. \quad (45)
$$

It is known that the solution of the semi-discrete Toda equation can be written in the compact form using the Casorati determinant

$$\tau_n^{(k)} = \begin{vmatrix} \phi_1(k,n) & \phi_1(k,n+1) & \cdots & \phi_1(k,n+N-1) \\ \phi_2(k,n) & \phi_2(k,n+1) & \cdots & \phi_2(k,n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(k,n) & \phi_N(k,n+1) & \cdots & \phi_N(k,n+N-1) \end{vmatrix}, \tag{46}$$

where $\phi_i(k,m)$ satisfies the following relations

$$\Delta_{-k}\phi_i(k,m) = \phi_i(k,m+1), \frac{\partial \phi_i(k,m)}{\partial t_1} = -\phi_i(k,m-1). \tag{47}$$

Here Δ_{-k} is the backwards difference operator defined by its action on functions of the discrete variable

$$\Delta_{-k}\phi_i(k,m) = \phi_i(k,m) - \phi_i(k-1,m).$$

A particular solution of (47) is obtained by choosing "exponential type" functions as

$$\phi_i(k,n) = c_i p_i^n (1-p_i)^{-k} e^{-p_i t} + d_i q_i^n (1-q_i)^{-k} e^{-q_i t}, \tag{48}$$

where $p_j, q_j, c_j, d_j (j = 1, \cdots, N)$ are arbitrary constants.

In [21], we have found the following Pfaffinized coupled semi-discrete Toda equation in bilinear form

$$D_{t_1}\tau_n^{(k)} \cdot \tau_n^{(k-1)} - \tau_{n+1}^{(k)}\tau_{n-1}^{(k-1)} + \tau_n^{(k)}\tau_n^{(k-1)} + \sigma_n^{(k)}\tilde{\sigma}_n^{(k-1)} = 0, \tag{49}$$
$$D_{t_1}\tau_n^{(k)} \cdot \tilde{\sigma}_{n-1}^{(k-1)} + \tau_n^{(k)}\tilde{\sigma}_{n-1}^{(k-1)} + \tau_{n-1}^{(k)}\tilde{\sigma}_n^{(k-1)} - \tau_{n-1}^{(k-1)}\tilde{\sigma}_n^{(k)} = 0, \tag{50}$$
$$D_{t_1}\sigma_n^{(k)} \cdot \tau_{n-1}^{(k-1)} + \tau_{n-1}^{(k-1)}\sigma_n^{(k)} + \tau_n^{(k-1)}\sigma_{n-1}^{(k)} - \tau_n^{(k)}\sigma_{n-1}^{(k-1)} = 0. \tag{51}$$

and shown that $\tau_n^{(k)}, \sigma_n^{(k)}$ and $\tilde{\sigma}_n^{(k)}$ given by

$$\tau_n^{(k)} = \mathrm{pf}(1,2,\cdots,N)_n^{(k)}, \quad \sigma_n^{(k)} = \mathrm{pf}(0,1,\cdots,N+1)_n^{(k)}, \quad \tilde{\sigma}_n^{(k)} = \mathrm{pf}(2,3,\cdots,N-1)_n^{(k)}$$

are a solution of (49)-(51) if the entries in our Pfaffian are chosen to satisfy

$$\Delta_{-k}\mathrm{pf}(i,j)_n^{(k)} = \mathrm{pf}(i+1,j)_n^{(k)} + \mathrm{pf}(i,j+1)_n^{(k)} - \mathrm{pf}(i+1,j+1)_n^{(k)}, \tag{52}$$
$$\frac{\partial}{\partial t_1}\mathrm{pf}(i,j)_n^{(k)} = -\mathrm{pf}(i-1,j)_n^{(k)} - \mathrm{pf}(i,j-1)_n^{(k)}, \tag{53}$$
$$\mathrm{pf}(i,j)_{n+1}^{(k)} = \mathrm{pf}(i+1,j+1)_n^{(k)}. \tag{54}$$

By the dependent variable transformation

$$u_n^{(k)} = \ln \tau_n^{(k)}, \quad v_n^{(k)} = \sigma_n^{(k)}/\tau_n^{(k)}, \quad w_n^{(k)} = \tilde{\sigma}_n^{(k)}/\tau_n^{(k)},$$

we may transform the Pfaffianized semi-discrete Toda lattice (16)-(18) into

$$u_{n,t_1}^{(k)} - u_{n,t_1}^{(k-1)} + v_n^{(k)}w_n^{(k-1)} + 1 - e^{(u_{n+1}^{(k)} + u_{n-1}^{(k-1)} - u_n^{(k-1)} - u_n^{(k)})} = 0, \tag{55}$$
$$v_{n,t_1}^{(k)} + v_n^{(k)}u_{n,t_1}^{(k)} - v_n^{(k)}u_{n-1,t_1}^{(k-1)} - v_{n-1}^{(k-1)} + v_n^{(k)} + v_{n-1}^{(k)}e^{(u_{n-1}^{(k)} + u_n^{(k-1)} - u_n^{(k)} - u_{n-1}^{(k-1)})} = 0, \tag{56}$$
$$\begin{aligned} &w_{n-1,t_1}^{(k-1)} - u_{n,t_1}^{(k)}w_{n-1}^{(k-1)} + w_{n-1}^{(k-1)}u_{n-1,t_1}^{(k-1)} \\ &\quad + w_n^{(k)} - w_{n-1}^{(k-1)} - w_n^{(k-1)}e^{(u_{n-1}^{(k)} + u_n^{(k-1)} - u_n^{(k)} - u_{n-1}^{(k-1)})} = 0. \end{aligned} \tag{57}$$

In the following, we firstly show that the matrix integral (4) with $\beta=2$ satisfies the semi-discrete Toda equation (47). In fact, the matrix integral (4) with $\beta=2$ can be rewritten as

$$Z_N^{(\beta=2)}(t_1,n,k)=\det\left[\int_0^\infty (1+x)^k x^{n+i+j-2}e^{\bar{\eta}(x,t_1)}dx\right]_{i,j=1,\cdots,N}. \tag{58}$$

If we choose

$$\phi_i(k,n)=\int_0^\infty (1+x)^k x^{n+i-1}e^{\bar{\eta}(x,t_1)}dx,$$

it is easy to verify that $\phi_i(k,n)$ given above satisfies

$$\frac{\partial\phi_i(k,n)}{\partial t_1}=\Delta\phi_i(k,n),\quad \Delta_{-k}\phi_i(k,n)=\phi_i(k,n+1),\quad \text{for}\quad i=1,2,\cdots,N. \tag{59}$$

Therefore, $Z_N^{(\beta=2)}(t_1,t_2,n)$ is a solution of (45).

We now turn to establish the relation between the matrix integral (2) with $\beta=1,4$ and the Pfaffianized semi-discrete Toda system (49)-(51). In order to do so, we need the formulae due to de Bruijn [17] again. Similar to [10], we can rewrite $Z_N^{(\beta=1)}(t_1,n,k)$ and $Z_N^{(\beta=1)}(t_1,n,k)$ in terms of Pfaffians as

$$\begin{aligned}&Z_N^{(\beta=1)}(t_1,n,k)\\&=\mathrm{Pf}\left[\int_0^\infty\int_x^\infty (x^{n+i-1}y^{n+j-1}-y^{n+i-1}x^{n+j-1})(1+x)^k(1+y)^k\exp[\bar{\eta}(x,t_1)+\bar{\eta}(y,t_1)]dydx\right]_{i,j=1,\cdots,N}\end{aligned} \tag{60}$$

$$Z_N^{(\beta=4)}(t_1,n,k)=\mathrm{Pf}\left[\int_0^\infty (i-j)x^{i+j-3}(1+x)^{2k}x^{2n}\exp[2\bar{\eta}(x,t_1)]dx\right]_{i,j=1,\cdots,2N}. \tag{61}$$

It is easy to verify that $\mathrm{pf}(i,j)_n$ defined by

$$\mathrm{pf}(i,j)_n^{(k)}=\int_0^\infty\int_x^\infty (x^{n+i-1}y^{n+j-1}-y^{n+i-1}x^{n+j-1})(1+x)^k(1+y)^k\exp[\bar{\eta}(x,t_1)+\bar{\eta}(y,t_1)]dydx,$$

$$\mathrm{pf}(i,j)_n^{(k)}=\int_0^\infty (i-j)x^{i+j-3}(1+x)^{2k}x^{2n}\exp[2\bar{\eta}(x,t_1)]dx,$$

respectively, satisfy (52)-(54). Therefore

$$\begin{aligned}\tau^{(\beta)}(n,k,t_1)&=Z_N^{(\beta)}(t_1,n,k),\\ \sigma^{(\beta)}(n,k,t_1)&=Z_{N+1+\delta_{1,\beta}}^{(\beta)}(t_1,n-1,k),\\ \tilde{\sigma}^{(\beta)}(n,k,t_1)&=Z_{N-1-\delta_{1,\beta}}^{(\beta)}(t_1,n+1,k)\end{aligned}$$

with $\beta=1,4$, satisfy (49)-(51).

5 Conclusion and discussions

In this paper, we have clarified the relations between the matrix integrals (2)-(4) with $\beta=1,2,4$ and their associated integrable differential-difference systems. It has been shown that these matrix integrals with special $\beta=2$ and $1,4$ satisfy the differential-difference KP equation, the two-dimensional Toda lattice, the semi-discrete Toda equation and their Pfaffianized systems, respectively. Finally, It is remaked that if we replace $\eta(x,t),\hat{\eta}(x,t_1,s_1)$ and $\bar{\eta}(x,t_1)$ in (2)-(4) by $\eta(x,t)=\sum_{m=1}^\infty x^m t_m+\eta_0(x)$, $\hat{\eta}(x,t,s)=\sum_{m=1}^\infty (x^m t_m+(-x)^{-m}s_m)+\hat{\eta}_0(x)$,$\bar{\eta}(x,t)=\sum_{m=1}^\infty x^m t_m+\bar{\eta}_0(x)$, it seems possible to link the corresponding matrix integrals with $\beta=2$ and $1,4$ with higher-order members of the hierarchies of the differential-difference KP, the two-dimensional Toda lattice, the semi-discrete Toda equation and their Pfaffianized systems. We hope that it will be solved.

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References

[1] G. Mathoux and M.L. Mehta, J. Phys. I France **1**(1991)1093.

[2] C.A. Tracy and H. Widom, J. Stat. Phys. **92**(1998)809-835.

[3] T. Nagao and M. Wadati, J. Phys. Soc. Japan **60**(1991)3298; **61**(1992)78; **61**(1992)1910.

[4] M. Adler, P.J. Forrester, T.Nagao and P. van Moerbeke, J. Stat. Phys. **99**(2000), 141.

[5] V.A. Kazakov, I.K. Kostov and N. Nekrasov, Nucl. Phys. B **557**(1999)413.

[6] M. Adler and P. van Moerbeke, Ann. of Math. **153**(2001)149.

[7] M. Adler, T. Shiota and P. van Moerbeke, Math. Ann. **322**(2002)423.

[8] M. Adler, V.B. Kuznetsov and P. van Moerbeke, Ergod. Th. Dynam. Sys. **22**(2002), 1365.

[9] M. Adler and P. van Moerbeke, Duke Math. J. **112**(2002), 1

[10] S. Kakei, J. Phys. Soc. Japan **68**(1999)2875.

[11] E. Date, M. Jimbo and T. Miwa, *J.Phys.Soc.Japan* **51**(1982), 4116.

[12] T. Tamizhmani,V.S. Kanaga and K.M. Tamizhmani, *J.Phys.A* **31**(1998) 7627.

[13] R.Hirota:*Direct method in soliton theory(In Japanese)*, (Iwanami Shoten, 1992); R.Hirota,: *Direct method in soliton theory(In English)*, (Edited and Translated by A.Nagai, J. Nimmo,and C.Gilson, ,Cambridge University Press, 2004.6).

[14] R. Hirota, Direct method in soliton theory, Solitons ed. R.K. Bullough and P. J. Caudrey (Berlin: Springer) 1980.

[15] R. Hirota and J. Satsuma,Prog. Theor. Phys. Suppl. **59**(1976), 64.

[16] J.X. Zhao, C.X. Li and X.B. Hu, J.Phys.Soc.Japan **73**(2004)3275.

[17] N.G. de Bruijn, J. Ind. Math. Soc. **19**(1955),133-151.

[18] Caieniello E R : *Combinatorics and Renomalization in Quantum Field Theory* (Benjamin, 1973).

[19] X.B. Hu, J.X. Zhao and H.W. Tam, J.Math.Anal.Appl. **296**(2004)256

[20] R. Inoue and K. Hikami, J. Phys. A **32**(1999)6853.

[21] C.X. Li and X.B. Hu, Phys.Lett.A **329**(2004)193