Two Nonconforming Quadrilateral Elements for the Reissner-Mindlin Plate

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We construct two low order nonconforming quadrilateral elements for the Reissner-Mindlin plate. The first one consists of a modified nonconforming rotated Q_1 element for one component of the rotation and the standard 4−node isoparametric element for the other component as well as for the the approximation of the transverse displacement, a modified rotated Raviart-Thomas interpolation operator is employed as the shear reduction operator. The second differs from the first only in the approximation of the rotation, which employs the modified rotated Q_1 element for both components of the rotation, and a jump term accounting the discontinuity of the rotation approximation is included in the variational formulation. Both elements give optimal error bounds uniform in the plate thickness with respect to the energy norm as well as the L^2 norm.

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1. Introduction

In the last two decades, extensive efforts have been devoted to the design and analysis of finite elements to resolve the Reissner-Mindlin (R-M) plate, which is one of the most widely used plate bending model. However, the elements for which a sound mathematical analysis exists are largely constricted to triangular and rectangular elements. Arnold, Boffi and Falk [3] checked the possible traps during the straightforward extension of rectangular elements to general quadrilateral meshes. Following the guideline of [3], it seems hopeful to analyze the classical quadrilateral MITC family [13] [35]. Durán, Hernández, Hervella-Nieto, Liberman and Rodríguez [20] recently proposed a new quadrilateral element (DL4) which is the same with MITC4 [8] except that a bubble enriched 4−node isoparametric element is used to approximate the rotation. They established the optimal H^1 error bound for a general quadrilateral mesh, while the optimal L^2 error bound is only proved for mildly distorted quadrilateral meshes[a]. Meanwhile, for the nested mesh with mildly distorted quadrilaterals, they derived optimal H^1 and L^2 error bounds for the classical MITC4. In [25], we proposed two elements that are also similar to MITC4 except that the rotation is approximated by the nonconforming rotated Q_1 element (NRQ_1) [34]. The optimal H^1 and L^2 error bounds are derived for mildly distorted quadrilateral meshes. Consequently, all the above elements cannot be regarded as strictly *locking-free* since they degrade over general quadrilateral meshes either in the L^2 norm or even in the energy norm.

In this paper, we present two new quadrilateral elements, which can be regarded as the quadrilateral extension of the rectangular elements in [28]. For the first element, we use the modified NRQ_1 to approximate one component of the rotation, and the 4−node isoparametric element to approximate the other component as well as the transverse displacement. A modified rotated Raviart-Thomas interpolation operator introduced in [31] is employed as the shear reduction operator. The second element differs from the first in the approximation of the rotation. The modified NRQ_1 is used to approximate both components of the rotation, and a jump term which accounts the discontinuity of the rotation approximation is included in the variational formulation. We prove optimal H^1 and L^2 error bounds uniform in the plate thickness over general quadrilateral meshes.

The main ingredient of our method is a new shear reduction operator, which is motivated by the observation due to [20] [31] [32], namely, the L^2 convergence rate deterioration originates from the the non-optimality of the following interpolation estimate for the rotated $\mathrm{RT}_{[0]}$ element [36]:

$$\|\operatorname{rot}(\boldsymbol{u}-\varPi\boldsymbol{u})\|_{L^2(\varOmega)} \le Ch|\operatorname{rot}\boldsymbol{u}|_{H^1(\varOmega)} + C\max_{K\in\mathcal{T}_h} d_K/h_K\,\|\operatorname{rot}\boldsymbol{u}\|_{L^2(\varOmega)},$$

where $\varPi$ is the rotated $\mathrm{RT}_{[0]}$ interpolation operator, and d_K is the distance between the midpoints of two diagonals of an element K of the triangulation $\mathcal{T}_h$ for a domain $\varOmega$. While the modified rotated $\mathrm{RT}_{[0]}$ element instead admits the optimal interpolation error estimate:

$$\|\operatorname{rot}(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\|_{L^2(\varOmega)} \le Ch|\operatorname{rot}\boldsymbol{u}|_{H^1(\varOmega)},$$

where $\boldsymbol{R}_h$ is the modified rotated $\mathrm{RT}_{[0]}$ interpolation operator. In the same spirit, the rotated $\mathrm{ABF}_{[0]}$ interpolation operator [4] could also be used as a shear reduction operator that would lead to the optimal L^2 error estimate. Actually, the relatively new $\mathrm{ABF}_{[0]}$ interpolation operator appeared in early 80's engineering literature in a disguised form as a kinematically linked interpolation operator. This interesting relation has recently been uncovered in [30] and [32]. Naturally, the L^2 error degradation of DL4 element could be cured by using either the modified rotated $\mathrm{RT}_{[0]}$ or the rotated $\mathrm{ABF}_{[0]}$ interpolation operator as the shear reduction operator.

The outline of the paper is as follows. We introduce the R-M model and recall some a priori and regularity estimates of the solutions in §2. In §3, we introduce the

[a]See [29] for the exact definition.

elements. Two *Korn's inequalities* for piecewise vector field of the approximation spaces of the rotation are established in §4. We derive error bounds for all variables in the energy norm and L^2 norm in the last section.

Throughout this paper, the generic constant C is assumed to be independent of the plate thickness t and the mesh size h.

2. Variational Formulation

Let Ω represent the mid-surface of the plate, which is assumed to be clamped along the boundary $\partial\Omega$. In the sequel, we assume that Ω is a convex polygon. Let $\boldsymbol{\phi}$ and ω be the rotation and the transverse displacement, respectively. In the R-M plate model, they are determined by the following variational formulation: Find $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega)$ and $\omega \in H_0^1(\Omega)$ such that

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi}, \nabla v - \boldsymbol{\psi}) = (g, v) \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega) \text{ and } v \in H_0^1(\Omega), \tag{2.1}$$

where $a(\boldsymbol{\eta}, \boldsymbol{\psi}) = \big(\boldsymbol{C}\mathcal{E}(\boldsymbol{\eta}), \mathcal{E}(\boldsymbol{\psi})\big)$ for any $\boldsymbol{\eta}, \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega)$, and $\mathcal{E}(\boldsymbol{\eta})$ is the symmetric part of the gradient of $\boldsymbol{\eta}$. Here $H_0^1(\Omega)$ denotes the standard Sobolev space, and $\boldsymbol{H}_0^1(\Omega)$ the corresponding space of $2-$vector-valued functions, this rule is applicable to other spaces and operators. Let g be the scaled transverse loading function, $t-$the plate thickness, $\lambda = E\kappa/[2(1+\nu)]$ with Young's modulus E, the Poisson ratio ν and the shear correction factor κ. For a 2×2 symmetric matrix $\boldsymbol{\tau}$, $\boldsymbol{C}\boldsymbol{\tau}$ is defined as $\boldsymbol{C}\boldsymbol{\tau} = D\,[(1-\nu)\boldsymbol{\tau} + \nu\,\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}]$ with the bending modulus $D = E/[12(1-\nu^2)]$, where $\boldsymbol{I}$ is a 2×2 identity matrix and $\mathrm{tr}(\boldsymbol{\tau})$ is the trace of $\boldsymbol{\tau}$.

For any domain D, the norm and semi-norm in $H^k(D)$ are denoted by $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$, the subscript D will be dropped if it is Ω.

Given $\boldsymbol{\phi}$ and ω, the shear stress $\boldsymbol{\gamma}$ is defined by

$$\boldsymbol{\gamma} = \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi}). \tag{2.2}$$

A proper space for the shear stress is $\boldsymbol{H}^{-1}(\mathrm{div}, \Omega)$, which is defined as the dual space of

$$\boldsymbol{H}_0(\mathrm{rot}, \Omega) = \{\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \mid \mathrm{rot}\,\boldsymbol{q} \in L^2(\Omega),\ \boldsymbol{q}\cdot\boldsymbol{t} = 0 \quad \text{on} \quad \partial\Omega \,\}$$

with $\boldsymbol{t}$ denoting the unit tangent to $\partial\Omega$ and $\mathrm{rot}\,\boldsymbol{q} = \mathrm{rot}(q_1, q_2) = \partial_x q_2 - \partial_y q_1$. It can be shown that

$$\boldsymbol{H}^{-1}(\mathrm{div}, \Omega) = \{\, \boldsymbol{q} \in \boldsymbol{H}^{-1}(\Omega) \ \mid\ \mathrm{div}\,\boldsymbol{q} \in H^{-1}(\Omega) \,\}$$

with $\mathrm{div}\,\boldsymbol{q} = \partial_x q_1 + \partial_y q_2$. Define

$$\boldsymbol{H}(\mathrm{div}, \Omega) = \{\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \ \mid\ \mathrm{div}\,\boldsymbol{q} \in L^2(\Omega) \,\},$$

and the norm in $\boldsymbol{H}(\mathrm{div}, \Omega)$ is given by

$$\|\boldsymbol{q}\|_{\boldsymbol{H}(\mathrm{div})} = (\|\boldsymbol{q}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\,\mathrm{div}\,\boldsymbol{q}\|^2_{L^2(\Omega)})^{1/2}.$$

The following a priori estimates and regularity results of the solution of (2.1) are essentially included in the Appendix of [6] and [17] as

$$\|\boldsymbol{\phi}\|_1+\|\omega\|_1+\|\boldsymbol{\gamma}\|_0 \le C\|g\|_{-1}, \tag{2.7}$$

$$\|\boldsymbol{\phi}\|_2 \le C\|g\|_{-1}, \quad \|\omega\|_2 \le C(\|g\|_{-1}+t^2\|g\|_0), \tag{2.8}$$

$$\|\operatorname{div}\boldsymbol{\gamma}\|_0 \le C\|g\|_0, \quad t\|\boldsymbol{\gamma}\|_1 \le C(\|g\|_{-1}+t\|g\|_0). \tag{2.9}$$

3. Finite Element Approximation

Let $\mathcal{T}_h$ be a partition of $\bar{\Omega}$ by convex quadrilaterals K with the diameter h_K and $h := \max_{K\in\mathcal{T}_h} h_K$. We assume that $\mathcal{T}_h$ is shape regular in the sense of Ciarlet-Raviart [18] . Namely, all quadrilaterals are convex and there exist constants $\sigma \ge 1$ and $0<\rho<1$ such that

$$h_K/\underline{h}_K \le \sigma, \qquad |\cos\theta_{i,K}| \le \rho, \quad i=1,2,3,4 \quad \forall K\in\mathcal{T}_h.$$

Here $\underline{h}_K$ and $\theta_{i,K}$ denote the shortest length of edges and the interior angles of K, respectively. The *quasi-uniformity* of $\mathcal{T}_h$ is *not* assumed.

Let $\hat{K}=(-1,1)^2$ be the reference square and the bilinear function $\boldsymbol{F}$ be an isomorphism from $\hat{K}\to K=\boldsymbol{F}(\hat{K})$. Let $D\boldsymbol{F}$ be the Jacobian matrix of the mapping $\boldsymbol{F}$ and J its determinant. Obviously, $J(\hat{\boldsymbol{x}})=J_0+J_1\hat{x}+J_2\hat{y}$.

For notation brevity, the inner products in $L^2(K)$ and $L^2(\Omega)$, and the dual pairing between $\boldsymbol{H}^{-1}(\mathrm{div},\Omega)$ and $\boldsymbol{H}_0(\mathrm{rot},\Omega)$ are all denoted by $(\cdot,\cdot)$. Denote by $⨍_{\Omega_1} f$ the mean value of a function f over the sub-domain Ω_1 of Ω.

We firstly use the standard 4−node isoparametric bilinear element space

$$W_h{:}=\{\, v\in H_0^1(\Omega) \mid v_{|K}\in Q_1(K) \quad \forall K\in\mathcal{T}_h \,\}$$

to approximate the transverse displacement, where

$$Q_1(K){:}=\{\, q\circ\boldsymbol{F}^{-1} \mid q\in \mathrm{Span}\{1,\hat{x},\hat{y},\hat{x}\hat{y}\}\,\}.$$

Denote by Π_1 the standard bilinear interpolation operator.

Next we define

$$\begin{aligned}N_h{:}=\{\, v\in L^2(\Omega) \mid v_{|K}\in \widehat{\mathcal{Q}}_1,\ v \text{ is continuous regarding } Q_e\\ \text{and } Q_e(v)=0 \text{ if } e\subset\partial\Omega\,\}\end{aligned}$$

with

$$\widehat{\mathcal{Q}}_1{:}=\{\, q\circ\boldsymbol{F}^{-1} \mid q\in\mathrm{Span}\langle 1,\hat{x},\hat{y},\hat{x}\hat{y},\hat{x}^2-\hat{y}^2\rangle\,\},$$

and $Q_e(v){:}=⨍_e v$ for all smooth function $v\colon K\to\mathbb{R}$ and $e\subset\partial K$. The five degrees of freedom associated with $\widehat{\mathcal{Q}}_1$ are give by the mean value of a function f over four edges and the integral $⨍_{\hat{K}} f\circ\boldsymbol{F}^{-1}\hat{x}\hat{y}$. Denote by Π_h the standard interpolation operator over N_h.

Remark 3.1. The finite element space N_h defined above is a modification of NRQ_1 by adding $\hat{x}\hat{y}$ in the basis function. It differs from the element introduced in [16] as

$$\widehat{\mathcal{Q}}_1 := \{\, q \circ \boldsymbol{F}^{-1} \mid q \in \mathrm{Span}\,\langle 1, \hat{x}, \hat{y}, \hat{x}\hat{y}, \theta_\ell(\hat{x}) - \theta_\ell(\hat{y}) \rangle \,\}$$

with

$$\theta_\ell(\hat{x}) = \begin{cases} \hat{x}^2 - \frac{5}{3}\hat{x}^4 & \ell = 1, \\ \hat{x}^2 - \frac{25}{6}\hat{x}^4 + \frac{7}{2}\hat{x}^6 & \ell = 2. \end{cases}$$

Define

$$\boldsymbol{V}_h = N_h \times W_h \quad \text{and} \quad \widetilde{\boldsymbol{V}}_h = N_h \times N_h$$

as the approximation space of the rotation. As $\boldsymbol{V}_h$ and $\widetilde{\boldsymbol{V}}_h$ are nonconforming, so when differential operators such as $\mathcal{E}$, rot and ∇ may be applied to functions in $\boldsymbol{V}_h$ or $\widetilde{\boldsymbol{V}}_h$, we shall write $\mathcal{E}_h, \mathrm{rot}_h$ and ∇_h in all these cases, which are defined piecewise on each element. The space $\boldsymbol{V}_h$ or $\widetilde{\boldsymbol{V}}_h$ is equipped with the piecewise semi-norm $|\boldsymbol{v}|_{1,h} = \|\nabla_h \boldsymbol{v}\|_0$ and the norm $\|\boldsymbol{v}\|_{1,h} = \|\boldsymbol{v}\|_0 + |\boldsymbol{v}|_{1,h}$. The same rule is applicable to the scalar functions in N_h.

Using the general theory in [2], we have the interpolation result

$$\|v - \Pi_h v\|_0 + h\|v - \Pi_h v\|_{1,h} \le Ch^2\|v\|_2 \quad \forall v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{3.8}$$

Finally, we define

$$\boldsymbol{\Gamma}_h := \{\, \boldsymbol{\chi} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathrm{rot}, \Omega) \mid \boldsymbol{\chi} = D\boldsymbol{F}^{-T}\hat{\boldsymbol{\chi}},\ \hat{\boldsymbol{\chi}} \in \boldsymbol{V}(\hat{K}) \quad \forall K \in \mathcal{T}_h \,\}.$$

Here $\boldsymbol{V}(\hat{K})$ is spanned by $(1,0)+\hat{\boldsymbol{b}}(\hat{x},\hat{y})$, $(0,1)+\hat{\boldsymbol{b}}(\hat{x},\hat{y})$, $(\hat{y},0)+\hat{\boldsymbol{b}}(\hat{x},\hat{y})$, $(0,\hat{x})+\hat{\boldsymbol{b}}(\hat{x},\hat{y})$ with $\hat{\boldsymbol{b}}(\hat{x},\hat{y}) = \big(\frac{J_2}{2|K|}(1-\hat{y}^2), \frac{J_1}{2|K|}(\hat{x}^2-1)\big)$. The interpolation operator $\boldsymbol{R}_h$ is defined as $\boldsymbol{R}_{h|_K} = \boldsymbol{R}_K$ with

$$\int_e (\boldsymbol{v} - \boldsymbol{R}_K\boldsymbol{v}) \cdot \boldsymbol{t}\, ds = 0, \quad \forall e \subset \partial K$$

for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathrm{rot}, \Omega)$. It is seen that

$$\boldsymbol{R}_K\boldsymbol{v} = \sum_{i=1}^{4} (\boldsymbol{b}_i + \boldsymbol{b}) \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{t}\, ds \tag{3.11}$$

with $\boldsymbol{b}_i(\boldsymbol{x}) = D\boldsymbol{F}^{-T}\hat{\boldsymbol{b}}_i(\hat{\boldsymbol{x}})$ and $\boldsymbol{b}(\boldsymbol{x}) = D\boldsymbol{F}^{-T}\hat{\boldsymbol{b}}(\hat{\boldsymbol{x}})$, where

$$\begin{aligned} \hat{\boldsymbol{b}}_1(\hat{\boldsymbol{x}}) &= \frac{1}{4}(0, \hat{x}-1), \quad & \hat{\boldsymbol{b}}_2(\hat{\boldsymbol{x}}) &= \frac{1}{4}(1-\hat{y}, 0), \\ \hat{\boldsymbol{b}}_3(\hat{\boldsymbol{x}}) &= \frac{1}{4}(0, \hat{x}+1), \quad & \hat{\boldsymbol{b}}_4(\hat{\boldsymbol{x}}) &= \frac{1}{4}(-1-\hat{y}, 0). \end{aligned}$$

It is easy to rewrite (3.11) into the following form:

$$\boldsymbol{R}_K\boldsymbol{v} = \sum_{i=1}^{4} \boldsymbol{b}_i \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{t}\, ds + \int_K \mathrm{rot}\, \boldsymbol{v}\, d\boldsymbol{x}\, \boldsymbol{b}. \tag{3.13}$$

A straightforward calculation gives

$$\begin{aligned}\operatorname{rot} \boldsymbol{R}_K \boldsymbol{v} &= \frac{1}{4J}\sum_{i=1}^{4}\int_{e_i}\boldsymbol{v}\cdot\boldsymbol{t}\,ds + \frac{1}{J}\Big(\frac{J}{|K|}-\frac{1}{4}\Big)\sum_{i=1}^{4}\int_{e_i}\boldsymbol{v}\cdot\boldsymbol{t}\,ds\\ &= \frac{1}{|K|}\sum_{i=1}^{4}\int_{e_i}\boldsymbol{v}\cdot\boldsymbol{t}\,ds = \fint_K \operatorname{rot}\boldsymbol{v}\,d\boldsymbol{x}. \end{aligned}\tag{3.14}$$

Next we prove a property of $\boldsymbol{R}_h$.

Lemma 3.1. *For any $u \in H^2(\Omega)\cap H_0^1(\Omega)$, we have*

$$\boldsymbol{R}_h \nabla u = \nabla \Pi_1 u. \tag{3.15}$$

Proof. Since $u \in H^2(\Omega)\cap H_0^1(\Omega)$, so $\Pi_1 u$ is well-defined. By the Sobolev imbedding theorem that $\boldsymbol{R}_h\nabla u$ is also well-defined. Using (3.13) with $\boldsymbol{v} = \nabla u$, we get

$$\boldsymbol{R}_K\nabla u = \sum_{i=1}^{4}\boldsymbol{b}_i\int_{e_i}\nabla u\cdot\boldsymbol{t}\,ds = \mathit{\Pi}\nabla u = \nabla\Pi_1 u,$$

where we have used Lemma 2.1 of [20] in the last identity. □

Lemma 3.2. *For any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\cap\boldsymbol{H}_0(\mathrm{rot},\Omega)$,*

$$\|\boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}\|_0 \le Ch|\boldsymbol{v}|_1. \tag{3.17}$$

If $\operatorname{rot}\boldsymbol{v} \in H^1(\Omega)$, then

$$\|\operatorname{rot}(\boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v})\|_0 \le Ch|\operatorname{rot}\boldsymbol{v}|_1. \tag{3.18}$$

Proof. Using (3.13), we have

$$\boldsymbol{R}_K\boldsymbol{v} = \mathit{\Pi}\boldsymbol{v} + \int_K \operatorname{rot}\boldsymbol{v}\,d\boldsymbol{x}\,\boldsymbol{b}.$$

Taking the rotation into account, it is proved in Theorem 7.1 of [24] that

$$\|\boldsymbol{v} - \mathit{\Pi}\boldsymbol{v}\|_0 \le Ch|\boldsymbol{v}|_1.$$

A straightforward calculation yields

$$\Big\|\int_K \operatorname{rot}\boldsymbol{v}\,d\boldsymbol{x}\,\boldsymbol{b}\Big\|_{0,K} \le C|K|^{1/2}\|\operatorname{rot}\boldsymbol{v}\|_{0,K}\|\hat{\boldsymbol{b}}\|_{0,\hat{K}} \le Ch_K\|\operatorname{rot}\boldsymbol{v}\|_{0,K}.$$

Combining the above three equations and adding up all $K \in \mathcal{T}_h$, we obtain (3.17).
The estimate (3.18) is a direct consequence of (3.14). □

Define by E_h all edges of $\mathcal{T}_h$ and $E_h^{'}$ all interior edges of $\mathcal{T}_h$. As in [15], for any piecewise vector $\boldsymbol{v} \in \Pi_{K\in\mathcal{T}_h}\boldsymbol{H}^1(K)$, we define the jump of $\boldsymbol{v}$ as

$$[\boldsymbol{v}] = (\boldsymbol{v}^+\otimes\boldsymbol{n}^+)_S + (\boldsymbol{v}^-\otimes\boldsymbol{n}^-)_S, \quad \forall e \in E_h^{'},$$

where $(\boldsymbol{v}\otimes\boldsymbol{n})_S$ denotes the symmetric part of the tensor product. On the boundary edge, we define the jump of a vector as $[\boldsymbol{v}] = (\boldsymbol{v}\otimes\boldsymbol{n})_S$, where $\boldsymbol{n}$ is the outward normal to $\partial\Omega$.

We introduce the first element which solves the following

Problem 3.1. Find $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$ and $\omega_h \in W_h$ such that

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + \lambda t^{-2}(\nabla\omega_h - \boldsymbol{R}_h\boldsymbol{\phi}_h, \nabla v - \boldsymbol{R}_h\boldsymbol{\psi}) = (g, v) \quad \forall\boldsymbol{\psi} \in \boldsymbol{V}_h \text{ and } v \in W_h,$$

where $a_h(\boldsymbol{u}, \boldsymbol{v}) := \big(\boldsymbol{C}\mathcal{E}_h(\boldsymbol{u}), \mathcal{E}_h(\boldsymbol{v})\big)$ for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_h$.

The shear stress is defined locally as

$$\boldsymbol{\gamma}_h := \lambda t^{-2}(\nabla\omega_h - \boldsymbol{R}_h\boldsymbol{\phi}_h).$$

The second element is defined as to solve the following

Problem 3.2. Find $\boldsymbol{\phi}_h \in \widetilde{\boldsymbol{V}}_h$ and $\omega_h \in W_h$ such that

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + \lambda t^{-2}(\nabla\omega_h - \boldsymbol{R}_h\boldsymbol{\phi}_h, \nabla v - \boldsymbol{R}_h\boldsymbol{\psi}) = (g, v) \quad \forall\boldsymbol{\psi} \in \widetilde{\boldsymbol{V}}_h \text{ and } v \in W_h,$$

where

$$a_h(\boldsymbol{u}, \boldsymbol{v}) := \big(\boldsymbol{C}\mathcal{E}_h(\boldsymbol{u}), \mathcal{E}_h(\boldsymbol{v})\big) + \sum_{e\in E_h} \kappa_e \fint_e [\boldsymbol{u}]\cdot[\boldsymbol{v}]\, ds \tag{3.26}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \widetilde{\boldsymbol{V}}_h$, where κ_e is a positive constant.

The shear stress is defined as in Problem 3.1.

Note that $\int_e \boldsymbol{\psi}\cdot\boldsymbol{t}\, ds$ is well-defined for any $\boldsymbol{\psi} \in \boldsymbol{V}_h$ or $\widetilde{\boldsymbol{V}}_h$, so $\boldsymbol{R}_h\boldsymbol{\psi}$ is also well-defined for any $\boldsymbol{\psi} \in \boldsymbol{V}_h$ or $\widetilde{\boldsymbol{V}}_h$.

Remark 3.2. It seems quite unusual at the first sight that the vector space $\boldsymbol{V}_h$ consists of two different finite element spaces. This is mainly due to the fact that the discrete *Korn's inequality* is invalid over $\widetilde{\boldsymbol{V}}_h$ as suggested in [28] by means of a counterexample. If we use $\widetilde{\boldsymbol{V}}_h$ to approximate the rotation in Problem 3.1, then the resulting method does not converge in the classic sense even over a rectangular mesh [28].

4. Korn's Inequality

In this section, we first prove *Korn's inequality* for $\boldsymbol{V}_h$, next we cite a weak *Korn's inequality* for $\widetilde{\boldsymbol{V}}_h$, which is the cornerstone for Problem 3.2.

We shall frequently use the following basic inequality: For any $K \in \mathcal{T}_h$ and $e \subset \partial K$, there exists a constant C only depending on the shape regularity constants σ and ρ, such that

$$\|v\|_{0,e} \le C(|e|^{-1/2}\|v\|_{0,K} + |e|^{1/2}|v|_{1,K}) \qquad \forall v \in H^1(K). \tag{4.1}$$

This inequality also holds for vector-valued functions $\boldsymbol{v} \in \boldsymbol{H}^1(K)$. We refer to [1] for a proof. Using the above inequality, we get

$$\|v - Q_e(v)\|_{0,e} \le C|e|^{1/2}|v|_{1,K} \qquad \forall v \in H^1(K). \tag{4.2}$$

Lemma 4.1. *For any $\boldsymbol{u} \in \boldsymbol{V}_h$,*

$$|\boldsymbol{u}|_{1,h} \leq \sqrt{2}\|\mathcal{E}_h(\boldsymbol{u})\|_0. \tag{4.3}$$

Proof. Let $\boldsymbol{u} = (u, v)$, then

$$\begin{aligned}\|\mathcal{E}_h(\boldsymbol{u})\|_0^2 = &\sum_{K\in\mathcal{T}_h}\int_K\Bigl(|\partial u/\partial x|^2 + \frac{1}{2}|\partial u/\partial y|^2\Bigr)\,d\boldsymbol{x}\\ &+\|\partial v/\partial y\|_0^2 + 1/2\|\partial v/\partial x\|_0^2 + \sum_{K\in\mathcal{T}_h}\int_K \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\,d\boldsymbol{x}.\end{aligned} \tag{4.4}$$

Green's formula yields

$$\sum_{K\in\mathcal{T}_h}\int_K \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\,d\boldsymbol{x} = \sum_{K\in\mathcal{T}_h}\int_K \frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\,d\boldsymbol{x} + \sum_{K\in\mathcal{T}_h}\int_{\partial K} u\frac{\partial v}{\partial \tau}\,ds. \tag{4.5}$$

For any edge $e \subset \partial K$, if it is on the boundary $\partial\Omega$, we have $\partial v/\partial\tau = 0$. If e is the common edge of two adjacent elements, then the summation of two integrals on e is

$$\int_e [u]\frac{\partial v}{\partial\tau}\,ds = \int_e [u]\,ds\frac{\partial v}{\partial\tau} = 0,$$

since $u \in N_h$ and $\partial v/\partial\tau$ is a common constant along e. Consequently, we obtain

$$\begin{aligned}\|\mathcal{E}_h(\boldsymbol{u})\|_0^2 = &\sum_{K\in\mathcal{T}_h}\int_K\Bigl(|\partial u/\partial x|^2 + \frac{1}{2}|\partial u/\partial y|^2\Bigr)\,d\boldsymbol{x}\\ &+\|\partial v/\partial y\|_0^2 + \frac{1}{2}\|\partial v/\partial x\|_0^2 + \sum_{K\in\mathcal{T}_h}\int_K \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\,d\boldsymbol{x}\\ \geq &\,\frac{1}{2}|\boldsymbol{u}|_{1,h}^2,\end{aligned}$$

which gives (4.3). □

Remark 4.1. An integration by parts gives

$$|\boldsymbol{u}|_1 \leq \sqrt{2}\|\mathcal{E}(\boldsymbol{u})\|_0 \quad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega),$$

while (4.3) indicates that this inequality is also valid for a piecewise $\boldsymbol{H}^1$ vector field.

Remark 4.2. Inequality (4.3) remains true for triangular meshes if N_h is replaced by the Crouzeix-Raviart element [19] and W_h is replaced by the conforming P_1 element. This inequality has been proven in [26] by a different method under a constraint on the mesh partition.

The next lemma concerns a weak *Korn's inequality* for a vector field in $\widetilde{\boldsymbol{V}}_h$.

Lemma 4.2. *For any $\boldsymbol{u} \in \widetilde{\boldsymbol{V}}_h$, there exists a constant C such that*

$$|\boldsymbol{u}|_{1,h} \leq C\Bigl(\|\mathcal{E}_h(\boldsymbol{u})\|_0 + \Bigl(\sum_{e\in E_h}⨍_e [\boldsymbol{u}]^2\,ds\Bigr)^{1/2}\Bigr). \tag{4.8}$$

The above inequality is a special case of the results in [11]. Notice that (4.3) cannot be directly deduced from (4.8).

Clearly, *Poincaré's inequality* for the function in $\boldsymbol{V}_h$ and $\widetilde{\boldsymbol{V}}_h$ hangs on *Poincaré's inequality* for the function in N_h.

Lemma 4.3. *There exists a constant C such that*

$$\|v\|_0 \le C|v|_{1,h} \qquad \forall v \in N_h.$$

The above inequality is well-known, see for instance, Remark 3.3 of [33] or see [12] for more general case.

Using (4.3), (4.8) and Lemma 4.3, it is straightforward to prove the coercivity of a_h.

Lemma 4.4. *There exists a constant C such that*

$$a_h(\boldsymbol{u},\boldsymbol{u}) \ge C\|\boldsymbol{u}\|_{1,h}^2 \qquad \textit{for all } \boldsymbol{u} \in \boldsymbol{V}_h. \tag{4.10}$$

If there exists a constant κ_0 such that $\kappa_e \ge \kappa_0$ for all $e \in E_h$, then

$$a_h(\boldsymbol{u},\boldsymbol{u}) \ge C\|\boldsymbol{u}\|_{1,h}^2 \qquad \textit{for all } \boldsymbol{u} \in \widetilde{\boldsymbol{V}}_h.$$

On the other hand, it follows from (4.1) that there exists a constant C such that

$$|a_h(\boldsymbol{u},\boldsymbol{v})| \le C\|\boldsymbol{u}\|_{1,h}\|\boldsymbol{v}\|_{1,h} \qquad \text{for all } \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}_h \text{ or } \widetilde{\boldsymbol{V}}_h. \tag{4.12}$$

5. Error Estimates

In this section, we shall derive the error bounds. Our approach is essentially the same as that in [21] and its generalization [22]. The main ingredient is a Fortin operator [23] constructed in next lemma.

Lemma 5.1. *Let M_h be a space consisting of piecewise constants on each element. Then there exists an operator $\boldsymbol{\Pi}$: $\boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h$ such that*

$$(\mathrm{rot}_h(\boldsymbol{v} - \boldsymbol{\Pi}\boldsymbol{v}), q) = 0 \qquad \forall q \in M_h \tag{5.1}$$

and

$$\|\boldsymbol{v} - \boldsymbol{\Pi}\boldsymbol{v}\|_0 \le Ch|\boldsymbol{v} - \boldsymbol{\Pi}\boldsymbol{v}|_{1,h}. \tag{5.2}$$

Moreover, if $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$, the following estimate holds:

$$|\boldsymbol{v} - \boldsymbol{\Pi}\boldsymbol{v}|_{1,h} \le Ch|\boldsymbol{v}|_2. \tag{5.3}$$

Proof. We consider the following auxiliary problem: find $(\boldsymbol{v}^h, p^h) \in \boldsymbol{V}_h \times M_h$ such that

$$\begin{aligned} (\nabla_h \boldsymbol{v}^h, \nabla_h \boldsymbol{z}) + (\mathrm{rot}_h\, \boldsymbol{z}, p^h) &= (\nabla \boldsymbol{v}, \nabla_h \boldsymbol{z}) \quad \forall \boldsymbol{z} \in \boldsymbol{V}_h, \\ (\mathrm{rot}_h\, \boldsymbol{v}^h, q) &= (\mathrm{rot}\, \boldsymbol{v}, q) \qquad \forall q \in M_h. \end{aligned} \tag{5.4}$$

Notice that $(\boldsymbol{V}_h, M_h)$ is a stable pair for the rot operator (see, e.g.,Theorem 4.5 of [28]), so the existence and uniqueness of $(\boldsymbol{v}^h, p^h)$ are the consequence of the classic mixed finite element method theory [14]. Denote $\boldsymbol{\Pi}_h = (\Pi_h, \Pi_h)$, we have

$$\begin{aligned}
&\big(\nabla_h(\boldsymbol{v}-\boldsymbol{v}^h), \nabla_h(\boldsymbol{v}^h - \boldsymbol{\Pi}_h\boldsymbol{v})\big) = (\mathrm{rot}_h(\boldsymbol{v}^h - \boldsymbol{\Pi}_h\boldsymbol{v}), p^h)\\
&= (\mathrm{rot}_h(\boldsymbol{v}^h - \boldsymbol{v}), p^h) + (\mathrm{rot}_h(\boldsymbol{v} - \boldsymbol{\Pi}_h\boldsymbol{v}), p^h)\\
&= (\mathrm{rot}_h(\boldsymbol{v} - \boldsymbol{\Pi}_h\boldsymbol{v}), p^h).
\end{aligned}$$

Using the *discrete B-B inequality* for $(\boldsymbol{V}_h, M_h)$, we obtain

$$C\|p^h\|_0 \le \sup_{\boldsymbol{z}\in\boldsymbol{V}_h} \frac{(\mathrm{rot}_h\,\boldsymbol{z}, p^h)}{\|\boldsymbol{z}\|_{1,h}} = \sup_{\boldsymbol{z}\in\boldsymbol{V}_h} \frac{(\nabla_h(\boldsymbol{v}-\boldsymbol{v}^h), \nabla\boldsymbol{z})}{\|\boldsymbol{z}\|_{1,h}} \le \|\nabla_h(\boldsymbol{v}-\boldsymbol{v}^h)\|_0.$$

A combination of the above two inequalities and using (3.8) give

$$\begin{aligned}
\|\nabla_h(\boldsymbol{v}-\boldsymbol{v}^h)\|_0^2 &= (\nabla_h(\boldsymbol{v}-\boldsymbol{v}^h), \nabla_h(\boldsymbol{v}-\boldsymbol{\Pi}_h\boldsymbol{v})) - (\mathrm{rot}_h(\boldsymbol{v}-\boldsymbol{\Pi}_h\boldsymbol{v}), p^h)\\
&\le Ch|\boldsymbol{v}|_2(\|\nabla_h(\boldsymbol{v}-\boldsymbol{v}^h)\|_0 + \|p^h\|_0)\\
&\le Ch|\boldsymbol{v}|_2\|\nabla_h(\boldsymbol{v}-\boldsymbol{v}^h)\|_0,
\end{aligned}$$

which implies (5.3).

A standard dual argument gives

$$\|\boldsymbol{v}-\boldsymbol{v}^h\|_0 \le Ch\|\nabla_h(\boldsymbol{v}-\boldsymbol{v}^h)\|_0.$$

Let $\boldsymbol{\Pi}\boldsymbol{v} = \boldsymbol{v}^h$, we complete the proof. □

Remark 5.1. The operator $\boldsymbol{\Pi}$ constructed in the above lemma is a type of Fortin operator. Such kind of operator is explicitly or implicitly exploited in many different settings (cf. [7 21 9 10]).

Remark 5.2. Let $\boldsymbol{\Pi} = (\Pi_h, \Pi_h)$. It is easy to see that $\boldsymbol{\Pi} : \boldsymbol{H}_0^1(\Omega) \to \widetilde{\boldsymbol{V}}_h$ and satisfies (5.1), (5.2) and (5.3).

Using Lemma 5.1 and Remark 5.1, we may construct a special interpolant for the shear stress as that in Lemma 3.1 of [21].

Lemma 5.2. *There exists $\omega^h \in W_h$ such that $\widehat{\boldsymbol{\gamma}} := \lambda t^{-2}(\nabla\omega^h - \boldsymbol{R}_h\boldsymbol{\Pi}\boldsymbol{\phi}) = \boldsymbol{R}_h\boldsymbol{\gamma}$.*

Proof. For any $q \in M_h$, it follows from the definition of $\boldsymbol{R}_h$ and (5.1) that

$$\int_\Omega \mathrm{rot}\,\boldsymbol{R}_h(\boldsymbol{\phi} - \boldsymbol{\Pi}\boldsymbol{\phi})\,q\,d\boldsymbol{x} = \int_\Omega \mathrm{rot}(\boldsymbol{\phi} - \boldsymbol{\Pi}\boldsymbol{\phi})\,q\,d\boldsymbol{x} = 0,$$

which together with (3.14) gives $\mathrm{rot}\,\boldsymbol{R}_K(\boldsymbol{\phi} - \boldsymbol{\Pi}\boldsymbol{\phi}) = 0$ over each element K. By Lemma 3.1, there exists $\omega_1 \in W_h$ such that

$$\boldsymbol{R}_h(\boldsymbol{\Pi}\boldsymbol{\phi} - \boldsymbol{\phi}) = \nabla\omega_1.$$

Define

$$\omega^h = \Pi_1\omega + \omega_1.$$

Using (3.15), we get $\nabla\Pi_1\omega = \boldsymbol{R}_h\nabla\omega$. Consequently,

$$\begin{aligned}\lambda t^{-2}(\nabla\omega^h - \boldsymbol{R}_h\boldsymbol{\Pi}\boldsymbol{\phi}) &= \lambda t^{-2}(\nabla\Pi_1\omega + \nabla\omega_1 - \boldsymbol{R}_h\boldsymbol{\Pi}\boldsymbol{\phi})\\ &= \lambda t^{-2}\boldsymbol{R}_h(\nabla\omega - \boldsymbol{\phi}) = \boldsymbol{R}_h\boldsymbol{\gamma}. \qquad \square\end{aligned}$$

Define the consistency error functional $e_h(\boldsymbol{u},\boldsymbol{v})$ for any $\boldsymbol{u}\in\boldsymbol{H}^2(\Omega)$ and $\boldsymbol{v}\in\boldsymbol{V}_h$ or $\widetilde{\boldsymbol{V}}_h$ as

$$e_h(\boldsymbol{u},\boldsymbol{v}) = \begin{cases}\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\boldsymbol{C}\mathcal{E}(\boldsymbol{u})\cdot\boldsymbol{n})_1 v_1\,ds,\\ \sum_{K\in\mathcal{T}_h}\int_{\partial K}\boldsymbol{C}\mathcal{E}(\boldsymbol{u})\cdot\boldsymbol{n}\boldsymbol{v}\,ds + \sum_{e\in E_h}⨍_e[\boldsymbol{u}]\cdot[\boldsymbol{v}]\,ds,\end{cases}$$

where v_1 is the first component of $\boldsymbol{v}$. Using (4.2), we estimate e_h as

$$|e_h(\boldsymbol{u},\boldsymbol{v})| \le Ch\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_{1,h}. \tag{5.11}$$

Theorem 5.1. *Let $(\boldsymbol{\phi}_h,\omega_h,\boldsymbol{\gamma}_h)$ be the solution of Problem 3.1 or Problem 3.2, and $(\boldsymbol{\phi},\omega,\boldsymbol{\gamma})$ be the solution of* (2.1) *and* (2.2)*, there holds*

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,h} + \|\nabla(\omega-\omega_h)\|_0 + t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 \le Ch(\|g\|_{-1} + t\|g\|_0). \tag{5.12}$$

Proof. For any $\boldsymbol{\psi}\in\boldsymbol{V}_h$ or $\widetilde{\boldsymbol{V}}_h$ and $v\in W_h$, we have the error equation for the solution

$$a_h(\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{\psi}) + (\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\nabla v - \boldsymbol{R}_h\boldsymbol{\psi}) = (\boldsymbol{\gamma},\boldsymbol{\psi}-\boldsymbol{R}_h\boldsymbol{\psi}) + e_h(\boldsymbol{\phi},\boldsymbol{\psi}),$$

from which we get

$$\begin{aligned}a_h(\boldsymbol{\Pi}\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{\psi}) + (\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_h,\nabla v-\boldsymbol{R}_h\boldsymbol{\psi}) &= a_h(\boldsymbol{\Pi}\boldsymbol{\phi}-\boldsymbol{\phi},\boldsymbol{\psi}) + (\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma},\nabla v-\boldsymbol{R}_h\boldsymbol{\psi})\\ &\quad + (\boldsymbol{\gamma},\boldsymbol{\psi}-\boldsymbol{R}_h\boldsymbol{\psi}) + e_h(\boldsymbol{\phi},\boldsymbol{\psi}). \qquad (5.14)\end{aligned}$$

Let $\boldsymbol{\psi} := \boldsymbol{\Pi}\boldsymbol{\phi}-\boldsymbol{\phi}_h$ and $v=\omega^h-\omega_h$. Applying Lemma 5.2, we conclude

$$\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_h = \lambda t^{-2}\big[\nabla(\omega^h-\omega_h) - \boldsymbol{R}_h(\boldsymbol{\Pi}\boldsymbol{\phi}-\boldsymbol{\phi}_h)\big] = \lambda t^{-2}(\nabla v - \boldsymbol{R}_h\boldsymbol{\psi}).$$

Substituting the above identity into (5.14), using (4.10), (4.12), (5.11) and (3.17), we obtain

$$\|\boldsymbol{\psi}\|_{1,h} + t\|\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_h\|_0 \le Ch(\|\boldsymbol{\phi}-\boldsymbol{\Pi}\boldsymbol{\phi}\|_{1,h} + t\|\boldsymbol{\gamma}-\boldsymbol{R}_h\boldsymbol{\gamma}\|_0) + Ch(\|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\gamma}\|_0).$$

Using (5.3), Remark 5.2, the interpolation estimate (3.17), the regularity estimates (2.7) and (2.9), we obtain

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,h} + t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 \le Ch(\|g\|_{-1} + t\|g\|_0). \tag{5.17}$$

It follows from

$$\nabla\omega_h = \boldsymbol{R}_h\boldsymbol{\phi}_h + \lambda t^2\boldsymbol{\gamma}_h, \qquad \nabla\omega = \boldsymbol{\phi} + \lambda t^2\boldsymbol{\gamma},$$

and (5.17) that the error bound (5.12) for ω holds. $\square$

We turn to the L^2 error estimate. To this end, we need the following lemma, which can be proved as that in Lemma 4.2 of [20].

Lemma 5.3. *For any* $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ *and* $\boldsymbol{\zeta} \in \boldsymbol{H}(\mathrm{div}, \Omega)$, *there exists a constant* C *such that*

$$|(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}, \boldsymbol{\zeta})| \le Ch^2|\boldsymbol{u}|_1\|\operatorname{div}\boldsymbol{\zeta}\|_0 + Ch\|\operatorname{rot}(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u})\|_0\|\boldsymbol{\zeta}\|_0. \tag{5.19}$$

Define an auxiliary problem as: find $(\boldsymbol{\psi}, z) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{m}, \psi) &+ \lambda t^{-2}(\nabla z - \boldsymbol{\psi}, \nabla n - \boldsymbol{m}) \\ &= (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{m}) + (\omega - \omega_h, n) \quad \forall(\boldsymbol{m}, n) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega). \end{aligned} \tag{5.20}$$

Define $\boldsymbol{s} = \lambda t^{-2}(\nabla z - \boldsymbol{\psi})$. Analog to (2.8) and (2.9), we have the regularity result of the above auxiliary problem as

$$\|\boldsymbol{\psi}\|_2 + \|z\|_3 + \|\boldsymbol{s}\|_{\boldsymbol{H}(\mathrm{div})} + t\|\boldsymbol{s}\|_1 \le C(\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\omega - \omega_h\|_0). \tag{5.21}$$

For the solutions of the above problem, using Lemma 5.2, there exists a function $z^h \in W_h$ such that

$$\lambda t^{-2}(\nabla z^h - \boldsymbol{R}_h\boldsymbol{\Pi}\boldsymbol{\psi}) = \boldsymbol{R}_h\boldsymbol{s}. \tag{5.22}$$

Exploiting the Aubin-Nitsche dual argument, we obtain the L^2 estimate as

Theorem 5.2. *The solutions of Problems 3.1 and 3.2 admit the error bounds*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\omega - \omega_h\|_0 \le Ch^2\|g\|_0. \tag{5.23}$$

Proof. Putting $\boldsymbol{m} = \boldsymbol{\phi} - \boldsymbol{\phi}_h$ and $n = \omega - \omega_h$ into the right-hand side of (5.20), we obtain

$$\begin{aligned} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0^2 + \|\omega - \omega_h\|_0^2 = a_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\psi}) &+ \big(\boldsymbol{s}, \nabla(\omega - \omega_h) - (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\big) \\ &+ e_h(\boldsymbol{\psi}, \boldsymbol{\phi} - \boldsymbol{\phi}_h). \end{aligned} \tag{5.24}$$

Obviously,

$$\nabla(\omega - \omega_h) - (\boldsymbol{\phi} - \boldsymbol{\phi}_h) = \lambda^{-1}t^2(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h) + \boldsymbol{\phi}_h - \boldsymbol{R}_h\boldsymbol{\phi}_h.$$

Using (5.22), we obtain

$$\begin{aligned} a_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\Pi}\boldsymbol{\psi}) + \lambda^{-1}t^2(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{R}_h\boldsymbol{s}) &= a_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\Pi}\boldsymbol{\psi}) - (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{R}_h\boldsymbol{\Pi}\boldsymbol{\psi}) \\ &= (\boldsymbol{\gamma}, \boldsymbol{\Pi}\boldsymbol{\psi} - \boldsymbol{R}_h\boldsymbol{\Pi}\boldsymbol{\psi}) + e_h(\boldsymbol{\phi}, \boldsymbol{\Pi}\boldsymbol{\psi}), \end{aligned}$$

where we have used $(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \nabla z^h) = 0$. By substituting the above two identities into (5.24) we obtain

$$\begin{aligned} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0^2 + \|\omega - \omega_h\|_0^2 &= a_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi}) + \lambda^{-1}t^2(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{s} - \boldsymbol{R}_h\boldsymbol{s}) \\ &\quad + (\boldsymbol{s}, \boldsymbol{\phi}_h - \boldsymbol{R}_h\boldsymbol{\phi}_h) + (\boldsymbol{\gamma}, \boldsymbol{\Pi}\boldsymbol{\psi} - \boldsymbol{R}_h\boldsymbol{\Pi}\boldsymbol{\psi}) \\ &\quad + e_h(\boldsymbol{\psi}, \boldsymbol{\phi} - \boldsymbol{\phi}_h) - e_h(\boldsymbol{\phi}, \boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi}) \\ &= I_1 + \cdots + I_6. \end{aligned}$$

Using (5.3) and the interpolation error estimate (3.17), we bound I_1 and I_2 as

$$|I_1| \le Ch\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,h}\|\boldsymbol{\psi}\|_2$$

and

$$|I_2| \le Cht\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0\, t\|\boldsymbol{s}\|_1.$$

We decompose I_3 into

$$I_3 = \big(\boldsymbol{s}, (\boldsymbol{I} - \boldsymbol{R}_h)(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\big) + (\boldsymbol{s}, \boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi}).$$

Using (3.17) for the first term and Lemma 5.3 for the second, we obtain

$$|I_3| \le Ch\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,h}\|\boldsymbol{s}\|_0 + Ch^2|\boldsymbol{\phi}|_1\|\operatorname{div}\boldsymbol{s}\|_0 + Ch\|\operatorname{rot}(\boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi})\|_0\|\boldsymbol{s}\|_0.$$

Similarly, I_4 is bounded as

$$|I_4| \le Ch^2\|\boldsymbol{\gamma}\|_0\|\boldsymbol{\psi}\|_2 + Ch^2\|\operatorname{div}\boldsymbol{\gamma}\|_0|\boldsymbol{\psi}|_1 + Ch\|\boldsymbol{\gamma}\|_0\|\operatorname{rot}(\boldsymbol{\psi} - \boldsymbol{R}_h\boldsymbol{\psi})\|_0.$$

The estimates for the last two consistency error functionals are standard as

$$|I_5| \le Ch\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,h}\|\boldsymbol{\psi}\|_2$$

and

$$|I_6| \le Ch^2\|\boldsymbol{\phi}\|_2\|\boldsymbol{\psi}\|_2.$$

Summing up all the above estimates, using (5.12), (3.18) and the regularity estimate (5.21), we obtain the desired estimate (5.23). □

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