

# Labeled Ballot Paths and the Springer Numbers

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**Abstract:** The Springer numbers are defined in connection with the irreducible root system of type  $B_n$  and also arise as the generalized Euler and class numbers introduced by Shanks. Combinatorial interpretations of the Springer numbers have been found by Purtill in terms of André signed permutations, and by Arnol'd in terms of snakes of type  $B_n$ . We introduce the inversion code of a snake of type  $B_n$  and establish a bijection between labeled ballot paths of length  $n$  and snakes of type  $B_n$ . Moreover, we obtain the bivariate generating function for the number  $B(n, k)$  of labeled ballot paths starting at  $(0, 0)$  and ending at  $(n, k)$ . Using our bijection, we find a statistic  $\alpha$  such that the number of snakes  $\pi$  of type  $B_n$  with  $\alpha(\pi) = k$  equals  $B(n, k)$ . We also show that our bijection specializes to a bijection between labeled Dyck paths of length  $2n$  and alternating permutations on  $[2n]$ .

**AMS Classification:** 05A05, 05A19.

**Keywords:** Springer number, snake of type  $B_n$ , labeled ballot path, labeled Dyck path, bijection

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