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The Minimum Number of Dependent Arcs and a Related Parameter of Generalized Mycielski Graphs

Hsin-Hao Lai
Department of Mathematics
National Kaohsiung Normal University
Yanchao, Kaohsiung 824, Taiwan
Email:hsinhaolai@nknucc.nknu.edu.tw

Ko-Wei Lih
Institute of Mathematics
Academia Sinica
Nankang, Taipei 115, Taiwan
Email:makwlih@sinica.edu.tw

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Abstract

Let D be an acyclic orientation of the graph G. An arc of D is *dependent* if its reversal creates a directed cycle. Let $d_{\min}(G)$ denote the minimum number of dependent arcs over all acyclic orientations of G. Let $G(V_0, E_0)$ be a graph with vertex set $V_0 = \{\langle 0,0\rangle, \langle 0,1\rangle, \ldots, \langle 0, n-1\rangle\}$ and edge set E_0. The *generalized Mycielski graph* $\mathsf{M}_m(G)$ of G, $m > 0$, has vertex set $V = V_0 \cup (\cup_{i=1}^m V_i) \cup \{u\}$, where $V_i = \{\langle i,j\rangle \mid 0 \leqslant j \leqslant n-1\}$ for $1 \leqslant i \leqslant m$, and edge set $E = E_0 \cup (\cup_{i=1}^m E_i) \cup \{\langle m,j\rangle u \mid 0 \leqslant j \leqslant n-1\}$, where $E_i = \{\langle i-1,j\rangle\langle i,k\rangle \mid \langle 0,j\rangle\langle 0,k\rangle \in E_0\}$ for $1 \leqslant i \leqslant m$. We generalize results concerning $d_{\min}(\mathsf{M}_1(G))$ in K. L. Collins, K. Tysdal, *J. Graph Theory* 46 (2004), 285-296, to $d_{\min}(\mathsf{M}_m(G))$. The underlying graph of a Hasse diagram is called a *cover graph*. Let $c(G)$ denote the the minimum number of edges to be deleted from a graph G to get a cover graph. Analogue results about $c(G)$ are also obtained.

Keyword. acyclic orientation, dependent arc, source-reversal, cover graph, generalized Mycielski graph

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1 Introduction

Graphs considered in this paper are finite, without loops, or multiple edges. We use $|G|$ and $\|G\|$, respectively, to denote the cardinalities of vertex set V and edge set E of a graph $G(V, E)$. The degree of a vertex v of G is denoted $d_G(v)$. An orientation D of G is obtained by assigning an arbitrary direction, either $x \to y$ or $y \to x$, on every edge xy of G. The original undirected graph is called the *underlying graph* of any such orientation. *Sources* (or *sinks*) are vertices with no ingoing (or outgoing) arcs. An orientation D is called *acyclic* if there does not exist any directed cycle.

Suppose that D is an acyclic orientation of G. An arc $u \to v$ of D, or its underlying edge, is called *dependent* (in D) if the new orientation $D' = (D - (u \to v)) \cup (v \to u)$ contains a directed cycle. Note that $u \to v$ is a dependent arc if and only if there exists a directed walk of length at least 2 from u to v. Let $d(D)$ denote the number of dependent arcs in D. Let $d_{\min}(G)$ and $d_{\max}(G)$, respectively, denote the minimum and maximum values of $d(D)$ over all acyclic orientations D of G. It is known ([3]) that $d_{\max}(G) = \|G\| - |G| + k$ for a graph G having k components.

Let $\chi(G)$ denote the *chromatic number* of G, i.e., the least number of colors to color the vertices of G so that adjacent vertices receive distinct colors. Let $g(G)$ denote the *girth* of G, i.e., the length of a shortest cycle of G if there is any, and ∞ if G possesses no cycles. Fisher et al. [3] showed that $d_{\min}(G) = 0$ when $\chi(G) < g(G)$. The Hasse diagram of a finite partially ordered set depicts the covering relation of elements; its underlying graph is called a *cover graph*. Pretzel [7] proved that $d_{\min}(G) = 0$ is equivalent to G being a cover graph.

A convenient tool for us is the source-reversal operation first introduced by Mosesian in the context of finite posets and extensively used by Pretzel dealing with cover graphs. We will summarize the main properties of this operation in Section 2. In Section 3, we will introduce another parameter $c(G)$ which lower bounds $d_{\min}(G)$ and show that $c(G) = 1$ if and only if $d_{\min}(G) = 1$. In Section 4, we will characterize the case $d_{\min}(\mathsf{M}_m(G)) \geqslant 1$. We give generalizations of results established by Collins and Tysdal in Section 5. In the final Section, we derive upper bounds for $c(\mathsf{M}_m(G))$.

2 Source-reversal

Let u be a source of the acyclic orientation D. A *source-reversal* operation applied to u reverses the direction of all outgoing arcs from u so that u becomes a sink. The new orientation remains acyclic. Note that, if there are no dependent arcs in D, neither will there be any after a source-reversal.

Theorem 1 *Let D be an acyclic orientation of a connected graph G. For any vertex u of G, there exists an orientation D' of G obtained from D by a sequence of source-reversals so that u becomes the unique source of D'.*

The above result originally appeared in Mosesian [6]. It was put to good use by Pretzel in a series of papers (for example, [7], [8], [9], and [10]).

Let D be an acyclic orientation of the graph G. For an undirected cycle C of G, we choose one of the two traversals of C as the positive direction. An arc is said to be *forward* if its orientation under D is along the positive direction of C, otherwise it is said to be *backward*. We use $(C, D)^+$ (or $(C, D)^-$) to denote the set of all forward (or backward) arcs of C with respect to D. The *flow difference* of C with respect to D, denoted $f_D(C)$, is defined to be $|(C,D)^+| - |(C,D)^-|$. The cycle C is called *k-good* if $|f_D(C)| \leqslant |C| - 2k$, i.e., C has at least k forward arcs and k backward arcs. An orientation D is called *k-good* if all undirected cycles of its underlying graph G are k-good. The set of acyclic orientations coincides with the set of 1-good orientations. A graph G has a 2-good orientation if and only if $d_{\min}(G) = 0$.

The *flow difference* of an orientation D is the mapping f from all cycles of G to integers such that $f(C) = f_D(C)$ for every cycle C. Let D and D' be two orientations of the graph G. We say that D is an *inversion* of D', and vice versa, if D and D' possess the same flow difference.

The following appeared in Pretzel [8].

Theorem 2 *If D and D' are two acyclic orientations of the graph G, then the following statements are equivalent.*

1. *D' is an inversion of D.*
2. *D' can be obtained from D by a sequence of source-reversals.*

3 The case for $d_{\min} = 1$

We denote by $c(G)$ the the minimum number of edges to be deleted from G so that the remaining graph is a cover graph, i.e.,

$$c(G) = \min\{|F| \mid F \subseteq E(G) \text{ and } G - F \text{ is a cover graph}\}.$$

Bollobás et al. [1] first introduced and studied this parameter. Their results were extended in Rödl and Thoma [11]. It was also one of the four parameters that give lower bounds to $d_{\min}(G)$ investigated in Lai and Lih [4]. It is straightforward to observe the following.

Fact 1. $c(G) \leqslant d_{\min}(G)$.

Fact 2. A sufficient and necessary condition for $c(G) = 0$ is $d_{\min}(G) = 0$.

Theorem 3 *A sufficient and necessary condition for* $c(G) = 1$ *is* $d_{\min}(G) = 1$.

Proof. It follows from Facts 1 and 2 that $d_{\min}(G) = 1$ implies $c(G) = 1$.

Now let us assume that $c(G) = 1$. Then there exists an edge $e = xy$ such that $G' = G - e$ has a 2-good orientation D'. We may assume that there is no directed path from y to x and extend D' to an acyclic orientation D of G by adding the arc $x \to y$.

Since G has no 2-good orientations, D must have at least one dependent arc. If D has only one dependent arc, then we are done. If D has at least two dependent arcs, then each of them must belong to a cycle containing e.

We claim that $x \to y$ can not be dependent in D. Suppose on the contrary that there exists a directed path $x, v_1, v_2, \ldots, v_s, y$, $s \geqslant 1$, from x to y in D. Since D has at least two dependent arcs, there is a dependent arc e' in D distinct from $x \to y$, and there exists a cycle $y, u_1, u_2, \ldots, u_t, x, y$ in G such that e' is the only backward arc in this cycle. Consider the closed walk $W = x, v_1, v_2, \ldots,$ $v_s, y, u_1, u_2, \ldots, u_t, x$. Reversing e' converts W into a closed directed walk. Hence, e' is a dependent arc in D' which contradicts the 2-goodness of D'. Therefore, $x \to y$ is not dependent in D.

By Theorems 1 and 2, we can find an inversion D^* of D such that D^* and D have the same flow difference and y is a source in D^*.

Let e^* be an arbitrary dependent arc in D^* and C^* be a cycle of G such that $(C^*, D^*)^- = \{e^*\}$. Then C^* must pass through the arc $y \to x$. Otherwise, $|(C^*, D^*)^-| = |(C^*, D)^-| = |(C^*, D')^-| = 1$ implies that e^* is a dependent arc in D', contradicting the 2-goodness of D'.

Suppose that e^* is different from the arc $y \to x$. Hence, $y \to x$ belongs to $(C^*, D^*)^+$. Then the arc $x \to y$ belongs to $(C^*, D)^-$. Since $x \to y$ is not dependent in D, we have $|(C^*, D)^-| \geqslant 2$. By Theorem 2, $2 = 2|(C^*, D^*)^-| = |C^*| - |(C^*, D)^+| + |(C^*, D)^-| > 2$, a contradiction. We conclude that e^* must be the arc $y \to x$ in D^*. Therefore, $d_{\min}(G) = d(D^*) = 1$. ∎

An immediate consequence of the above Theorem is the following.

Corollary 4 *If $d_{\min}(G) = 2$, then $c(G) = 2$.*

4 Non-cover Mycielski graphs

Let $G(V_0, E_0)$ be a graph with vertex set $V_0 = \{\langle 0,0\rangle, \langle 0,1\rangle, \ldots, \langle 0,n-1\rangle\}$ and edge set E_0. For $m > 0$, the *generalized Mycielski* graph $\mathsf{M}_m(G)$ of G has vertex set $V = V_0 \cup (\cup_{i=1}^{m} V_i) \cup \{u\}$, where $V_i = \{\langle i,j\rangle \mid 0 \leqslant j \leqslant n-1\}$ for $1 \leqslant i \leqslant m$, and edge set $E = E_0 \cup (\cup_{i=1}^{m} E_i) \cup \{\langle m,j\rangle u \mid 0 \leqslant j \leqslant n-1\}$, where $E_i = \{\langle i-1,j\rangle\langle i,k\rangle \mid \langle 0,j\rangle\langle 0,k\rangle \in E_0\}$ for $1 \leqslant i \leqslant m$. We note that $\mathsf{M}_1(G)$ is commonly known as the *Mycielskian* $M(G)$ of G. It is easy to see that if H is a subgraph of G, then $\mathsf{M}_m(H)$ is a subgraph of $\mathsf{M}_m(G)$. The following was proved in Lih et al. [5].

Theorem 5 *Let $n \geqslant 3$. Then $\mathsf{M}_m(C_n)$ is a cover graph if and only if n is even.*

This can be generalized as follows.

Theorem 6 $d_{\min}(\mathsf{M}_m(G)) \geqslant 1$ *if and only if G is not bipartite.*

Proof. If G has no edge, then obviously $\mathsf{M}_m(G)$ is a cover graph. Let G be a bipartite graph with at least one edge. Then $\chi(\mathsf{M}_m(G)) = 3 < g(\mathsf{M}_m(G))$. Hence $\mathsf{M}_m(G)$ is a cover graph. If G is not bipartite, then G contains an odd cycle C of length at least 3. By Theorem 5, $\mathsf{M}_m(C)$ is not a cover graph. Since $\mathsf{M}_m(G)$ is a supergraph of $\mathsf{M}_m(C)$, it is not a cover graph. ∎

Corollary 7 $c(\mathsf{M}_m(G)) \geqslant 1$ *if and only if G is not bipartite.*

We are going to construct examples to show that equality can hold in Theorem 6.

Theorem 8 *Let $G(V_0, E_0)$ be a triangle-free graph that is not bipartite. Suppose that there exists some vertex $\langle 0, v\rangle$ of G such that $G - \langle 0, v\rangle$ is a bipartite graph whose two parts are denoted by X and Y. If $\langle 0, v\rangle$ has precisely one neighbor in X and at least one neighbor in Y, then $d_{\min}(\mathsf{M}_m(G)) = 1$.*

Proof. By Theorem 6, we know $d_{\min}(\mathsf{M}_m(G)) \geqslant 1$. It suffices to construct an acyclic orientation of $\mathsf{M}_m(G)$ possessing a unique dependent arc.

Step 1. Define an orientation D_1 of G as follows.

(1) If xy is an edge in $G - \langle 0, v\rangle$, $x \in X$ and $y \in Y$, then let $x \to y$.

(2) If $\langle 0, v'\rangle$ is the unique neighbor of $\langle 0, v\rangle$ in X, then let $\langle 0, v'\rangle \to \langle 0, v\rangle$.

(3) If $\langle 0, v''\rangle$ is any neighbor of $\langle 0, v\rangle$ in Y, then let $\langle 0, v\rangle \to \langle 0, v''\rangle$.

Obviously, each vertex in X is a source, each vertex in Y is a sink, and $\langle 0, v\rangle$ is neither a source nor a sink. It follows that D_1 is an acyclic orientation. Moreover, if P is a directed path of length at least 2 in D_1, then $\langle 0, v'\rangle$ must be the initial vertex of P and the length of P is precisely 2. Since G is triangle-free, D_1 has no dependent arc.

Step 2. Let D_2 be the extension of D_1 into $\mathsf{M}_m(G) - u$ by defining $\langle i, w_1\rangle \to \langle i-1, w_2\rangle$ and $\langle i-1, w_1\rangle \to \langle i, w_2\rangle$ if $\langle 0, w_1\rangle \to \langle 0, w_2\rangle$ in D_1 and $1 \leqslant i \leqslant m$.

If $\langle i_1, v_1\rangle, \langle i_2, v_2\rangle, \ldots, \langle i_t, v_t\rangle, \langle i_1, v_1\rangle$ is a directed cycle in D_2, then $\langle 0, v_1\rangle$, $\langle 0, v_2\rangle$, $\ldots$, $\langle 0, v_t\rangle, \langle 0, v_1\rangle$ is a directed closed walk in D_1, contradicting the acyclicity of D_1. Similarly, D_2 has no dependent arc since D_1 has none.

Step 3. Let D_3 be the extension of D_2 into $\mathsf{M}_m(G)$ by defining $\langle m, w\rangle \to u$ for every $\langle 0, w\rangle$.

Since D_2 is acyclic and u is a sink in D_3, D_3 is acyclic. If e is a dependent arc in D_3, then e must be some $\langle m, w\rangle \to u$. If $\langle 0, w\rangle \neq \langle 0, v'\rangle$, then there is a directed path P' from $\langle m, w\rangle$ to a certain $\langle m, w'\rangle$ in D_3. Since there is no edge between $\langle m, w\rangle$ and $\langle m, w'\rangle$ in $\mathsf{M}_m(G)$, P' must have length at least 2. Hence, we can find a directed path of length at least 2 in D_1 and $\langle 0, v'\rangle$ is not the initial vertex of that path. This is a contradiction.

Let us consider the arc $\langle m, v'\rangle \to u$. Let $\langle 0, v''\rangle$ be a neighbor of $\langle 0, v\rangle$ in Y. The cycle $u, \langle m, v'\rangle, \langle m-1, v\rangle, \langle m, v''\rangle, u$ shows that $\langle m, v'\rangle \to u$ is a unique dependent arc in D_3. ∎

A graph G satisfying Theorem 8 can be constructed as follows. Let v be a fixed vertex. Let X be a set of $p \geqslant 2$ vertices and Y be a set of $q \geqslant 2$ vertices. Choose a vertex v' in X and a nonempty proper subset Y' of Y. Add edges vv' and vv'' for all $v'' \in Y'$. Add a path of length at least 3 from v' to some vertex z in Y' which alternately uses vertices in X and Y and uses no vertex in Y' except the terminal vertex z.

However, the problem of characterizing graphs G that satisfy $d_{\min}(\mathsf{M}_m(G)) = 1$ remains open.

5 Generalizing a theorem of Collins and Tysdal

The following appeared in Collins and Tysdal [2].

Theorem 9 *Let G be a triangle-free graph. Then the following statements hold.*

1. *If $d_{\min}(G) \geqslant 1$, then $d_{\min}(M(G)) \geqslant 3$.*
2. *If $d_{\min}(G) \geqslant 2$, then $d_{\min}(M(G)) \geqslant 4$.*
3. *If $d_{\min}(G) \geqslant 3$, then $d_{\min}(M(G)) \geqslant 6$.*

Let S be a set of vertices of the graph $G(V_0, E_0)$. We use S' to denote the set of vertices $\{\langle 1, j\rangle \mid \langle 0, j\rangle \in S\}$ and $G-S+S'$ to denote the subgraph of $M(G)$ induced by the set of vertices $(V_0 \setminus S) \cup S'$ in $M(G)$.

Lemma 10 *If S is an independent set of G, then the subgraph $G - S + S'$ of $M(G)$ is isomorphic to G.*

Proof. The mapping $\sigma : V(G) \to V(G - S + S')$ defined below is an isomorphism. $\sigma(\langle 0, i\rangle) = \langle 1, i\rangle$ if $\langle 0, i\rangle \in S$ and $\sigma(\langle 0, i\rangle) = \langle 0, i\rangle$ if $\langle 0, i\rangle \notin S$. ∎

Proofs of Lemmas 11 and 14 are modeled after ideas used in Collins and Tysdal [2].

Lemma 11 *Let $G(V_0, E_0)$ be a triangle-free graph with at least two edges. For any two edges e_1, e_2 in $M(G) - u$, $M(G) - u - \{e_1, e_2\}$ contains a subgraph isomorphic to G.*

Proof. If none of e_1 and e_2 is an edge in E_0, we are done. Hence, we assume that $e_1 = \langle 0, x_1\rangle\langle 0, y_1\rangle \in E_0$ and consider the subgraph G' of $M(G)$ induced by $(V_0 \setminus \{\langle 0, x_1\rangle\}) \cup \{\langle 1, x_1\rangle\}$. The graph G' is isomorphic to G. If e_2 is not an edge in G', we are done. Assume that e_2 is an edge in G'.

Case 1. The edge e_2 is not incident to $\langle 1, x_1\rangle$. Since G is triangle-free, $\langle 0, x_1\rangle$ can not be adjacent to both endpoints of e_2. Suppose that $\langle 0, x_2\rangle$ is an endpoint of e_2 and not adjacent to $\langle 0, x_1\rangle$. Let $S = \{\langle 0, x_1\rangle, \langle 0, x_2\rangle\}$.

Case 2. The vertex $\langle 1, x_1\rangle$ is an endpoint of e_2. Let $S = \{\langle 0, y_1\rangle\}$.

In each case, S is an independent set. By Lemma 10, $G - S + S'$ is a subgraph of $M(G) - u - \{e_1, e_2\}$ that is isomorphic to G. ∎

Theorem 12 *If a graph G is triangle-free with at least two edges and $d_{\min}(G) \geqslant 1$, then $d_{\min}(\mathsf{M}_m(G)) \geqslant d_{\min}(G) + 2$.*

Proof. By assumption, $d_{\min}(M(G)-u) \geqslant d_{\min}(G) \geqslant 1$. Let F be the set of dependent arcs of an acyclic orientation D of $M(G)-u$ that satisfies $d(D)=d_{\min}(M(G)-u)$, hence $|F| \geqslant 1$. Pick an edge e_1 from F and another edge $e_2 \neq e_1$ of $M(G)-u$. By Lemma 11, $M(G)-u-\{e_1,e_2\}$ contains a subgraph isomorphic to G. Thus $d_{\min}(M(G)-u) \geqslant d_{\min}(G)+1 \geqslant 2$, and hence we can find two distinct edges e_1' and e_2' from F. By Lemma 11 again, $M(G)-u-\{e_1',e_2'\}$ contains a subgraph isomorphic to G. It follows that $d_{\min}(M(G)-u) \geqslant d_{\min}(G)+2$. Finally, $d_{\min}(\mathsf{M}_m(G)) \geqslant d_{\min}(M(G)-u) \geqslant d_{\min}(G)+2$. ∎

If we replace the set F in the above proof by a set F' of edges of $M(G)-u$ such that $M(G)-u-F'$ is a cover graph and $|F'| = c(M(G)-u)$, then we can use the same argument to get the following.

Corollary 13 *If a graph G is triangle-free and $c(G) \geqslant 1$, then $c(\mathsf{M}_m(G)) \geqslant c(G)+2$.*

Lemma 14 *Let $G(V_0,E_0)$ be a triangle-free graph with $\|G\| \geqslant 3$. For any three edges e_1,e_2,e_3 in E_0, $M(G)-u-\{e_1,e_2,e_3\}$ contains a subgraph isomorphic to G.*

Proof. Let G' be the subgraph of G induced by $\{e_1,e_2,e_3\}$.

Case 1. If G' is a star, then let x be the vertex of degree 3 and let $S=\{x\}$.

Case 2. If G' is a path $v_0v_1v_2v_3$ of length 3, then let $S=\{v_0,v_2\}$. Since G is triangle-free, v_0 and v_2 are not adjacent.

Case 3. If G' consists of the disjoint union of a path P_3 of length 2 and an edge P_2, then one endpoint y of P_2 is not adjacent to the center vertex x of P_3 because G is triangle-free. Let $S=\{x,y\}$.

Case 4. Let the three edges $e_1=x_1y_1$, $e_2=x_2y_2$, and $e_3=x_3y_3$ be mutually non-incident. Since G is triangle-free, at least one endpoint of e_2, say x_2, is not adjacent to x_1. Similarly, at least one endpoint of e_3, say x_3, is not adjacent to x_1.

If the vertices x_2 and x_3 are not adjacent, then let $S = \{x_1,\ x_2,\ x_3\}$.

If x_2 and x_3 are adjacent and x_1 is not adjacent to y_i, $i = 2$ or 3, then y_i and x_{5-i} are not adjacent. Let $S = \{x_1, y_i, x_{5-i}\}$.

If x_2 and x_3 are adjacent and both y_2 and y_3 are adjacent to x_1, then $\{y_1, y_2, y_3\}$ is an independent set. Let $S = \{y_1, y_2, y_3\}$.

In all cases, S so defined is an independent set. By Lemma 10, $G - S + S'$ is a subgraph of $M(G) - u - \{e_1, e_2, e_3\}$ isomorphic to G. ∎

Theorem 15 *If a graph G is triangle-free with at least three edges and $d_{\min}(G) \geqslant 3$, then $d_{\min}(\mathsf{M}_m(G)) \geqslant d_{\min}(G) + 3$.*

Proof. Let F be the set of dependent arcs of an acyclic orientation D of G that satisfies $d(D) = d_{\min}(G)$, hence $|F| \geqslant 3$. Pick three edges e_1, e_2, e_3 from F. By Lemma 14, $M(G) - u - \{e_1, e_2, e_3\}$ contains a subgraph isomorphic to G. It follows that $d_{\min}(\mathsf{M}_m(G)) \geqslant d_{\min}(M(G) - u) \geqslant d_{\min}(G) + 3$. ∎

Corollary 16 *If a graph G is triangle-free with at least three edges and $c(G) \geqslant 3$, then $c(\mathsf{M}_m(G)) \geqslant c(G) + 3$.*

6 Upper bounds of $c(\mathsf{M}_m(G))$

In this section, we derive upper bounds for $c(\mathsf{M}_m(G))$. Since $\chi(G) < g(G)$ implies that G is a cover graph, we have the following inequality.

$$c(G) \leqslant \min\{\|G\| - \|H\| \mid H \text{ is a subgraph of } G \text{ and } \chi(H) < g(H)\}.$$

Let $e_k(G)$ be the maximum number of edges in a k-colorable subgraph of G. Since the girth of a subgraph is never smaller than that of the given graph, the above inequality implies the following.

$$c(G) \leqslant \|G\| - e_{k-1}(G) \text{ if } g(G) \geqslant k. \tag{1}$$

Let G be a triangle-free graph. If $H = (X, Y)$ is a bipartite subgraph of G, then the following inequality holds by the above inequality.

$$c(G) \leqslant \|G\| - \|H\| - e_2(G[X]). \tag{2}$$

Proof. Let $X' = (X_1, X_2)$ be a bipartite subgraph of $G[X]$ with $e_2(G[X])$ edges. Consider the subgraph $G' = (V(H), E(H) \cup E(X'))$ of G. Obviously, G' is 3-colorable. Hence, $e_3(G) \geq \|H\| + \|X'\| = \|H\| + e_2(G[X])$. By inequality (1), we are done. ∎

Theorem 17 *If G is a graph and m is a positive integer, then $c(\mathsf{M}_m(G)) \leqslant \|G\|$. Moreover, if G is a triangle-free graph, then $c(\mathsf{M}_m(G)) \leqslant \|G\| - e_2(G)$.*

Proof. Obviously, $\mathsf{M}_m(G) - E(G)$ is bipartite. We have $e_2(\mathsf{M}_m(G)) \geqslant \|\mathsf{M}_m(G)\| - \|G\|$. By inequality (1), $c(\mathsf{M}_m(G)) \leq \|G\|$. If G is triangle-free, so is $\mathsf{M}_m(G)$. It is easy to see that $\mathsf{M}_m(G) - E(G)$ is a bipartite graph with bipartition (X, Y) such that the vertices of G belong to the same partite set, say X. Since G has a bipartite subgraph with $e_2(G)$ edges, $e_2(\mathsf{M}_m(G)[X]) \geqslant e_2(G)$. By inequality (2), $c(\mathsf{M}_m(G)) \leqslant \|\mathsf{M}_m(G)\| - \|\mathsf{M}_m(G) - E(G)\| - e_2(\mathsf{M}_m(G)[X]) \leqslant \|G\| - e_2(G)$. ∎

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