Zeta Functions and the Log-behavior of Combinatorial Sequences

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Abstract: In this paper, we use the Riemann zeta function $\zeta(x)$ and the Bessel zeta function $\zeta_\mu(x)$ to study the log-behavior of combinatorial sequences. We prove that $\zeta(x)$ is log-convex for $x>1$. As a consequence, we deduce that the sequence $\{|B_{2n}|/(2n)!\}_{n\geqslant 1}$ is log-convex, where B_n is the n-th Bernoulli number. We introduce the function $\theta(x)=(2\zeta(x)\Gamma(x+1))^{1/x}$, where $\Gamma(x)$ is the gamma function, and we show that $\log\theta(x)$ is strictly increasing for $x\geqslant 6$. This confirms a conjecture of Sun stating that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geq 1}$ is strictly increasing. Amdeberhan, Moll and Vignat defined the numbers $a_n(\mu)=2^{2n+1}(n+1)!(\mu+1)_n\zeta_\mu(2n)$ and conjectured that the sequence $\{a_n(\mu)\}_{n\geqslant 1}$ is log-convex for $\mu=0$ and $\mu=1$. By proving that $\zeta_\mu(x)$ is log-convex for $x>1$ and $\mu>-1$, we show that the sequence $\{a_n(\mu)\}_{n\geqslant 1}$ is log-convex for any $\mu>-1$. We introduce another function $\theta_\mu(x)$ involving $\zeta_\mu(x)$ and the gamma function $\Gamma(x)$ and we show that $\log\theta_\mu(x)$ is strictly increasing for $x>8e(\mu+2)^2$. This implies that $\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)}$ for $n> 4e(\mu+2)^2$. Based on Dobinski's formula, we prove that $\sqrt[n]{B_n} < \sqrt[n+1]{B_{n+1}}$ for $n\geqslant 1$, where B_n is the n-th Bell number. This confirms another conjecture of Sun. We also establish a connection between the increasing property of $\{\sqrt[n]{B_n}\}_{n\geq 1}$ and Hölder's inequality in probability theory.

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Keywords: log-convexity, Riemann zeta function, Bernoulli number, Bell number, Bessel zeta function, Narayana number, Hölder's inequality

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Return