Variational Bounds in Turbulent Convection

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June 5, 2000

1 Introduction

When sufficient energy is steadily supplied to a fluid, the ensuing dynamical behavior involves many spatial and temporal scales and energy is dissipated efficiently. For instance when sufficiently strong heat is supplied against the pull of gravity to a fluid, the heat flux due to fluid flow convection exceeds the heat flux due to molecular diffusion. The average of heat flux is quantified in the Nusselt number N. Numerous experiments and numerical simulations ([1]) under a variety of conditions report power-law behavior

$$N \sim R^q$$

where the Rayleigh number R is proportional to the amount of heat supplied externally. The exponent q is very robust and most experiments give $q = \frac{2}{7}$, while some situations produce $q = \frac{1}{3}$ for large R.

Mathematically, the description is based on the three dimensional Boussinesq equations for Rayleigh-Bénard convection ([2]), a system of equations coupling the three dimensional Navier-Stokes equations to a heat advection-diffusion equation. The only known rigorous upper bound for N ([3]) at large

*Partially supported by NSF/DMS-9501062.

R is of the order $R^{\frac{1}{2}}$; the bound is valid for all weak solutions (the global existence of smooth solutions is not known). This bound is not only a mathematical uper bound: there are physical reasons why $q = \frac{1}{2}$ might be the true asymptotic value at exceedingly high R (in conditions perhaps difficult to achieve in the laboratory; nevertheless, a few recent experimental results hint also at $q = \frac{1}{2}$).

Although one can describe conditions that imply $N \sim R^q$ with $q = \frac{1}{3}$ and even $q = \frac{2}{7}$ for a range of R in the full Boussinesq system ([4]), there is yet no rigorous derivation of the exponent $2/7$ as the unconditional limit for large R in a different non-trivial model of convection. The scaling exponents have been discussed by several authors using physical reasoning and dimensional analysis ([5]) and in particular the exponent $2/7$ has been derived in several physical fashions involving somewhat different predictions.

Variational methods for bounding bulk dissipation in turbulence are a classical subject. Ideas of Malkus from the fifties were followed by Howard's flux maximization results [6] and subsequently were developed by Busse [7] and many others. This classical approach starts from the Reynolds equations and assumes certain statistical symmetries.

In recent years another general variational method has been developed and applied to estimate bulk dissipation quantities in systems in which energy is supplied by boundary conditions ([8] - [15]). A connection between the classical method of Howard and Busse and a version of the background field method has been established in ([16]). The method starts by translating the equation in function space by a background - a time independent function that obeys the driving boundary conditions. A quadratic form is associated naturally to each background, and the method consists in selecting those backgrounds for which this quadratic form is positive semi-definite and then minimizing a certain integral of the background. The set of selected backgrounds is convex. The method has certain advantages over the classical approach – in particular, there is no need for statistical assumptions. The method is flexible enough to accommodate more partial differential information. The partial differential equation confers special properties to the functions that represent long-lived solutions. These functions belong to a large but finite dimensional set, the attractor associated to the PDE at the given values of the parameters. If one can find certain quantitative features of functions belonging to this attractor one can incorporate them in a judicious variational problem. This is how a rigorous upper bound of the form

$$N \leq 1 + C_1 R^{\frac{1}{3}} (1 + \log_+(R))^{\frac{2}{3}}$$

for arbitrary R was derived recently [17] for the three dimensional equations for Rayleigh-Bénard convection obtained in the limit of infinite Prandtl number. The Prandtl number is the ratio of the fluid's viscosity to the fluid's heat conduction coefficient. These equations are an example of active scalars ([18]); they are easier to analyze and simulate numerically than the full Boussinesq system. In the infinite Prandtl number example one can obtain more information about the long time behavior of solutions than in the finite Prandtl number equations. The additional information concerns higher derivatives. In order to exploit this additional information and deduce a better upper bound one needs to modify substantially the background field method: the quadratic form is no longer required to be semidefinite. Instead, the additional information coming from the evolution equation is incorporated in the constraints of a mini-max procedure.

There are several other examples of active scalars for which one can obtain interesting rigorous bounds for the bulk dissipation. For instance, recent results ([19]) on convection in a porous layer employ an improvement of the background field method ([20]) and agree remarkably well with the experimental data.

In this paper we will confine ourselves to the effects of rotation on heat transfer in the infinite Prandtl number cases. Not only are these systems more amenable to analysis but also the variety of physical phenomena poses a challenge to the background flow method as originally formulated. Indeed, in its original formulation the method seems insensitive to linear low order anti-symmetric perturbations such as rotation. The physical effect of very rapid rotation is to stratify the flow and to totally suppress convective heat transport. This effect has been proved recently at large but finite rotation rates in infinite Prandtl number convection in ([21]) using the background field methodology. The limit of slow rotation is not singular. At fixed rotation one can recover the large R rigorous logarithmic 1/3 upper bound ([22]). The situation is complicated though: moderate rotation rates may effectively increase the heat transfer. This experimental fact ([23]) is consistent with the fact that the logarithmic 1/3 upper bound diverges at very high rotation rates; the best known rigorous uniform upper bound valid for all rotation rates has a higher exponent (2/5) than the bound found in the absence of rotation. The uniform bound

$$N \leq\sim R^{\frac{2}{5}}$$

will be derived in this work. We start with the non-rotating case.

2 Infinite Prandtl Number Equations

The infinite Prandtl number equations for Rayleigh-Bénard convection in the Boussinesq approximation are a system of five equations for velocities (u, v, w), pressure p and temperature T in three spatial dimensions. The temperature is advected and diffuses according to the active scalar equation

$$(\partial_t + \mathbf{u} \cdot \nabla) T = \Delta T \tag{1}$$

where $\mathbf{u} = (u, v, w)$. The velocity and pressure are determined from the temperature by solving time independent non-local equations of state:

$$-\Delta u + p_x = 0, \tag{2}$$

together with

$$-\Delta v + p_y = 0 \tag{3}$$

and

$$-\Delta w + p_z = RT. \tag{4}$$

R represents the Rayleigh number. The velocity is divergence-free

$$u_x + v_y + w_z = 0. \tag{5}$$

The horizontal independent variables (x, y) belong to a basic square $Q \subset \mathbf{R}^2$ of side L. Sometimes we will drop the distinction between x and y and denote both horizontal variables x. The vertical variable z belongs to the interval $[0, 1]$. The non-negative variable t represents time. The boundary conditions are as follows: all functions $((u, v, w), p, T)$ are periodic in x and y with period L; u, v, and w vanish for $z = 0, 1$, and the temperature obeys $T = 0$ at $z = 1$, $T = 1$ at $z = 0$.

We will write

$$\|f\|^2 = \frac{1}{L^2} \int_0^1 \int_Q |f(x, y, z)|^2 dz dx dy$$

for the (normalized) L^2 norm on the whole domain. We denote by Δ_D the Laplacian with periodic-Dirichlet boundary conditions. We will denote by Δ_h the Laplacian in the horizontal directions x and y. We will use $< \cdots >$ for long time average:

$$\langle f \rangle = \limsup_{t \to \infty} \frac{1}{t} \int_0^t f(s) ds.$$

We will denote horizontal averages by an overbar:

$$\overline{f(\cdot, z)} = \frac{1}{L^2} \int_Q f(x, y, z) dx dy.$$

We will also use the notation for scalar product

$$(f, g) = \frac{1}{L^2} \int_0^1 \int_Q (fg)(x, y, z) dx dy dz.$$

The Nusselt number is

$$N = 1 + \langle (w, T) \rangle . \tag{6}$$

One can prove using the equation (1) and the boundary conditions that

$$N = \left\langle \|\nabla T\|^2 \right\rangle \tag{7}$$

and using the equations of state (2 - 4) that

$$\left\langle \|\nabla u\|^2 \right\rangle = R(N - 1). \tag{8}$$

This defines a Nusselt number that depends on the choice of initial data; we take the supremum of all these numbers. The system has global smooth solutions for arbitrary smooth initial data. The solutions exist for all time and approach a finite dimensional set of functions. If we think in terms of this dynamical system picture then the Nusselt number represents the maximal long time average distance from the origin on trajectories. Because all invariant measures can be computed using trajectories the Nusselt number is also the maximal expected dissipation, when one maximizes among all invariant measures.

3 Bounding the heat flux

We take a function $\tau(z)$ that satisfies $\tau(0) = 1$, $\tau(1) = 0$, and write $T = \tau + \theta(x, y, z, t)$. The role of τ is that of a convenient background; there is no implied smallness of θ, but of course θ obeys the same homogeneous boundary conditions as the velocity. The equation obeyed by θ is

$$(\partial_t + u \cdot \nabla - \Delta)\, \theta = -\tau'' - w\tau' \tag{9}$$

where we used $\tau' = \frac{d\tau}{dz}$. We are interested in the function $b(z,t)$ defined by

$$b(z,t) = \frac{1}{L^2} \int_Q w(\cdot, z) T(\cdot, z) dx.$$

Its average is related to the Nusselt number:

$$N - 1 = \left\langle \int_0^1 b(z) dz \right\rangle.$$

Note that

$$T - \overline{T} = \theta - \overline{\theta}$$

Also note that from the boundary conditions and incompressibility

$$\overline{w}(z,t) = 0$$

and therefore

$$b(z,t) = \frac{1}{L^2} \int_Q w(\cdot, z) \theta(\cdot, z) dx.$$

From the equation (9) it follows that

$$N = \left\langle -2 \int_0^1 \tau'(z) b(z) dz - \|\nabla \theta\|^2 \right\rangle + \int_0^1 (\tau'(z))^2\, dz. \tag{10}$$

Now we are in a position to explain the variational method and some previous results. Consider a choice of the background τ that is "admissible" in the sense that

$$\left\langle -2 \int_0^1 \tau'(z) b(z) dz - \|\nabla \theta\|^2 \right\rangle \leq 0$$

holds *for all* functions θ. Then of course

$$N \leq \int_0^1 (\tau'(z))^2\, dz.$$

The set of admissible backgrounds is not empty, convex and closed in the H^1 topology. The background method, as originally applied, is then to seek the admissible background that achieves the minimum $\int_0^1 (\tau'(z))^2\,dz$. Such an approach would predict $N \le cR^{\frac{1}{2}}$ for this active scalar, just as in the case of the full Boussinesq system. One can do better. Let us write

$$b(z,t) = \frac{1}{L^2}\int_Q \int_0^z \int_0^{z_1} w_{zz}(x, z_2, t)\theta(x,z)dxdzdz_2dz_1. \qquad (11)$$

It follows that

$$|b(z,t)| \le z^2\left(1 + \|\tau\|_{L^\infty}\right)\|w_{zz}\|_{L^\infty(dz;L^1(dx))}. \qquad (12)$$

Now we will use two a priori bounds. First, one can prove using (9) and (8) that there exists a positive constant C_Δ such that

$$\left\langle \|\Delta\theta\|^2 \right\rangle \le C_\Delta \left\{ RN + \int_0^1 \left[(\tau''(z))^2 + Rz(\tau'(z))^2 \right] dz \right\} \qquad (13)$$

holds. Secondly, one has the basic logarithmic bound ([17])

$$\|w_{zz}\|_{L^\infty} \le CR(1 + \|\tau\|_{L^\infty})[1 + \log_+(R\|\Delta\theta\|)]^2. \qquad (14)$$

We will describe briefly how to obtain (13) and (14) in the next section. Using (14) together with (13) in (12) one deduces from (10)

$$N \le \int_0^1 (\tau'(z))^2 dz + CR(1 + \|\tau\|_{L^\infty})^2 \left[\int_0^1 z^2|\tau'|dz\right]$$

$$\left[1 + \log_+ \left\{ RN + \int_0^1 \left[(\tau''(z))^2 + Rz(\tau'(z))^2 \right] dz \right\}\right] \qquad (15)$$

Choosing τ to be a smooth approximation of $\tau(z) = \frac{1-z}{\delta}$ for $0 \le z \le \delta$ and $\tau = 0$ for $z \ge \delta$ and optimizing in δ one obtains

Theorem 1 *There exists a constant C_0 such that the Nusselt number for the infinite Prandtl number equation is bounded by*

$$N \le N_0(R)$$

where

$$N_0(R) = 1 + C_0 R^{1/3}(1 + \log_+ R)^{\frac{2}{3}}$$

The associated optimization procedure consists in the mini-max suggested by (10) for functions θ that obey the constraint (13).

Theorem 2 *The Nusselt number for the infinite Prandtl number equation is bounded by the constrained mini-max procedure*

$$N \leq \inf_{\tau} \sup_{\theta \in C_\tau} \left\{ \left\langle -\|\nabla\theta\|^2 + 2\int_0^1 -\tau'(z) b(z) dz \right\rangle + \int_0^1 (\tau'(z))^2 \right\}$$

where C_τ is the set of smooth, time dependent functions θ that obey periodic-homogeneous Dirichlet boundary conditions and the inequality

$$\left\langle \|\Delta\theta\|^2 \right\rangle \leq C_\Delta \left\{ RN_0(R) + \int_0^1 \left[(\tau''(z))^2 + Rz(\tau'(z))^2 \right] dz \right\}.$$

The functions $b(z,t)$ are computed via

$$b(z,t) = \frac{1}{L^2} \int \int_Q w(x,y,z,t)\theta(x,y,z,t) dxdy$$

and the functions $w(x,y,z,t)$ are computed by solving

$$\Delta^2 w = -R\Delta_h \theta$$

with periodic-homogeneous Dirichlet and Neumann boundary conditions.

4 Two inequalities

The inequalities (13) and (14) played an important role. We present here the ingredients needed to prove them because they are of more general use.

In order to prove (13) using only the bound (8) on the velocity we use the interpolation inequality

$$\|\nabla\theta\|^2_{L^4(dx)} \leq 3\|\theta\|_{L^\infty} \|\Delta\theta\|_{L^2(dx)}$$

that is valid in all dimensions (and can be proved directly by integration by parts). Multiplying (9) by $-\Delta\theta$, integrating by parts in the convective term and using the divergence-free condition one obtains after long time average the bound (13).

In order to obtain (14) we write first the equation obeyed by the pressure in view of (5):

$$\Delta p = RT_z.$$

Differentiating and substituting, the equation (4) becomes

$$\Delta^2 w = -R\Delta_h T. \tag{16}$$

In view of the incompressibility condition, the boundary conditions are

$$w(x,y,0) = w'(x,y,0) = w(x,y,1) = w'(x,y,1) = 0. \tag{17}$$

Denote by $(\Delta_{DN}^2)^{-1}f$ the solution $w = (\Delta_{DN}^2)^{-1}f$ of

$$\Delta^2 w = f$$

with horizontally periodic and vertically Dirichlet and Neumann boundary conditions $w = w' = 0$. Thus, in the infinite Prandtl number system

$$w_{zz} = -RB\theta$$

where

$$B = \frac{\partial^2}{\partial z^2}(\Delta_{DN}^2)^{-1}\Delta_h.$$

The inequality (14) was proved as a consequence of the logarithmic L^∞ estimate for the operator B ([17]) given below.

Theorem 3 *For any $\alpha \in (0,1)$ there exists a positive constant C_α such that every Hölder continuous function θ that is horizontally periodic and vanishes at the vertical boundaries satisfies*

$$\|B\theta\|_{L^\infty} \le C_\alpha \|\theta\|_{L^\infty}\left(1 + \log_+ \|\theta\|_{C^{0,\alpha}}\right)^2. \tag{18}$$

The spatial $C^{0,\alpha}$ norm is defined as

$$\|\theta\|_{C^{0,\alpha}} = \sup_{X=(x,y,z)\in Q\times[0,1]} |\theta(X,t)| + \sup_{X\neq Y} \frac{|\theta(X,t)-\theta(Y,t)|}{|X-Y|^\alpha}$$

The proof ([17]) is based on a decomposition

$$B\theta = (I - B_1 + B_2 + B_3)B_1\theta$$

where

$$B_1(\theta) = \Delta_h \left(\Delta_D\right)^{-1} \theta$$

and B_2 and B_3 are certain singular integral operators. One proves for B_j, $j = 1, 2, 3$ the estimates

$$\|B_j\theta\|_{L^\infty} \le C_\alpha \|\theta\|_{L^\infty} \left(1 + \log_+ \|\theta\|_{C^{0,\alpha}}\right). \tag{19}$$

These estimates are well-known for singular integral operators of the classical Calderon-Zygmund type. The operators B_j are not translationally invariant. They have kernels K_j,

$$B_1(\theta)(x,z) = L^{-2} \int_Q \int_0^1 K_1(x-y,z,\zeta)\left(\theta(y,\zeta) - \theta(x,z)\right) dy d\zeta$$

and

$$B_2(\theta)(x,z) = L^{-2} \int_Q \int_0^1 K_2(x-y,z,\zeta)\left(\theta(y,\zeta) - \theta(y,1)\right) dy d\zeta$$

and

$$B_3(\theta)(x,z) = L^{-2} \int_Q \int_0^1 K_3(x-y,z,\zeta)\left(\theta(y,\zeta) - \theta(y,0)\right) dy d\zeta.$$

The kernels K_j can be written as oscillatory sums of exponentials. The Poisson summation formula and Poisson kernel are used to derive inequalities of the type

$$|K_1(x-y,z,\zeta)| \le C\left(|x-y|^2 + |z-\zeta|^2\right)^{-\frac{3}{2}} \tag{20}$$

and

$$|K_2(x-y,z,\zeta)| \le C\left(|x-y|^2 + |1-\zeta|^2\right)^{-\frac{3}{2}} \tag{21}$$

and similarly

$$|K_3(x-y,z,\zeta)| \le C\left(|x-y|^2 + |\zeta|^2\right)^{-\frac{3}{2}}. \tag{22}$$

The inequalities (20, 21, 22) are the heart of the matter; once they are proved, the estimates (19) follow in a straightforward manner.

5 Rotation

We assume that the domain D rotates at a uniform angular rate around the z axis, and we place ourselves in a frame rotating with the domain. We will still consider the infinite Prandtl number case. The boundary conditions and the equation (1) for the temperature are the same as in the non-rotating case. In the presence of rotation the velocity is determined by the temperature through the Poincaré-Stokes equation of state:

$$\begin{aligned} -\Delta u - E^{-1}v + p_x = 0 \\ -\Delta v + E^{-1}u + p_y = 0 \\ -\Delta w + p_z = RT. \end{aligned} \tag{23}$$

Here E is the Ekman number. The non-rotating case corresponds formally to $E = \infty$. The incompressibility condition (5) is maintained. We denote by ζ the vertical component of vorticity

$$\zeta = v_x - u_y. \tag{24}$$

Taking the divergence of (23) to obtain the equation for the pressure:

$$\Delta p - E^{-1}\zeta = RT_z. \tag{25}$$

Eliminating the pressure we obtain the analogue of (16)

$$\Delta^2 w - E^{-1}\zeta_z = -R\Delta_h T \tag{26}$$

together with

$$-\Delta\zeta - E^{-1}w_z = 0. \tag{27}$$

Incompressibility is used to deduce the boundary conditions

$$\begin{aligned} w(x,y,0,t) = w(x,y,1,t) = 0 \\ w_z(x,y,0,t) = w_z(x,y,1,t) = 0 \\ \zeta(x,y,0,t) = \zeta(x,y,1,t) = 0. \end{aligned} \tag{28}$$

From (26) and (27) it is easy to obtain ([21]) bounds for the velocity and pressure that are uniform for all rotation rates E^{-1}:

$$\|\Delta w\|^2 + 2\|\nabla\zeta\|^2 \leq R^2, \tag{29}$$

$$\|p_z\|^2 \leq 4R^2 \tag{30}$$

$$\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2 \leq R^2. \tag{31}$$

These inequalities hold pointwise in time and are valid in the non-rotating case as well. Notice that the uniform bound (29) has a very important consequence for strongly rotating (small Ekman number) systems: the vertical acceleration w_z is suppressed. Indeed from (27) it follows that

$$(\Delta_D)^{-1} w_z = -E\zeta$$

and thus w_z tends to zero in H^{-1} as $E \to 0$ at fixed R. In order to take advantage of this observation we need to control the growth of the full gradients of the horizontal components of velocity at the boundaries. This is achieved ([21]) in the following manner. First we differentiate the equation for u in (23) with respect to z

$$-\Delta u_z = E^{-1} v_z - p_{zx}, \tag{32}$$

we multiply by u and integrate horizontally:

$$-\overline{u\Delta u_z} = E^{-1}\overline{v_z u} + \overline{p_z u_x}. \tag{33}$$

Secondly, we observe that

$$-\overline{u\Delta u_z} = \frac{d}{dz}\left(\frac{1}{2}\overline{|\nabla u|_2^2}\right) - \frac{d}{dz}\overline{u u_{zz}}. \tag{34}$$

Integrating (33, 34) vertically on $[0, z]$ using the Dirichlet boundary condition on u we obtain

$$\frac{1}{2}\overline{|\nabla u(\cdot, 0)|_2^2} = \frac{1}{2}\overline{|\nabla u(\cdot, z)|_2^2} - \overline{u u_{zz}} - E^{-1}\int_0^z \overline{v_z u} - \int_0^z \overline{p_z u_x},$$

and integrating again with respect to z from 0 to 1 we deduce

$$\frac{1}{2}\overline{|\nabla u(\cdot, 0)|^2} \leq \frac{1}{2}\|\nabla u\|^2 + \|u_z\|^2 + E^{-1}\|v_z\|\|u\| + \|p_z\|\|u_x\|. \tag{35}$$

Now from (35) using the bounds (30), (31) and the Poincare inequality we obtain

$$\overline{|\nabla u(\cdot, 0)|^2} \leq C(1 + E^{-1})R^2. \tag{36}$$

Similar inequalities hold for v and the other boundary $z = 1$.

6 Heat flux in a rotating system

We recall (10)

$$N = \left\langle -2\int_0^1 \tau'(z)b(z)dz - \|\nabla\theta\|^2 \right\rangle + \int_0^1 (\tau'(z))^2\, dz \tag{37}$$

and write

$$\int_0^1 \tau'(z)b(z)dz = -(w_z, \Theta)$$

where Θ is

$$\Theta(x,y,z,t) = \int_0^z \tau'(s)\theta(x,y,s,t)ds. \tag{38}$$

Now we replace w_z using (27) in order to exhibit the small parameter E

$$\int_0^1 \tau'(z)b(z)dz = E(\Delta\zeta.\Theta) \tag{39}$$

We need to integrate by parts once and consider a boundary term:

$$\int_0^1 \tau'(z)b(z)dz = I + II \tag{40}$$

where

$$I = -E(\nabla\zeta, \nabla\Theta) \tag{41}$$

and

$$II = E\overline{\zeta_z(\cdot,1,t)\Theta(\cdot,1,t)}. \tag{42}$$

It is easy to show that

$$\|\nabla\Theta\| \le g\|\nabla\theta\| \tag{43}$$

where

$$g = \left[\int_0^1 (1-z)\,(\tau'(z))^2\, dz\right]^{\frac{1}{2}}. \tag{44}$$

The first term in (40) is bounded in view of (29)

$$|I| = E|(\nabla\zeta, \nabla\Theta)| \le \frac{Eg}{\sqrt{2}} R\|\nabla\theta\|. \tag{45}$$

The second term can be written after one horizontal integration by parts as

$$II = E\overline{(u_z(\cdot,1,t)\Theta_y(\cdot,1,t) - v_z(\cdot,1,t)\Theta_x(\cdot,1,t))}. \tag{46}$$

Because Θ is an integral of θ it is easy to see that

$$\|\nabla_h \Theta(\cdot, 1, t)\|_h \leq G\|\nabla\theta\|$$

where

$$G = \sup_z |\tau'(z)| \tag{47}$$

and

$$\|\nabla_h \Theta(\cdot, 1, t)\|_h^2 = \overline{|\nabla_h \Theta(\cdot, 1, t)|^2}.$$

is the normalized horizontal L^2 norm. Using the boundary bound (36) on u_z and v_z we deduce that the contribution of the second term is estimated

$$|II| \leq CG\sqrt{E^2 + E}R\|\nabla\theta\|. \tag{48}$$

Gathering (45) and (48) we obtain

$$\left|\int_0^1 \tau'(z)b(z)dz\right| \leq C\left\{Eg + G\sqrt{E^2 + E}\right\} R\|\nabla\theta\|. \tag{49}$$

We deduce

$$\left|\int_0^1 \tau'(z)b(z)dz\right| \leq C_1\left\{g^2E^2 + G^2(E^2 + E)\right\} R^2 + \frac{1}{2}\|\nabla\theta\|^2 \tag{50}$$

On the other hand, it is not difficult to see using (29) and $0 \leq T \leq 1$ (maximum principle) in (11) that

$$\left|\int_0^1 \tau'(z)b(z)dz\right| \leq C_2R\int_0^1 z^{\frac{3}{2}}|\tau'(z)|dz \tag{51}$$

This observation allows us to improve the results of ([21]). For any τ we may choose to apply either the bound (50) or (51) in the Nusselt number calculation (37). Let us set

$$\Gamma_\tau(E, R) = \min\left\{2C_1\left[g^2E^2 + G^2\left(E^2 + E\right)\right] R^2; 2C_2MR\right\} \tag{52}$$

where

$$M = \int_0^1 z^{\frac{3}{2}}|\tau'(z)|dz. \tag{53}$$

Consequently we obtain

$$N \leq \int_0^1 (\tau'(z))^2\, dz + \Gamma_\tau(E, R). \tag{54}$$

If one chooses τ to be a smooth approximation of $\tau = (1-z)\delta^{-1}$ for $0 \leq z \leq \delta$ and $\tau = 0$ for $\delta \leq z \leq 1$ then $g = O(\delta^{-\frac{1}{2}})$, $G = O(\delta^{-1})$ and $M = O(\delta^{\frac{3}{2}})$.

Optimizing in τ ([21], [22]) one obtains

Theorem 4 *The Nusselt number for rotating infinite Prandtl-number convection is bounded by*

$$N - 1 \leq \min\left\{c_1 R^{\frac{2}{5}};\ (c_2 E^2 + c_3 E) R^2\right\}.$$

7 Discussion

Rotation has a non-trivial effect on heat transfer in the infinite Prandtl number convection. The equation determining the vertical velocity from the temperature is

$$\left(\Delta^2 + E^{-2}\partial_z \Delta_D^{-1} \partial_z\right) w = -R\Delta_h T$$

The operator $\partial_z \Delta_D^{-1} \partial_z$ is a low order perturbation of Δ^2 and both operators are non-negative in L^2. In the absence of rotation ($E = \infty$) one has a rigorous upper bound of the type $N \leq\sim R^{\frac{1}{3}} (\log R)^{\frac{2}{3}}$. However, the presently known rotation independent uniform bound has a higher exponent, $N \leq\sim R^{\frac{2}{5}}$. If rotation is increased sufficiently ($ER^{\frac{8}{5}} << 1$) for fixed R, then its effect is to dramatically laminarize the flow and the the heat transfer is due then exclusively to molecular diffusion: $N \to 1$. On the other hand, for fixed E one can recover the logarithmic 1/3 bound for large R ([22]), but the bound diverges for $E \to 0$; the envelope is finite nevertheless because of the uniform 2/5 bound. These rigorous results capture some of the complexity of the phenomena.

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