

# Expansions of Step-transition Operators of Multi-step Methods and Order Barriers for Dahlquist Pairs<sup>1</sup>

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## ABSTRACT

Using least parameters, we expand the step-transition operator of any linear multi-step method (**LMSM**) up to  $O(\tau^{s+5})$  with order  $s = 1$  and rewrite the expansion of the step-transition operator for  $s = 2$  (obtained by the second author in a former paper). We prove that in the Dahlquist relation  $G_3^{\lambda\tau} \circ G_1^\tau = G_2^\tau \circ G_3^{\lambda\tau}$  with  $G_1$  being an **LMSM**, (1) the order of  $G_2$  can not be higher than that of  $G_1$ ; (2) if  $G_3$  is also an **LMSM** and  $G_2$  is a symplectic *B-series*, then the order of both  $G_1$  and  $G_2$  must be 2.

**Keywords:** *Linear Multi-Step Method; Step-Transition Operator; B-series; Dahlquist (Conjugate) Relation; Symplecticity*

## 1. Introduction

For an ordinarily differential equation (**ODE**)

$$\frac{d}{dt}Z = f(Z), \quad Z \in R^p, \quad (1)$$

any compatible linear  $m$ -step difference scheme (**DS**)

$$\sum_{k=0}^m \alpha_k Z_k = \tau \sum_{k=0}^m \beta_k f(Z_k) \quad \left( \sum_{k=0}^m \beta_k \neq 0 \right) \quad (2)$$

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can be characterized by a step-transition operator (**STO**)  $G$  (also denoted by  $G^\tau$ ):  $\mathbb{R}^p \rightarrow \mathbb{R}^p$  satisfying

$$\sum_{k=0}^m \alpha_k G^k = \tau \sum_{k=0}^m \beta_k f \circ G^k, \quad (3)$$

where  $G^k$  stands for  $k$ -time composition of  $G$ :  $G \circ G \cdots \circ G$  (refer to [2,3,5,6,7]). This operator  $G^\tau$  can be represented as a power series in  $\tau$  with first term equal to *identity*  $I$ . More precisely, one can expand<sup>[9]</sup> the **STO**  $G^\tau(Z)$  of any linear multi-step method (**LMSM**)<sup>3</sup> of form (2) with order  $s \geq 2$  up to  $O(\tau^{s+5})$ :

$$G^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{s+1} A(Z) + \tau^{s+2} B(Z) + \tau^{s+3} C(Z) + \tau^{s+4} D(Z) + O(\tau^{s+5}) \quad (4)$$

(where  $Z^{[0]} = Z$ ,  $Z^{[1]} = f(Z)$ ,  $Z^{[k+1]} = \frac{\partial Z^{[k]}}{\partial Z} Z^{[1]} = Z_z^{[k]} Z^{[1]}$  for  $k = 1, 2, \dots$ ) with complete formulae for calculation of  $A(Z)$ ,  $B(Z)$ ,  $C(Z)$  and  $D(Z)$ .

Thus, the **STO**  $G^\tau$  satisfying equation (3) completely characterizes the **LMSM** (2) as:  $Z_1 = G^\tau(Z_0)$ ,  $\dots$ ,  $Z_m = G^\tau(Z_{m-1}) = [G^\tau]^m(Z_0)$ ,  $\dots$ .

When equation (1) is a hamiltonian system, i.e.,  $p = 2n$  and  $f(Z) = J\nabla H(Z)$ , where  $J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$ ,  $\nabla$  stands for the gradient operator, and  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}^1$  is a smooth function, (1), (2) and (3) become

$$\frac{dZ}{dt} = J\nabla H(Z), \quad Z \in \mathbb{R}^{2n}, \quad (5)$$

$$\sum_{k=0}^m \alpha_k Z_k = \tau \sum_{k=0}^m \beta_k J\nabla H(Z_k) \quad \left( \sum_{k=0}^m \beta_k \neq 0 \right), \quad (6)$$

$$\sum_{k=0}^m \alpha_k G^k = \tau \sum_{k=0}^m \beta_k J\nabla H \circ G^k, \quad (7)$$

and we can rewrite

$$\begin{aligned} Z^{[0]} &= Z, \\ Z^{[1]} &= J\nabla H, \\ Z^{[2]} &= JH_{zz} J\nabla H = Z_z^{[1]} Z^{[1]}, \\ Z^{[3]} &= Z_{z^2}^{[1]} (Z^{[1]})^2 + Z_z^{[1]} Z^{[2]}, \\ Z^{[4]} &= Z_{z^3}^{[1]} (Z^{[1]})^3 + 3Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} + Z_z^{[1]} Z^{[3]}, \end{aligned} \quad (8)$$

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<sup>3</sup>More generally, one can use an **STO** to characterize any **DS** compatible with (1), and obviously the **STO** can be written in form (4).

$$Z^{[5]} = Z_{z^4}^{[1]} (Z^{[1]})^4 + 6Z_{z^3}^{[1]} (Z^{[1]})^2 Z^{[2]} + 3Z_{z^2}^{[1]} (Z^{[2]})^2 + 4Z_{z^2}^{[1]} Z^{[1]} Z^{[3]} + Z_z^{[1]} Z^{[4]},$$

and generally,

$$Z^{[r+1]} = \sum_{j=1}^r \sum_{i_1+i_2+\dots+i_j=r; i_u \geq 1} \frac{r! \Omega(i_1, i_2, \dots, i_j)}{j! i_1! i_2! \dots i_j!} J(\nabla H)_{z^j} Z^{[i_1]} Z^{[i_2]} \dots Z^{[i_j]}$$

where  $i_1 \leq i_2 \leq \dots \leq i_j$ ,  $\Omega(i_1, i_2, \dots, i_j)$  is the number of all different permutations of  $\{i_1, i_2, \dots, i_j\}$ , and  $(\nabla H)_{z^j} Z^{[i_1]} Z^{[i_2]} \dots Z^{[i_j]}$  stands for the multi-linear form

$$\sum_{1 \leq t_1, \dots, t_j \leq 2n} \frac{\partial^j (\nabla H)}{\partial Z_{(t_1)} \dots \partial Z_{(t_j)}} Z_{(t_1)}^{[i_1]} \dots Z_{(t_j)}^{[i_j]},$$

$Z_{(t_u)}^{[i_u]}$  stands for the  $t_u$ -th component of the  $2n$ -dim vector  $Z^{[i_u]}$ .

The expansion of **STO** (4) has been used to study the *symplecticity* of **LMSM** (refer to [3], [7]), and also the *symplecticity* of Dahlquist pair (refer to [8]).

**Definition 1.** (due to K. Feng<sup>[2],[7]</sup>) *An LMSM is said to be symplectic for Hamiltonian system (5) iff its STO  $G$  defined by (7) is symplectic, i.e.,*

$$\left[ \frac{\partial G(Z)}{\partial Z} \right]^\top J \left[ \frac{\partial G(Z)}{\partial Z} \right] = J \quad (9)$$

for any hamiltonian function  $H$  and any sufficiently small step-size  $\tau$ .

**Definition 2.** *If three B-serieses<sup>4</sup>  $G_1^\tau$ ,  $G_2^\tau$  and  $G_3^\tau$  in form (4) compatible with equation (1) satisfy*

$$G_3^{\lambda\tau} \circ G_1^\tau = G_2^\tau \circ G_3^{\lambda\tau} \quad (10)$$

for some real number  $\lambda$  and for any smooth function  $f$  and any sufficiently small step-size  $\tau$ , then  $G_1^\tau$  and  $G_2^\tau$  are said to be a Dahlquist<sup>5</sup> pair or a conjugate pair via  $G_3^\tau$ , and we call equation (10) a Dahlquist relation or a conjugate relation. A Dahlquist pair  $G_1^\tau$  and  $G_2^\tau$  is said to be symplectic if  $G_1^\tau$  or  $G_2^\tau$  is symplectic. In this case when one of  $G_1^\tau$  and  $G_2^\tau$  is symplectic, we also call the other conjugate-symplectic.

In the present paper, for any linear multi-step method (**LMSM**) with order  $s = 1$ , using 6 parameters we obtain the expansion of its step-transition operator in form (4) up to  $O(\tau^6)$ ; and using 5 parameters we rewrite the expansion of the step-transition operator for  $s = 2$  (obtained by Tang in a former paper [9] where 9 parameters are used) (in Section

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<sup>4</sup>For the details about  $B$ -series, one can refer to [4]. We would like to thank Ernst Hairer for the suggestion that the case when  $G_3^\tau$  is a  $B$ -series should be considered in the conjugate relation.

<sup>5</sup>It was G. Dahlquist<sup>[1]</sup> who found that the trapezoid rule and the mid-point rule are a conjugate pair via the Euler-forward scheme.

2). We prove that in Dahlquist relation (10) with  $G_1$  being an **LMSM**, (1) the order of  $G_2$  can not be higher than that of  $G_1$  (that means, conjugation will not improve the order of any **LMSM**); (2) if  $G_3$  is also an **LMSM** and  $G_2$  is a symplectic  $B$ -series, then the order of both  $G_1$  and  $G_2$  must be 2 (in Section 3).

## 2. Expansion of Step-Transition Operator

**Theorem 1.** *If scheme (2) is of order  $s = 1$ , then the corresponding step-transition operator defined by (3) has the following expansion:*

$$G(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^2 C(Z) + \tau^3 D(Z) + \tau^4 E(Z) + \tau^5 F(Z) + O(\tau^6), \quad (11)$$

where  $C(Z), D(Z), E(Z), F(Z)$  can be determined by 6 parameters  $\omega, \rho, \delta, \sigma, \eta, \nu$ :

$$C = \omega Z^{[2]}, \quad \omega = \frac{\sum_{k=0}^m \left( k\beta_k - \frac{k^2}{2}\alpha_k \right)}{\sum_{k=0}^m k\alpha_k}; \quad (12.1)$$

$$D = \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) Z^{[3]} + \omega \left( \omega + \frac{1}{2} \right) Z_z^{[1]} Z^{[2]}, \quad (12.2)$$

$$\rho = \frac{\sum_{k=0}^m \left[ k^2\beta_k - \frac{k^3}{3}\alpha_k \right]}{\sum_{k=0}^m k\alpha_k}, \quad \delta = \frac{\sum_{k=0}^m k^2\alpha_k}{\sum_{k=0}^m k\alpha_k};$$

$$E = \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) Z^{[4]} \quad (12.3)$$

$$+ \left( \rho\omega - \frac{3\delta\omega^2}{4} + \frac{3\omega^2}{4} + \frac{\omega}{3} \right) Z_{z^2}^{[1]} Z^{[1]} Z^{[2]}$$

$$+ \left( -\delta\omega^2 + \frac{\omega^2}{2} + \rho\omega - \frac{\omega\delta}{4} + \frac{\omega}{6} + \frac{\rho}{4} \right) Z_z^{[1]} Z^{[3]}$$

$$+ \left( \omega^3 + \frac{3\omega^2}{4} - \frac{\omega^2\delta}{4} + \frac{\omega\rho}{2} + \frac{\omega}{6} \right) Z_z^{[1]} Z_z^{[1]} Z^{[2]},$$

$$\sigma = \frac{\sum_{k=0}^m \left( \frac{k^3}{2}\beta_k - \frac{k^4}{8}\alpha_k \right)}{\sum_{k=0}^m k\alpha_k} \omega, \quad \eta = \frac{\sum_{k=0}^m k^3\alpha_k}{\sum_{k=0}^m k\alpha_k};$$

$$F = \nu Z^{[5]} \quad (12.4)$$

$$+ \left\{ -\frac{3\omega^2\delta}{4} + \frac{5\omega^2\delta^2}{8} - \frac{5\eta\omega^2}{12} + \frac{13\omega^2}{24} - \frac{3\omega\rho\delta}{4} + \frac{3\omega\rho}{4} + \sigma\omega + \frac{\omega}{8} \right\} Z_{z^3}^{[1]} (Z^{[1]})^2 Z^{[2]}$$

$$+ \left\{ \frac{\omega^3}{2} - \frac{\delta\omega^3}{2} + \frac{5\delta^2\omega^2}{8} - \frac{3\delta\omega^2}{4} + \frac{\rho\omega^2}{2} - \frac{5\eta\omega^2}{12} + \frac{17\omega^2}{24} - \frac{3\delta\rho\omega}{4} + \frac{3\omega\rho}{4} + \sigma\omega + \frac{\omega}{8} \right\}$$

$$Z_{z^2}^{[1]} (Z^{[2]})^2$$

$$\begin{aligned}
& + \left\{ \frac{9\delta^2\omega^2}{8} - \frac{5\delta\omega^2}{4} - \frac{5\eta\omega^2}{12} + \frac{13\omega^2}{24} - \frac{7\delta\rho\omega}{4} + \frac{5\rho\omega}{4} + \sigma\omega - \frac{\omega\delta}{6} + \frac{\omega}{8} + \frac{\rho^2}{2} + \frac{\rho}{6} \right\} \\
& Z_{z^2}^{[1]} Z^{[1]} Z^{[3]} \\
& + \left\{ -\delta\omega^3 + \omega^3 + \frac{5\omega^2\delta^2}{8} - \frac{3\omega^2\delta}{4} + \rho\omega^2 - \frac{5\eta\omega^2}{12} + \frac{7\omega^2}{8} - \frac{3\delta\rho\omega}{4} + \frac{3\rho\omega}{4} + \sigma\omega + \frac{\omega}{8} \right\} \\
& Z_{z^2}^{[1]} Z^{[1]} Z_z^{[1]} Z^{[2]} \\
& + \left\{ -\frac{\eta\omega^2}{3} + \frac{3\omega^2\delta^2}{4} - \frac{\omega^2\delta}{2} + \frac{\omega^2}{6} + \frac{\delta^2\omega}{8} - \delta\rho\omega + \frac{2\sigma\omega}{3} + \frac{\rho\omega}{2} - \frac{\omega\delta}{12} - \frac{\eta\omega}{12} + \frac{\omega}{24} \right. \\
& \quad \left. + \frac{\rho^2}{4} + \frac{\rho}{12} + \frac{\sigma}{6} - \frac{\delta\rho}{8} \right\} Z_z^{[1]} Z^{[4]} \\
& + \left\{ -\frac{7\delta\omega^3}{4} + \omega^3 - \frac{3\delta\omega^2}{4} + \frac{3\delta^2\omega^2}{8} + \frac{7\rho\omega^2}{4} - \frac{\eta\omega^2}{3} + \frac{7\omega^2}{12} - \frac{\delta\rho\omega}{2} + \frac{2\sigma\omega}{3} + \rho\omega + \frac{\omega}{12} \right\} \\
& Z_z^{[1]} Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\
& + \left\{ -\frac{3\delta\omega^3}{2} + \frac{\omega^3}{2} + \frac{3\delta^2\omega^2}{8} - \delta\omega^2 + \frac{3\rho\omega^2}{2} - \frac{\omega^2\eta}{12} + \frac{3\omega^2}{8} - \frac{3\delta\rho\omega}{4} + \frac{\sigma\omega}{3} + \rho\omega + \frac{\omega}{24} \right. \\
& \quad \left. + \frac{\rho^2}{4} + \frac{\rho}{12} - \frac{\omega\delta}{12} \right\} Z_z^{[1]} Z_z^{[1]} Z^{[3]} \\
& + \left\{ \omega^4 - \frac{5\delta\omega^3}{4} + \frac{3\omega^3}{2} + \frac{\delta^2\omega^2}{4} - \frac{\eta\omega^2}{4} + \frac{5\rho\omega^2}{4} - \frac{\delta\omega^2}{2} + \frac{5\omega^2}{8} - \frac{\delta\rho\omega}{4} + \frac{\rho\omega}{2} + \frac{\delta\omega}{8} \right. \\
& \quad \left. - \frac{\eta\omega}{12} + \frac{\sigma\omega}{3} \right\} Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[2]},
\end{aligned}$$

$$\begin{aligned}
\nu = \frac{1}{\sum_{k=0}^m k\alpha_k} \sum_{k=0}^m \left\{ \frac{k^4}{4!} \beta_k - \left[ \frac{k^5}{5!} + \frac{k^4 - 2k^3 + k^2}{24} \omega + \frac{2k^3 - 3k^2 + k}{12} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \right. \\
\left. \left. + \frac{k^2 - k}{2} \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) \right] \alpha_k \right\}.
\end{aligned}$$

Here we use the notation for example,

$$Z_{z^3}^{[1]} (Z^{[1]})^2 Z^{[2]} = \sum_{i,j,k=1}^p \frac{\partial^3 Z^{[1]}}{\partial z_i \partial z_j \partial z_k} [Z^{[1]}]_{(i)} [Z^{[1]}]_{(j)} [Z^{[2]}]_{(k)}$$

where  $z_i$  is the  $i$ -th component of  $p$ -dim vector  $Z$ , and  $[Z^{[r]}]_{(j)}$  stands for the  $j$ -th component of  $p$ -dim vector  $Z^{[r]}$ .

The proof of Theorem 1 is tedious but straightforward calculation, and similar to that for  $s \geq 2$  given in [9]. A difference is that we here try to use least parameters in expressing  $C(Z)$ ,  $D(Z)$ ,  $E(Z)$  and  $F(Z)$ . We give the complete proof of Theorem 1 later in **Appendix 1**.

Similar result for  $s \geq 2$  is already given in [9], where 9 parameters  $\lambda$ ,  $\mu$ ,  $\nu$ ,  $\rho$ ,  $\xi$ ,  $\sigma$ ,  $\chi$ ,  $\eta$  and  $\zeta$  are used for expressing  $A(Z)$ ,  $B(Z)$ ,  $C(Z)$  and  $D(Z)$  in (4). Using 5 parameters

$\omega_2, \rho_2, \delta_2, \sigma_2$  and  $\nu_2$ , we rewrite the result for  $s = 2$  as follows:

**Theorem 2.** *If scheme (2) is of order  $s = 2$ , then the step-transition operator decided by equation (3) has the following expansion:*

$$G(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^3 C(Z) + \tau^4 D(Z) + \tau^5 E(Z) + \tau^6 F(Z) + O(\tau^7), \quad (13)$$

where  $C(Z), D(Z), E(Z), F(Z)$  can be expressed by 5 parameters  $\omega_2, \rho_2, \delta_2, \sigma_2$  and  $\nu_2$ :

$$C = \omega_2 Z^{[3]}, \quad \omega_2 = \frac{\sum_{k=0}^m \left\{ \frac{k^2}{2} \beta_k - \frac{k^3}{6} \alpha_k \right\}}{\sum_{k=0}^m k \alpha_k}; \quad (14.1)$$

$$D = \left( \rho_2 - \frac{1}{2} \delta_2 \omega_2 + \frac{\omega_2}{2} \right) Z^{[4]} + \frac{\omega_2}{2} Z_z^{[1]} Z^{[3]}, \quad (14.2)$$

$$\rho_2 = \frac{\sum_{k=0}^m \left[ \frac{k^3}{6} \beta_k - \frac{k^4}{24} \alpha_k \right]}{\sum_{k=0}^m k \alpha_k}, \quad \delta_2 = \frac{\sum_{k=0}^m k^2 \alpha_k}{\sum_{k=0}^m k \alpha_k};$$

$$E = \sigma_2 Z^{[5]} + \left( \omega_2^2 + \frac{\omega_2}{6} \right) Z_z^{[1]} Z_z^{[1]} Z^{[3]} \quad (14.3)$$

$$+ \left( \omega_2^2 - \frac{1}{4} \delta_2 \omega_2 + \frac{\omega_2}{6} + \frac{\rho_2}{2} \right) Z_z^{[1]} Z^{[4]} + \left( 2\omega_2^2 + \frac{\omega_2}{3} \right) Z_{z^2}^{[1]} Z^{[1]} Z^{[3]},$$

$$\sigma_2 = \frac{\sum_{k=0}^m \left[ \frac{k^4}{24} \beta_k - \frac{k^5}{120} \alpha_k - \left\{ \frac{2k^3 - 3k^2 + k}{12} \omega_2 + \frac{k^2 - k}{2} \left( \rho_2 - \frac{1}{2} \delta_2 \omega_2 + \frac{\omega_2}{2} \right) \right\} \alpha_k \right]}{\sum_{k=0}^m k \alpha_k};$$

$$F = \nu_2 Z^{[6]} \quad (14.4)$$

$$+ \left\{ -\delta_2 \omega_2^2 + \frac{\omega_2^2}{2} + 2\rho_2 \omega_2 + \frac{1}{24} \delta_2 \omega_2 - \frac{\omega_2}{24} - \frac{\rho_2}{12} + \frac{\sigma_2}{2} \right\} Z_z^{[1]} Z^{[5]}$$

$$+ \left\{ -\delta_2 \omega_2^2 + \omega_2^2 + 2\rho_2 \omega_2 - \frac{1}{12} \delta_2 \omega_2 + \frac{\omega_2}{24} + \frac{\rho_2}{6} \right\} Z_z^{[1]} Z_z^{[1]} Z^{[4]}$$

$$+ \left\{ -\frac{5}{2} \delta_2 \omega_2^2 + \frac{3}{2} \omega_2^2 + 5\rho_2 \omega_2 - \frac{1}{6} \delta_2 \omega_2 + \frac{\omega_2}{8} + \frac{\rho_2}{3} \right\} Z_{z^2}^{[1]} Z^{[1]} Z^{[4]}$$

$$+ \left\{ -\frac{1}{2} \delta_2 \omega_2^2 + \omega_2^2 + \rho_2 \omega_2 + \frac{\omega_2}{24} \right\} Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[3]}$$

$$+ \left\{ -\delta_2 \omega_2^2 + 2\omega_2^2 + 2\rho_2 \omega_2 + \frac{\omega_2}{12} \right\} Z_z^{[1]} Z_{z^2}^{[1]} Z^{[1]} Z^{[3]}$$

$$+ \left\{ -\frac{3}{2} \delta_2 \omega_2^2 + \frac{3}{2} \omega_2^2 + 3\rho_2 \omega_2 + \frac{\omega_2}{8} \right\} Z_{z^2}^{[1]} Z^{[1]} (Z_z^{[1]} Z^{[3]})$$

$$+ \left\{ -\frac{3}{2} \delta_2 \omega_2^2 + \frac{3}{2} \omega_2^2 + 3\rho_2 \omega_2 + \frac{\omega_2}{8} \right\} Z_{z^2}^{[1]} Z^{[2]} Z^{[3]}$$

$$+ \left\{ -\frac{3}{2} \delta_2 \omega_2^2 + \frac{3}{2} \omega_2^2 + 3\rho_2 \omega_2 + \frac{\omega_2}{8} \right\} Z_{z^3}^{[1]} (Z^{[1]})^2 Z^{[3]},$$

$$\nu_2 = \frac{\sum_{k=0}^m \left\{ \frac{k^5}{5!} \beta_k - \left[ \frac{k^6}{6!} + \frac{k^4 - 2k^3 + k^2}{24} \omega_2 + \frac{2k^3 - 3k^2 + k}{12} \left( \rho_2 - \frac{\delta_2 \omega_2}{2} + \frac{\omega_2}{2} \right) + \frac{k^2 - k}{2} \sigma_2 \right] \alpha_k \right\}}{\sum_{k=0}^m k \alpha_k}.$$

### 3. Order Barriers for STOs in Dahlquist Relation

**Theorem 3.** *In Dahlquist relation (10), if B-series  $G_1$  stands for an LMSM, then the order of  $G_2$  can not be higher than that of  $G_1$ .*

*Proof.* Supposing the order of  $G_1^\tau$ ,  $G_2^\tau$  and  $G_3^\tau$  are  $u$ ,  $v$  and  $w - 1$  respectively, we write their expansions as follows:

$$G_1^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{u+1} A(Z) + O(\tau^{u+2}) \quad (15)$$

$$G_2^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{v+1} M(Z) + O(\tau^{v+2}) \quad (16)$$

$$G_3^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^w B(Z) + O(\tau^{w+1}) \quad (17)$$

where  $A(Z) \neq 0$ ,  $B(Z) \neq 0$  and  $M(Z) \neq 0$ .

Provided  $v > u$ , there are three cases:

Case 1.  $w > u$ , expanding both sides of (10) and comparing the terms in  $\tau^{u+1}$  we have

$$A(Z) = 0. \quad (18)$$

Case 2.  $w = u$ , expanding both sides of (10) and comparing the terms in  $\tau^{u+1}$  we have

$$\lambda^w B_z Z^{[1]} + A(Z) = \lambda^w Z_z^{[1]} B. \quad (19)$$

Case 3.  $w < u$ , expanding both sides of (10) and comparing the terms in  $\tau^{w+1}$  we have

$$\lambda^w B_z Z^{[1]} = \lambda^w Z_z^{[1]} B. \quad (20)$$

From Theorem 1, Theorem 2 in Section 2 above, and Lemma 1 in [7], we know that in fact  $A(Z) = aZ^{[u+1]}$  for some  $a \neq 0$ . Since B-series  $G_3^\tau$  is compatible with (3),  $w \geq 2$ . When  $\lambda \neq 0$ , it's easy to check that any of the cases (18), (19) and (20) is impossible; when  $\lambda = 0$ , equation (10) becomes into  $G_1^\tau(Z) = G_2^\tau(Z)$  which contradicts  $v > u$ .

So the only possible case should be  $v \leq u$ .  $\square$

**Theorem 4.** *In Dahlquist relation (10), if both  $G_1$  and  $G_3$  stand for LMSMs, and  $G_2$  is a symplectic  $B$ -series<sup>6</sup>, then the orders of  $G_1^\tau$ ,  $G_2^\tau$  and  $G_3^\tau$  are 2, 2 and 1 respectively.*

The result in Theorem 4 is a little different from that in Theorem 1 in [8], and the proof of the former will also be based on the latter.

*Proof of Theorem 4.* Supposing the order of  $G_1^\tau$ ,  $G_2^\tau$  and  $G_3^\tau$  are  $u$ ,  $v$  and  $w - 1$  respectively. Since they are compatible with (3),  $u \geq 1$ ,  $v \geq 1$  and  $w \geq 2$ . We write their expansions as (15-17). According to Theorem 1 in [8], if  $G_2$  is symplectic, then the order of  $G_1^\tau$  can not be greater than 2. So  $1 \leq u \leq 2$ . And according to Theorem 3 above, we know  $v \leq u$ . Let's discuss all the cases as follows:

Case 1. If  $u = 1$ , then  $v = 1$ . Expanding both sides of (10) and comparing the terms in  $\tau^2$  we have

$$A(Z) = M(Z). \quad (21)$$

Case 2. If  $u = 2$ ,  $v = 1$ . Expanding both sides of (10) and comparing the terms in  $\tau^2$  we have

$$0 = M(Z). \quad (22)$$

Case 3. If  $u = 2$ ,  $v = 2$  and  $w = 2$ . Expanding both sides of (10) and comparing the terms in  $\tau^3$  we have

$$\lambda^w B_z Z^{[1]} + A(Z) = M(Z) + \lambda^w Z_z^{[1]} B. \quad (23)$$

Case 4. If  $u = 2$ ,  $v = 2$  and  $w > 2$ . Expanding both sides of (10) and comparing the terms in  $\tau^3$  we have

$$A(Z) = M(Z). \quad (24)$$

Since  $A(Z) = aZ^{[u+1]}$  for some  $a \neq 0$ , and  $G_2$  is a symplectic  $B$ -series with order  $v$ , cases (21), (22) and (24) are impossible. So the only possible case is (23), i.e.,  $u = v = w = 2$ .

□

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<sup>6</sup>For the details about symplectic  $B$ -series, one can refer to [4].



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## Appendix 1. Proof of Theorem 1

When we set

$$G^k(Z) = \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} Z^{[i]} + \tau^2 C_k(Z) + \tau^3 D_k(Z) + \tau^4 E_k(Z) + \tau^5 F_k(Z) + O(\tau^6), \quad (25)$$

then

$$\begin{aligned} & \sum_{i=0}^{+\infty} \frac{(k+1)^i \tau^i}{i!} Z^{[i]} + \tau^2 C_{k+1}(Z) + \tau^3 D_{k+1}(Z) \\ & \quad + \tau^4 E_{k+1}(Z) + \tau^5 F_{k+1}(Z) + O(\tau^6) \\ = & G^{k+1}(Z) = G^k[G(Z)] \quad (26) \\ = & \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} [G(Z)]^{[i]} + \tau^2 C_k[G(Z)] + \tau^3 D_k[G(Z)] \\ & \quad + \tau^4 E_k[G(Z)] + \tau^5 F_k[G(Z)] + O(\tau^6) \\ \equiv & \widetilde{I} + \widetilde{II} + \widetilde{III} + \widetilde{IV} + \widetilde{V} + O(\tau^6), \end{aligned}$$

and

$$\begin{aligned} \widetilde{I} = & \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \left[ \sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right. \quad (26.1) \\ & \left. + \tau^2 C_1(Z) + \tau^3 D_1(Z) + \tau^4 E_1(Z) + \tau^5 F_1(Z) + O(\tau^6) \right]^{[i]} \\ = & \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \left[ \sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right]^{[i]} + \tau^2 C_1 + \tau^3 D_1 + \tau^4 E_1 + \tau^5 F_1 \\ & + \frac{k\tau}{1!} \left\{ Z_z^{[1]} \circ \left[ \sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] * (\tau^2 C_1 + \tau^3 D_1 + \tau^4 E_1) + \frac{1}{2!} Z_{z^2}^{[1]} \circ \left[ \sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] * (\tau^2 C_1)^2 \right\} \\ & + \frac{k^2 \tau^2}{2!} \left\{ Z_z^{[2]} \circ \left[ \sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] * (\tau^2 C_1 + \tau^3 D_1) \right\} \\ & + \frac{k^3 \tau^3}{3!} \left\{ Z_z^{[3]} \circ \left[ \sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] * (\tau^2 C_1) \right\} + O(\tau^6) \\ = & \sum_{l=0}^{+\infty} \frac{(k+1)^l \tau^l}{l!} Z^{[l]} + \tau^2 C_1 + \tau^3 \{ D_1 + k Z_z^{[1]} C_1 \} \\ & + \tau^4 \left\{ E_1 + k Z_z^{[1]} D_1 + k Z_{z^2}^{[1]} Z^{[1]} C_1 + \frac{k^2}{2} Z_z^{[2]} C_1 \right\} \\ & + \tau^5 \left\{ F_1 + k Z_z^{[1]} E_1 + k Z_{z^2}^{[1]} Z^{[1]} D_1 + \frac{k}{2} Z_{z^2}^{[1]} Z^{[2]} C_1 + \frac{k}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 C_1 \right. \\ & \left. + \frac{k}{2} Z_{z^2}^{[1]} (C_1)^2 + \frac{k^2}{2} Z_z^{[2]} D_1 + \frac{k^2}{2} Z_{z^2}^{[2]} Z^{[1]} C_1 + \frac{k^3}{6} Z_z^{[3]} C_1 \right\} + O(\tau^6); \end{aligned}$$

$$\begin{aligned}
\widetilde{II} &= \tau^2 C_k \circ \left( Z + \tau Z^{[1]} + \frac{\tau^2}{2} Z^{[2]} + \frac{\tau^3}{6} Z^{[3]} + \tau^2 C_1 + \tau^3 D_1 \right) + O(\tau^6) \quad (26.2) \\
&= \tau^2 C_k + \tau^3 (C_k)_z Z^{[1]} + \tau^4 \left\{ \frac{1}{2} (C_k)_z Z^{[2]} + (C_k)_z C_1 + \frac{1}{2} (C_k)_{z^2} [Z^{[1]}]^2 \right\} \\
&\quad + \tau^5 \left\{ \frac{1}{6} (C_k)_z Z^{[3]} + (C_k)_z D_1 + (C_k)_{z^2} Z^{[1]} C_1 + \frac{1}{2} (C_k)_{z^2} [Z^{[1]} Z^{[2]}] \right. \\
&\quad \left. + \frac{1}{6} (C_k)_{z^3} [Z^{[1]}]^3 \right\} + O(\tau^6);
\end{aligned}$$

$$\begin{aligned}
\widetilde{III} &= \tau^3 D_k \circ \left( Z + \tau Z^{[1]} + \frac{\tau^2}{2} Z^{[2]} + \tau^2 C_1 \right) + O(\tau^6) \quad (26.3) \\
&= \tau^3 D_k + \tau^4 (D_k)_z Z^{[1]} \\
&\quad + \tau^5 \left\{ \frac{1}{2} (D_k)_z Z^{[2]} + (D_k)_z C_z + \frac{1}{2} (D_k)_{z^2} [Z^{[1]}]^2 \right\} + O(\tau^6);
\end{aligned}$$

$$\begin{aligned}
\widetilde{IV} &= \tau^4 E_k \circ (Z + \tau Z^{[1]}) + O(\tau^6) \quad (26.4) \\
&= \tau^4 E_k + \tau^5 (E_k)_z Z^{[1]} + O(\tau^6);
\end{aligned}$$

$$\widetilde{V} = \tau^5 F_k + O(\tau^6). \quad (26.5)$$

From (26), (26.1)–(26.5), we obtain

$$C_{k+1} = C_1 + C_k; \quad (27.1)$$

$$D_{k+1} = D_1 + k Z_z^{[1]} C_1 + (C_k)_z Z^{[1]} + D_k; \quad (27.2)$$

$$\begin{aligned}
E_{k+1} &= E_1 + k Z_z^{[1]} D_1 + k Z_{z^2}^{[1]} [Z^{[1]} C_1] + \frac{k^2}{2} Z_z^{[2]} C_1 + \frac{1}{2} (C_k)_z Z^{[2]} \quad (27.3) \\
&\quad + (C_k)_z C_1 + \frac{1}{2} (C_k)_{z^2} [Z^{[1]}]^2 + (D_k)_z Z^{[1]} + E_k;
\end{aligned}$$

$$\begin{aligned}
F_{k+1} &= F_1 + k Z_z^{[1]} E_1 + k Z_{z^2}^{[1]} [Z^{[1]} D_1] + \frac{k}{2} Z_{z^2}^{[1]} [Z^{[2]} C_1] \quad (27.4) \\
&\quad + \frac{k}{2} Z_{z^3}^{[1]} [(Z^{[1]})^2 C_1] + \frac{k}{2} Z_{z^2}^{[1]} (C_1)^2 + \frac{k^2}{2} Z_z^{[2]} D_1 \\
&\quad + \frac{k^2}{2} Z_{z^2}^{[2]} [Z^{[1]} C_1] + \frac{k^3}{6} Z_z^{[3]} C_1 + \frac{1}{6} (C_k)_z Z^{[3]} + (C_k)_z D_1 \\
&\quad + \frac{1}{2} (C_k)_{z^2} [Z^{[1]} Z^{[2]}] + (C_k)_{z^2} Z^{[1]} C_1 + \frac{1}{6} (C_k)_{z^3} [Z^{[1]}]^3 \\
&\quad + \frac{1}{2} (D_k)_z Z^{[2]} + (D_k)_z C_1 + \frac{1}{2} (D_k)_{z^2} [Z^{[1]}]^2 + (E_k)_z Z^{[1]} \\
&\quad + F_k.
\end{aligned}$$

From (3), we have

$$\begin{aligned}
& \sum_{k=0}^m \alpha_k \left[ \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} Z^{[i]} + \tau^2 C_k(Z) + \tau^3 D_k(Z) + \tau^4 E_k(Z) + \tau^5 F_k(Z) + O(\tau^6) \right] \\
&= \tau \sum_{k=0}^m \beta_k f \left( \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} Z^{[i]} + \tau^2 C_k(Z) + \tau^3 D_k(Z) + \tau^4 E_k(Z) + O(\tau^5) \right) \\
&= \tau \sum_{k=0}^m \beta_k \left\{ f \circ \left[ \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} \right] + f_z \circ \left[ \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} \right] * (\tau^2 C_k + \tau^3 D_k + \tau^4 E_k) \right. \\
&\quad \left. + \frac{1}{2} f_{z^2} \circ \left[ \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} \right] * (\tau^2 C_k)^2 \right\} + O(\tau^6) \tag{28} \\
&= \sum_{l=0}^{+\infty} \sum_{k=0}^m \beta_k \frac{k^l \tau^{l+1}}{l!} Z^{[l+1]} + \tau^3 \sum_{k=0}^m \beta_k Z_z^{[1]} C_k + \tau^4 \sum_{k=0}^m \beta_k \left\{ Z_z^{[1]} D_k + k Z_{z^2}^{[1]} Z^{[1]} C_k \right\} \\
&\quad + \tau^5 \sum_{k=0}^m \beta_k \left\{ Z_z^{[1]} E_k + k Z_{z^2}^{[1]} Z^{[1]} D_k + \frac{k^2}{2} Z_{z^2}^{[1]} Z^{[2]} C_k + \frac{k^2}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 C_k \right. \\
&\quad \left. + \frac{1}{2} Z_{z^2}^{[1]} (C_k)^2 \right\} + O(\tau^6),
\end{aligned}$$

comparing the coefficients of  $\tau^2$ ,  $\tau^3$ ,  $\tau^4$  and  $\tau^5$  respectively on both sides of (28) we obtain

$$\sum_{k=0}^m \alpha_k C_k = \sum_{k=0}^m \left\{ k \beta_k - \frac{k^2}{2!} \alpha_k \right\} Z^{[2]}; \tag{29.1}$$

$$\sum_{k=0}^m \alpha_k D_k = \sum_{k=0}^m \left\{ \frac{k^2}{2!} \beta_k - \frac{k^3}{3!} \alpha_k \right\} Z^{[3]} + \sum_{k=0}^m \beta_k Z_z^{[1]} C_k; \tag{29.2}$$

$$\sum_{k=0}^m \alpha_k E_k = \sum_{k=0}^m \left\{ \frac{k^3}{3!} \beta_k - \frac{k^4}{4!} \alpha_k \right\} Z^{[4]} \tag{29.3}$$

$$+ \sum_{k=0}^m \beta_k \left\{ Z_z^{[1]} D_k + k Z_{z^2}^{[1]} Z^{[1]} C_k \right\};$$

$$\begin{aligned}
\sum_{k=0}^m \alpha_k F_k &= \sum_{k=0}^m \left\{ \frac{k^4}{4!} \beta_k - \frac{k^5}{5!} \alpha_k \right\} Z^{[5]} + \sum_{k=0}^m \beta_k \left\{ Z_z^{[1]} E_k + k Z_{z^2}^{[1]} Z^{[1]} D_k \right. \\
&\quad \left. + \frac{k^2}{2} Z_{z^2}^{[1]} Z^{[2]} C_k + \frac{k^2}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 C_k + \frac{1}{2} Z_{z^2}^{[1]} (C_k)^2 \right\}. \tag{29.4}
\end{aligned}$$

From relations (27.1) and (29.1) we deduce directly

$$C_k = k C_1 \equiv k C, \tag{30}$$

and (12.1). Substituting (30) into (27.2), we obtain

$$D_k = k D_1 + \frac{k^2 - k}{2} \omega (Z_z^{[1]} Z^{[s+1]} + Z^{[3]}), \tag{31}$$

substituting (31) and (30) into (29.2), we obtain

$$\begin{aligned} \left( \sum_{k=0}^m k\alpha_k \right) D_1 &= \sum_{k=0}^m \left\{ \frac{k^2}{2!} \beta_k - \frac{k^3}{3!} \alpha_k - \frac{k^2 - k}{2} \omega \alpha_k \right\} Z^{[3]} \\ &\quad + \sum_{k=0}^m \left\{ k\omega \beta_k - \frac{k^2 - k}{2} \omega \alpha_k \right\} Z_z^{[1]} Z^{[2]}. \end{aligned}$$

Then we get (12.2), and

$$D_k = \left[ \frac{k^2 \omega}{2} + k\omega^2 \right] Z_z^{[1]} Z^{[2]} + \left[ \frac{k^2 - k}{2} \omega + k \left( \frac{\rho}{2} - \frac{\omega \delta}{2} + \frac{\omega}{2} \right) \right] Z^{[3]}. \quad (32)$$

Substituting (30) and (32) into (27.3), we have

$$\begin{aligned} E_{k+1} &= E_1 + \frac{k^2 \omega + 3k\omega^2 + k\omega}{2} Z_z^{[1]} Z_z^{[1]} Z^{[2]} \\ &\quad + \left[ k \left( \frac{\rho}{2} - \frac{\omega \delta}{2} + \frac{\omega}{2} \right) + k\omega^2 + \frac{k^2 \omega}{2} \right] Z_z^{[1]} Z^{[3]} \\ &\quad + \frac{2k^2 \omega + 3k\omega^2 + k\omega}{2} Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &\quad + \left[ k \left( \frac{\rho}{2} - \frac{\omega \delta}{2} + \frac{\omega}{2} \right) + \frac{k^2 \omega}{2} \right] Z^{[4]} + E_k, \end{aligned}$$

and then

$$\begin{aligned} E_k &= kE_1 + \left[ \frac{k^2 - k}{4} (3\omega^2 + \omega) + \frac{2k^3 - 3k^2 + k}{12} \omega \right] Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &\quad + \left[ \frac{k^2 - k}{2} \left( \frac{\rho}{2} - \frac{\omega \delta}{2} + \frac{\omega}{2} \right) + \frac{k^2 - k}{2} \omega^2 + \frac{2k^3 - 3k^2 + k}{12} \omega \right] Z_z^{[1]} Z^{[3]} \\ &\quad + \left[ \frac{2k^3 - 3k^2 + k}{6} \omega + \frac{k^2 - k}{2} \left( \frac{3\omega^2}{2} + \omega \right) \right] Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &\quad + \left[ \frac{k^2 - k}{2} \left( \frac{\rho}{2} - \frac{\omega \delta}{2} + \frac{\omega}{2} \right) + \frac{2k^3 - 3k^2 + k}{12} \omega \right] Z^{[4]} \end{aligned} \quad (33)$$

Substituting (33), (30) and (32) into (29.3), we obtain

$$\begin{aligned} \left( \sum_{k=0}^m k\alpha_k \right) E_1 &= \\ &\sum_{k=0}^m \left\{ \frac{k^3}{3!} \beta_k - \frac{k^4}{4!} \alpha_k - \left[ \frac{k^2 - k}{2} \left( \frac{\rho}{2} - \frac{\omega \delta}{2} + \frac{\omega}{2} \right) + \frac{2k^3 - 3k^2 + k}{12} \omega \right] \alpha_k \right\} Z^{[4]} \\ &\quad + \sum_{k=0}^m \left\{ k^2 \omega \beta_k - \frac{2k^3 - 3k^2 + k}{6} \omega \alpha_k - \frac{k^2 - k}{2} \left( \frac{3\omega^2}{2} + \omega \right) \alpha_k \right\} Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &\quad + \sum_{k=0}^m \left\{ \left[ k \left( \frac{\rho}{2} - \frac{\omega \delta}{2} + \frac{\omega}{2} \right) + \frac{k^2 - k}{2} \omega \right] \beta_k - \left[ \frac{k^2 - k}{2} \left( \frac{\rho}{2} - \frac{\omega \delta}{2} + \frac{\omega}{2} \right) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{k^2 - k}{2} \omega^2 + \frac{2k^3 - 3k^2 + k}{12} \omega \Big] \alpha_k \Big\} Z_z^{[1]} Z^{[3]} \\
& + \sum_{k=0}^m \left\{ \left( k\omega^2 + \frac{k^2\omega}{2} \right) \beta_k - \left[ \frac{k^2 - k}{4} (3\omega^2 + \omega) + \frac{2k^3 - 3k^2 + k}{12} \omega \right] \alpha_k \right\} Z_z^{[1]} Z_z^{[1]} Z^{[2]},
\end{aligned}$$

and we have (12.3), and

$$\begin{aligned}
E_k = & \left[ \frac{2k^3 - 3k^2 + k}{12} \omega + \frac{k^2 - k}{2} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + k \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} \right. \right. \\
& \left. \left. - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) \right] Z^{[4]} \\
& + \left[ \frac{k^3\omega}{3} + \frac{3k^2\omega^2}{4} + k\rho\omega - \frac{3k\omega^2\delta}{4} \right] Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\
& + \left[ \frac{k^2}{2} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{k^2 - k}{2} \omega^2 + \frac{2k^3 - 3k^2}{2} \omega + k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \\
& \left. + \frac{k\rho\omega}{2} - \frac{k\delta\omega^2}{2} \right] Z_z^{[1]} Z^{[3]} \\
& + \left[ \frac{k^3\omega}{6} + \frac{3k^2\omega^2}{4} + \frac{k\rho\omega}{2} + k\omega^3 - \frac{k\omega^2\delta}{4} \right] Z_z^{[1]} Z_z^{[1]} Z^{[2]}.
\end{aligned} \tag{34}$$

Substituting (30), (32) and (34) into (27.4), we obtain

$$\begin{aligned}
F_{k+1} = & F_1 + F_k \\
& + \left\{ \frac{k^2\omega^2}{2} + \frac{k^3}{6} \omega + \frac{k^2}{2} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + k \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) \right. \\
& \left. + k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{k\omega\rho}{2} - \frac{k\delta\omega^2}{2} \right\} Z_z^{[1]} Z^{[4]} \\
& + \left\{ k\omega^3 + \frac{3k^2}{4} \omega^2 + \frac{2k^3 - k}{12} \omega + 2k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + k\rho\omega + \frac{k^2 + k}{2} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \\
& \left. - \frac{3k\omega^2\delta}{4} \right\} Z_z^{[1]} Z_z^{[1]} Z^{[3]} \\
& + \left\{ \frac{5k^2 + 2k}{4} \omega^2 + \frac{2k^3 + k^2}{4} \omega + 2k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{3}{2} k\rho\omega - \frac{5k\delta\omega^2}{4} \right. \\
& \left. + (k^2 + k) \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right\} Z_{z^2}^{[1]} Z^{[1]} Z^{[3]} \\
& + \left\{ 3k\omega^3 + \frac{6k^2 + 5k}{4} \omega^2 + \frac{k^3 + 3k^2 + k}{6} \omega + k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{k\omega\rho}{2} - \frac{k\delta\omega^2}{4} \right\} \\
& Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[2]} \\
& + \left\{ 2k\omega^3 + \frac{7k^2 + 5k}{4} \omega^2 + \frac{2k^3 + 3k^2 + 2k}{6} \omega + \frac{3}{2} k\rho\omega + k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) - k\delta\omega^2 \right\} \\
& Z_z^{[1]} Z_z^{[1]} Z^{[1]} Z^{[2]} \\
& + \left\{ 2k\omega^3 + \frac{9k^2 + 6k}{4} \omega^2 + \frac{2k^3 + 3k^2 + 2k}{4} \omega + 2k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{k\rho\omega}{2} - \frac{k\delta\omega^2}{4} \right\}
\end{aligned}$$

$$\begin{aligned}
& Z_{z^2}^{[1]} Z^{[1]} [Z_z^{[1]} Z^{[2]}] \\
& + \left\{ k\omega^3 + \frac{7k^2 + 6k}{4}\omega^2 + \frac{2k^3 + 3k^2 + 2k}{4}\omega + k\rho\omega + k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) - \frac{3k\delta\omega^2}{4} \right\} \\
& Z_{z^2}^{[1]} (Z^{[2]})^2 \\
& + \left\{ \frac{5k^2 + 4k}{4}\omega^2 + \frac{2k^3 + 3k^2 + 2k}{4}\omega - \frac{3k\delta\omega^2}{4} + k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + k\rho\omega \right\} \\
& Z_{z^3}^{[1]} (Z^{[1]})^2 Z^{[2]} \\
& + \left\{ \frac{k^3}{6}\omega + \frac{k^2}{2} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + k \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) \right\} Z^{[5]},
\end{aligned}$$

and then

$$\begin{aligned}
F_k &= kF_1 \tag{35} \\
& + \left\{ \frac{2k^3 - 3k^2 + k}{12}\omega^2 + \frac{k^4 - 2k^3 + k^2}{24}\omega + \frac{2k^3 - 3k^2 + k}{12} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \\
& \quad + \frac{k^2 - k}{2} \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) + \frac{k^2 - k}{2}\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \\
& \quad \left. + \frac{k^2 - k}{4}\rho\omega - \frac{k^2 - k}{4}\delta\omega^2 \right\} Z_z^{[1]} Z^{[4]} \\
& + \left\{ \frac{k^2 - k}{2}\omega^3 + \frac{2k^3 - 3k^2 + k}{8}\omega^2 + \frac{k^4 - 2k^3 + k}{24}\omega + (k^2 - k)\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \\
& \quad \left. + \frac{k^2 - k}{2}\rho\omega + \frac{k^3 - k}{6} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) - \frac{3(k^2 - k)}{8}\delta\omega^2 \right\} Z_z^{[1]} Z_z^{[1]} Z^{[3]} \\
& + \left\{ \frac{10k^3 - 9k^2 - k}{24}\omega^2 + \frac{3k^4 - 4k^3 + k}{24}\omega + (k^2 - k)\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{3(k^2 - k)}{4}\rho\omega \right. \\
& \quad \left. - \frac{5(k^2 - k)}{8}\delta\omega^2 + \frac{k^3 - k}{3} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right\} Z_{z^2}^{[1]} Z^{[1]} Z^{[3]} \\
& + \left\{ \frac{3(k^2 - k)}{2}\omega^3 + \frac{4k^3 - k^2 - 3k}{8}\omega^2 + \frac{k^4 + 2k^3 - 3k^2}{24}\omega + \frac{k^2 - k}{2}\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \\
& \quad \left. + \frac{k^2 - k}{4}\rho\omega - \frac{k^2 - k}{8}\delta\omega^2 \right\} Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[2]} \\
& + \left\{ (k^2 - k)\omega^3 + \frac{7k^3 - 3k^2 - 4k}{12}\omega^2 + \frac{k^4 - k}{12}\omega + \frac{3(k^2 - k)}{4}\rho\omega - \frac{k^2 - k}{2}\delta\omega^2 \right. \\
& \quad \left. + \frac{k^2 - k}{2}\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right\} Z_z^{[1]} Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\
& + \left\{ (k^2 - k)\omega^3 + \frac{6k^3 - 3k^2 - 3k}{8}\omega^2 + \frac{k^4 - k}{8}\omega + (k^2 - k)\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \\
& \quad \left. + \frac{k^2 - k}{4}\rho\omega - \frac{k^2 - k}{8}\delta\omega^2 \right\} Z_{z^2}^{[1]} Z^{[1]} [Z_z^{[1]} Z^{[2]}] \\
& + \left\{ \frac{k^2 - k}{2}\omega^3 + \frac{14k^3 - 3k^2 - 11k}{24}\omega^2 + \frac{k^4 - k}{8}\omega + \frac{k^2 - k}{2}\rho\omega - \frac{3(k^2 - k)}{8}\delta\omega^2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{k^2 - k}{2} \omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \left. \right\} Z_{z^2}^{[1]} (Z^{[2]})^2 \\
& + \left\{ \frac{10k^3 - 3k^2 - 7k}{24} \omega^2 + \frac{k^4 - k}{8} \omega - \frac{3(k^2 - k)}{8} \delta\omega^2 + \frac{k^2 - k}{2} \omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \\
& \quad \left. + \frac{k^2 - k}{2} \rho\omega \right\} Z_{z^3}^{[1]} (Z^{[1]})^2 Z^{[2]} \\
& + \left\{ \frac{k^4 - 2k^3 + k^2}{24} \omega + \frac{k^2 - k}{2} \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) \right. \\
& \quad \left. + \frac{2k^3 - 3k^2 + k}{12} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right\} Z^{[5]}.
\end{aligned}$$

Substituting (30), (32), (34) and (35) into (29.4) we have

$$\begin{aligned}
& \left( \sum_{k=0}^m k\alpha_k \right) F_1 = \\
& \sum_{k=0}^m \left\{ \frac{k^4}{4!} \beta_k - \frac{k^5}{5!} \alpha_k - \frac{k^4 - 2k^3 + k^2}{24} \alpha_k \omega - \frac{2k^3 - 3k^2 + k}{12} \alpha_k \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \\
& \quad \left. - \frac{k^2 - k}{2} \alpha_k \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) \right\} Z^{[5]} \\
& + \sum_{k=0}^m \left\{ \left[ \frac{2k^3 - 3k^2 + k}{12} \omega + \frac{k^2 - k}{2} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + k \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} \right. \right. \right. \\
& \quad \left. \left. + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) \right] \beta_k - \left[ \frac{2k^3 - 3k^2 + k}{12} \omega^2 + \frac{2k^3 - 3k^2 + k}{12} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \\
& \quad \left. + \frac{k^4 - 2k^3 + k^2}{24} \omega + \frac{k^2 - k}{2} \left( \frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4} \right) \right. \\
& \quad \left. + \frac{k^2 - k}{2} \omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{k^2 - k}{4} \rho\omega - \frac{k^2 - k}{4} \delta\omega^2 \right] \alpha_k \left. \right\} Z_z^{[1]} Z^{[4]} \\
& + \sum_{k=0}^m \left\{ \left[ \frac{k^3\omega}{6} + \frac{3k^2\omega^2}{4} + \frac{k\rho\omega}{2} + k\omega^3 - \frac{k\delta\omega^2}{4} \right] \beta_k - \left[ \frac{3(k^2 - k)}{2} \omega^3 \right. \right. \\
& \quad \left. \left. + \frac{4k^3 - k^2 - 3k}{8} \omega^2 + \frac{k^4 + 2k^3 - 3k^2}{24} \omega + \frac{k^2 - k}{2} \omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \right. \\
& \quad \left. \left. + \frac{k^2 - k}{4} \rho\omega - \frac{k^2 - k}{8} \delta\omega^2 \right] \alpha_k \right\} Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[2]} \\
& + \sum_{k=0}^m \left\{ \left[ \frac{k^3\omega}{3} + \frac{3k^2}{4} \omega^2 + k\rho\omega - \frac{3k}{4} \delta\omega^2 \right] \beta_k - \left[ (k^2 - k) \omega^3 + \frac{7k^3 - 3k^2 - 4k}{12} \omega^2 \right. \right. \\
& \quad \left. \left. + \frac{k^4 - k}{12} \omega + \frac{3(k^2 - k)}{4} \rho\omega + \frac{k^2 - k}{2} \omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) - \frac{k^2 - k}{2} \delta\omega^2 \right] \alpha_k \right\} \\
& \quad Z_z^{[1]} Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\
& + \sum_{k=0}^m \left\{ \left[ \frac{k^2 - k}{2} \omega^2 + \frac{2k^3 - 3k^2}{12} \omega + \frac{k^2}{2} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + k\omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right. \right.
\end{aligned}$$



$$\begin{aligned}
& + \frac{k\rho\omega}{2} - \frac{k\delta\omega^2}{2} \left] \beta_k - \left[ \frac{k^2 - k}{2} \omega^3 + \frac{2k^3 - 3k^2 + k}{8} \omega^2 + \frac{k^4 - 2k^3 + k}{24} \omega \right. \right. \\
& + (k^2 - k) \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{k^2 - k}{2} \rho\omega + \frac{k^3 - k}{6} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \\
& \left. \left. - \frac{3(k^2 - k)}{8} \delta\omega^2 \right] \alpha_k \right\} Z_z^{[1]} Z_z^{[1]} Z^{[3]} \\
& + \sum_{k=0}^m \left\{ \left[ k^2 \omega^2 + \frac{k^2}{2} \omega + \frac{k^3 - k^2}{2} \omega \right] \beta_k - \left[ (k^2 - k) \omega^3 + \frac{6k^3 - 3k^2 - 3k}{8} \omega^2 \right. \right. \\
& \left. \left. + \frac{k^4 - k}{8} \omega + (k^2 - k) \omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{k^2 - k}{4} \rho\omega - \frac{k^2 - k}{8} \delta\omega^2 \right] \alpha_k \right\} \\
& Z_{z^2}^{[1]} Z^{[1]} (Z_z^{[1]} Z^{[2]}) \\
& + \sum_{k=0}^m \left\{ \left[ \frac{k^3 - k^2}{2} \omega + k^2 \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right] \beta_k - \left[ \frac{10k^3 - 9k^2 - k}{24} \omega^2 \right. \right. \\
& \left. \left. + \frac{3k^4 - 4k^3 + k}{24} \omega + (k^2 - k) \omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{3(k^2 - k)}{4} \rho\omega \right. \right. \\
& \left. \left. - \frac{5(k^2 - k)}{8} \delta\omega^2 + \frac{k^3 - k}{3} \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) \right] \alpha_k \right\} Z_{z^2}^{[1]} Z^{[1]} Z^{[3]} \\
& + \sum_{k=0}^m \left\{ \left[ \frac{k^3}{2} \omega + \frac{k^2}{2} \omega^2 \right] \beta_k - \left[ \frac{k^2 - k}{2} \omega^3 + \frac{14k^3 - 3k^2 - 11k}{24} \omega^2 + \frac{k^4 - k}{8} \omega \right. \right. \\
& \left. \left. + \frac{k^2 - k}{2} \rho\omega + \frac{k^2 - k}{2} \omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{3(k^2 - k)}{8} \delta\omega^2 \right] \alpha_k \right\} Z_{z^2}^{[1]} (Z^{[2]})^2 \\
& + \sum_{k=0}^m \left\{ \frac{k^3}{2} \beta_k \omega - \left[ \frac{k^4 - k}{8} \omega + \frac{10k^3 - 3k^2 - 7k}{24} \omega^2 - \frac{3(k^2 - k)}{8} \delta\omega^2 \right. \right. \\
& \left. \left. + \frac{k^2 - k}{2} \omega \left( \frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2} \right) + \frac{k^2 - k}{2} \rho\omega \right] \alpha_k \right\} Z_{z^3}^{[1]} (Z^{[1]})^2 Z^{[2]},
\end{aligned}$$

and we obtain (12.4).  $\square$