Yangian and Applications

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The Yangian relations are tremendiously simplified for $SU(2)$, $SU(3)$, $SO(5)$ and $SO(6)$ based on RTT relations that much benifits the realization of Yangian in Physics. The Physical meaning and some applications of Yangian have been shown.

(I) Yangian and RTT Relations

The Yangian algebras $Y(SL(n))$ associated with $SL(n)$ were given by Drinfeld (1985). For a given Lie algebraic generators I_μ the new generators J_ν were introduced to satisfy

(1)$[I_\lambda, I_\mu] = C_{\lambda\mu\nu} I_\nu, \lambda, \mu, \nu = 1, 2, 3, \cdots,$

where $C_{\lambda\mu\nu}$ structure constants.

(2)$[I_\lambda, J_\mu] = C_{\lambda\mu\nu} J_\nu, \lambda, \mu, \nu = 1, 2, 3, \cdots,$

and for $n \geq 3$:

(3)$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = a_{\lambda\mu\nu\alpha\beta\gamma}\{I_\alpha, I_\beta, I_\gamma\},$

$a_{\lambda\mu\nu\alpha\beta\gamma} = \frac{1}{4!} C_{\lambda\alpha\sigma} C_{\mu\beta\tau} C_{\nu\gamma\rho},$

$$\{x_1, x_2, x_3\} = \sum_{\substack{i,j,k=1,2,3 \\ i \neq j \neq k}} x_i x_j x_k.$$

which is symmetric summation over $x_i's$.

or, for $n = 2$:

(4) $[[J_\lambda, J_\mu], [I_\sigma, J_\tau]] + [[J_\sigma, J_\tau], [I_\lambda, J_\mu]] = (a_{\lambda\mu\nu\alpha\beta\gamma} C_{\sigma\tau\nu} + a_{\sigma\tau\nu\alpha\beta\gamma} C_{\lambda\mu\nu}\{I_\alpha, I_\beta, J_\gamma\}$

When $C_{\lambda\mu\nu} = i\varepsilon_{\lambda\mu\nu}(\lambda, \mu, \nu = 1, 2, 3)$, Eq(3) is identically satisfied based on the Jacobian identities. Besides the commutation relations there are co-products.

Further, the Yangian can be derived through RTT relations where R is rational solution of Yang-Baxter eq (YBE). (Drinfeld, Faddeev and his school).

After lengthy calculations we found (Ge, Xue and Zhang), the independent relations for $Y(SU(2))$, $Y(SU(3))$, $(Y(SO(5))$ and $Y(SO(6))$ by expanding the RTT relations and also checked through (1) — (4) by substituting the structure constants. RTT relation (Faddeev, Reshetikhin, Takhtajan — RFT) satisfies

$$\check{R}(u-v)(T(u)\otimes 1)(1\otimes T(v)) = (1\otimes T(v))(T(u)\otimes 1)\check{R}(u-v)$$

(1)$Y(SU(2))$

$$\check{R}_{12}(u) = PR_{12}(u) = uP_{12} + I. \qquad (P_{12} = Permutation)$$

$$T(u) = I + \sum_{n=1}^{\infty} u^{-n} \begin{bmatrix} T_{11}^{(n)} & T_{12}^{(n)} \\ T_{21}^{(n)} & T_{22}^{(n)} \end{bmatrix} = I + \sum_{n=1}^{\infty} u^{-n} \begin{bmatrix} \frac{1}{2}(T_0^{(n)} + T_3^{(n)}), & T_+^{(n)} \\ T_-^{(n)}, & \frac{1}{2}(T_0^{(n)} - T_3^{(n)}) \end{bmatrix}$$

Substituting $T(u)$ into RTT relation it turns out that only

$$I_\pm = T_\pm^{(1)}, I_3 = \frac{1}{2} T_3^{(1)}$$

$$J_\pm = T_\pm^{(2)}, J_3 = \frac{1}{2}T_3^{(2)}$$

are independent ones. The quantum determinant

$$detT(u) = T_{11}(u)T_{22}(u-1) - T_{12}(u)T_{21}(u-1) = C_0 + \sum_{n=1}^{\infty} u^{-n}C_n$$

gives

$$\begin{aligned} C_0 = 1, \quad C_1 = T_0^{(1)} = trT^{(1)} \\ C_2 = T_0^{(2)} - \mathbf{I}^2 + T_0^{(1)}(1 + \frac{1}{2}T_0^{(1)}) \\ \cdots \end{aligned}$$

The independent commutation relations of $Y(SU(2))$ are:

$$\begin{aligned} &[I_\lambda, I_\mu] = i\epsilon_{\lambda\mu\nu}I_\nu \qquad (\lambda, \mu, \nu = 1, 2, 3) \\ &[I_\lambda, J_\mu] = i\epsilon_{\lambda\mu\nu}J_\nu \end{aligned}$$

and $(A_\pm = A_1 \pm iA_2]$

$$[J_3, [J_+, J_-]] = (J_-J_+ - I_-J_+)I_3$$

that can be checked to generate all of relations of Eqs(1),(2) and (4).

The co-product is given through (RFT)

$$\Delta T_{ab} = \sum_c T_{ac} \otimes T_{cb}$$

The simplest realization of $Y(SU(2)$ is

$$\mathbf{I} = \sum_{i=1}^{N} \mathbf{I}_i \quad (i : \text{lattice indices})$$
$$\mathbf{J} = \sum_{i=1}^{N} \mu_i \mathbf{I}_i + \sum_{i<j}^{N} w_{ij} \mathbf{I}_i \times \mathbf{I}_j$$

where

$$W_{ij} = \begin{cases} 1 & i < j \\ 0 & i = j \\ -1 & i > j \end{cases} \qquad \texttt{(for any representation of} SU(2)\texttt{)}$$

or

$$W_{jk} = i \cot \frac{(j-k)\pi}{N} \qquad \texttt{(only for spin } \frac{1}{2}, \texttt{ Haldane-Shastry model)},$$

and μ_i arbitrary constants. Noting that μ_i plays important role for the representation theory of $Y(SU(2))$ (Chari-Pressley, 1990, 1991).

The big difference between representations of Lie algebra and Yangian is in that in Yangian there appear free parameters μ_i dependent on models.

Another example for single particle is finite w-algebra (Sorba-Ragoucy 1997). Denoting by **L** and **B** angular momentum and lorentz boost, respectively, as well as D the dilitation operator, the set of **L** and **J** satisfies $Y(SU(2))$ where (Sorba-Ragoucy 1998, Ge, Xue 1999)

$$\mathbf{I} = \mathbf{L}$$
$$\mathbf{J} = \mathbf{I} \times \mathbf{B} - i(D-1)\mathbf{B}$$

and

$$[J_\alpha, J_\beta] = i\epsilon_{\alpha\beta\gamma}(2\mathbf{I}^2 - c_2' - 4)\mathbf{I}_\gamma$$
$$c_2' \text{ casimir of } SO(4,2).$$

The Hamiltonian commuting with $Y(SU(2))$:

- Two component NSE eq (Wadati, $\cdots$)
- One-dimensional Hubbard model (Uglov,Korepin)

Essler and Korepin found the complete solutions (1991) and excitation spectrum (1994) of 1-D Hubbard model.

• Haldane-Shastry model(Haldane) whose Hamiltonian is given by the quantum determinant (Wang, Ge, Xue)

• Hydrogen atom (with and without monopole, Ge, Xue,Bai)

• Super YM($N=4$): $Y(SO(6)$ (Dolan, Nappi, Witten)

(2) $Y(SU(3))$

Independent relations

$$[I_\lambda, I_\mu] = if_{\lambda\mu\nu}I_\nu, \quad [I_\lambda, J_\mu] = if_{\lambda\mu\nu}J_\nu \quad (\lambda, \mu, \nu = 1, \cdots, 8)$$

Define

$$I_\pm^{(1)} = I_1 \pm iI_2, \; U_\pm^{(1)} = I_6 \pm iI_7, \; V_\pm^{(1)} = I_4 \mp iI_5, \; \frac{\sqrt{3}}{2}I_8^{(1)} = I_8$$

and the corresponding operator for $I_\pm^{(2)}, U_\pm^{(2)}, V_\pm^{(2)}$ and $I_8^{(2)}, I_3^{(2)}$ that represent J_μ, after lenthy calculation one finds there is only one additional relation for $Y(SU(3))$

$$[I_8^{(2)}, I_3^{(2)}] = \frac{1}{3!}(\{I_+^{(1)}, U_+^{(1)}, V_+^{(1)}\} - \{I_-^{(1)}, U_-^{(1)}, V_-^{(1)}\})$$

where $\{\cdots\}$ stands for symmetric summation. The conclusion can be verified through both the Drinfeld formula ($C_{\lambda\mu\nu} = if_{\lambda\mu\nu}$) and RTT relations with the replacment of P_{12} in $SU(2)$ by

$$P_{12} = \frac{1}{3}I + \frac{1}{2}\sum_{\mu} \lambda_{\mu}\lambda_{\mu}$$

where λ_{μ} are the Gell-mann matries.

$$T(u) = \sum_{n=0}^{\infty} u^{-n} T(n)$$

$$T^{(n)} = \begin{bmatrix} \frac{1}{3}T_0^{(n)} + T_3^{(n)} + \frac{1}{\sqrt{3}}T_8^{(n)} & T_1^{(n)} - iT_2^{(n)} & T_4^{(n)} - iT_5^{(n)} \\ T_1^{(n)} + iT_2^{(n)} & \frac{1}{3}T_0^{(n)} - T_3^{(n)} + \frac{1}{\sqrt{3}}T_8^{(n)} & T_6^{(n)} - iT_7^{(n)} \\ T_4^{(n)} + iT_5^{(n)} & T_6^{(n)} + iT_7^{(n)} & \frac{1}{3}T_0^{(n)} - \frac{2}{\sqrt{3}}T_8^{(n)} \end{bmatrix}$$

and the co-product, for example,

$$\begin{aligned} \Delta I_{\pm}^{(2)} &= I_{\pm}^{(2)} \otimes 1 + 1 \otimes I_{\pm}^{(2)} \\ &\pm 2(I_3^{(1)} \otimes I_{\pm}^{(1)} - I_{\pm}^{(1)} \otimes I_3^{(1)} + \frac{1}{2}(V_{\mp}^{(1)} \otimes U_{\mp}^{(1)} \\ &- U_{\mp}^{(1)} \otimes V_{\mp}^{(1)} \end{aligned}$$

and others.

An example of realization of $Y(SU(3))$ is the generalization of Haldane-Shastry:

$$
\begin{aligned}
I_\mu &= \sum_i F_i^\mu \\
J_\mu &= \sum_i \mu_i F_i^\mu + \lambda f_{\mu\lambda\nu} \sum_{i \neq j} \omega_{ij} F_i^\nu F_j^\lambda
\end{aligned}
$$

Where ω_{ij} satisfies the same relation as in HS model and F^μ the Gell-mann matrices.

(3) $Y(SO(5))$

For $SO(N)$ it holds

$$
\begin{aligned}
[L_{ij}, L_{kl}] &= i C_{ij,kl}^{st} L_{st} \\
C_{ij,kl}^{st} &= \delta_{ik}\delta_{js}\delta_{lt} - \delta_{il}\delta_{js}\delta_{kt} - \delta_{jk}\delta_{is}\delta_{lt} + \delta_{jl}\delta_{is}\delta_{kt}
\end{aligned}
$$

The rational solutions of YBE for $SO(N)$ were firstly given by Zamolodchikov's (1972), also rederived by taking the rational limit of the trigonometric R-Matrix:

$$
\breve{R}(u) = f(u)[u^2 P + u(A - I - \frac{3}{2}P)\xi + \frac{3}{2}I\xi^2]
$$

where u stands for spectral parameter and ξ the other free parameter (Cheng, Ge, Xue, 1991; Ge, Xue, 1992). The elements of $\breve{R}(u)$ are $(a,b,c,d=-2,-1,0,1,2)$

$$[\breve{R}(u)]^{ab}_{cd} = u^2\delta_{ab}\delta_{bc} + u(\delta_{a-b}\delta_{c-d} - \delta_{ac}\delta_{bd} - \frac{3}{2}\delta_{ad}\delta_{bc})\xi + \frac{3}{2}\delta_{ac}\delta_{bd}\xi^2$$

For $SO(5)$ we introduce

$$T^{(1)} = \xi\begin{bmatrix} E_3-\frac{3}{2} & U_+ & E_+ & V_+ & 0 \\ U_- & F_3-\frac{3}{2} & F_+ & 0 & -V_+ \\ E_- & F_- & -\frac{3}{2} & -F_+ & -E_+ \\ V_- & 0 & -F_- & -F_3-\frac{3}{2} & -U_+ \\ 0 & -V_- & -E_- & -U_- & -E_3-\frac{3}{2} \end{bmatrix}$$

$$E_3 = E_{22} - E_{-2,-2}, \quad F_3 = E_{11} - E_{-1-1}, \quad U_+ = E_{21} - E_{-1-2}, \quad V_+ = E_{2-1} - E_{1-2}$$
$$E_+ = E_{20} - E_{0,-2}, \quad F_+ = E_{10} - E_{0-1}, \quad U_- = E_{12} - E_{-2-1}, \quad V_- = E_{-12} - E_{-2}$$
$$E_- = E_{02} - E_{-20}, \quad F_- = E_{01} - E_{-10}$$

$$T^{(2)}_{ab} = \frac{3}{2}\xi^2 E^{(2)}_{ab} \qquad (a,b=-2,-1,0,1,2)$$

Substituting $T^{(n)}$ (only $n = 1, 2$ are needed to be considered) into RTT relation there appears 35 relations for J_μ besides the Jocobi indentities. However , a leathy computation shows that besides

$$\begin{array}{l} [I_\alpha, I_\beta] = C^\gamma_{\alpha\beta} I_\gamma \\ [I_\alpha, I_\beta] = C^\gamma_{\alpha\beta} J_\gamma \end{array} \qquad (\alpha = ij)$$

there is only one independent relation

$$[E_3^{(2)}, F_3^{(2)}] = \frac{1}{4!}(\{U_-, E_+, F_-\} - \{U_+, E_-, F_+\} - \{V_+, E_-, F_-\} + \{V_-, E_+, F_+\})$$

where again $\{\ \ \}$ stands for the symmetric summation. A realization of $Y(SO(5))$:

$$I_{ab}(x) = \frac{1}{2}\psi^+_\alpha(x)(I^{ab})_{\alpha\beta}\psi_\beta(x) \qquad (a, b = -2, -1, 0, 1, 2)$$

$$\{\psi^+_\alpha(x), \psi_\beta(y)\}_+ = \delta(x - y)\delta_{\alpha\beta}$$

$$\begin{aligned} I_{ab} &= \sum_x L_{ab}(x) \\ J_{ab} &= \sum_{\substack{x, y \\ c \neq a; b}} \epsilon(x - y) I_{ac}(x) I_{cb}(y) \end{aligned}$$

satisfies the commuting relations for $Y(SO(5))$. The following Hamiltonian of ladder model not only commutes with I_{ab}, i.e. possesses $SO(5)$ symmetry, but also commutes with J_{ab}.

$$
\begin{aligned}
H &= H_1 + \sum_x H_2(x) + \sum_x H_3(x) \\
H_1 &= 2t_1 \sum_{<x,y>} [c_\sigma^+(x) c_\sigma(y) + d_\sigma^+(x) d_\sigma(y) + H.C.] \\
H_2(x) &= U(n_{c\uparrow} - \frac{1}{2})(n_{c\downarrow} - \frac{1}{2}) + (c \to d) + V(n_c - 1)(n_d - 1) + J\mathbf{S}_c \cdot \mathbf{S}_d \\
&= \frac{J}{4} \sum_{a<b} I_{ab}^2 + (\frac{1}{8}J + \frac{1}{2}U)(\psi_\alpha^+ \psi_\alpha - 2) \\
H_3(x) &= -2t_3(c_\sigma^+(x) d_\sigma(x) + H.C.)
\end{aligned}
$$

For $SO(6) \simeq SU(4)$ we introduce (15 generators)

$$
\begin{aligned}
T_{ab}^{(1)} &= I_{ab} \\
T_{ab}^{(2)} &= I_{ab}^{(2)} \\
(a, b &= 1, 2, \ldots, 6.)
\end{aligned}
$$

and the $\check{R}(u)$-matrix reads

$$\check{R}(u) = f(u)[u^2 P + u\xi(A - 2P - I) + 2\xi^2 I]$$

The RTT gives $4 + 4 + 441 + 315 + 225$ more relations. After careful calculations one find (Zhang, Ge, Xue) the independent relations for J_{ab} themselves:

$$\begin{aligned}
[I_{12}^{(2)}, I_{34}^{(2)}] &= \frac{i}{24}(\{I_{23}, I_{16}, I_{46}\} + \{I_{23}, I_{15}, I_{45}\} + \{I_{14}, I_{25}, I_{35}\} + \{I_{14}, I_{26}, I_{36}\} \\
&\quad -\{I_{13}, I_{26}, I_{46}\} - \{I_{13}, I_{25}, I_{45}\} - \{I_{24}, I_{15}, I_{35}\} - \{I_{24}, I_{16}, I_{36}\}) \\
[I_{12}^{(2)}, I_{56}^{(2)}] &= \frac{i}{24}(\{I_{15}, I_{23}, I_{36}\} + \{I_{15}, I_{24}, I_{46}\} + \{I_{26}, I_{13}, I_{35}\} + \{I_{26}, I_{14}, I_{45}\} \\
&\quad -\{I_{25}, I_{13}, I_{36}\} - \{I_{25}, I_{14}, I_{46}\} - \{I_{16}, I_{23}, I_{35}\} - \{I_{16}, I_{24}, I_{45}\}) \\
[I_{34}^{(2)}, I_{56}^{(2)}] &= \frac{i}{24}(\{I_{45}^{(1)}, I_{13}^{(1)}, I_{16}^{(1)}\} + \{I_{45}^{(1)}, I_{23}^{(1)}, I_{26}^{(1)}\} + \{I_{36}^{(1)}, I_{14}^{(1)}, I_{16}^{(1)}\} + \{I_{36}^{(1)}, I_{24}^{(1)}, I_{26}^{(1)}\} \\
&\quad -\{I_{35}^{(1)}, I_{14}^{(1)}, I_{16}^{(1)}\} - \{I_{35}^{(1)}, I_{24}^{(1)}, I_{26}^{(1)}\} - \{I_{46}^{(1)}, I_{13}^{(1)}, I_{16}^{(1)}\} - \{I_{46}^{(1)}, I_{23}^{(1)}, I_{26}^{(1)}\})
\end{aligned}$$

II. Applications of Yangian

The first example was given by Belavin (1992)in deriving the spectrum of nonlinear σ model.

(1)Reduction of $Y(SU(2))$

The simplest realization of $Y(SU(2))$ is made for two-spin system with $\mathbf{S_1}$ and $\mathbf{S_2}$ (any dimensional reps of SU(2)):

$$\mathbf{J}' = \frac{\mathbf{1}}{\mu+\nu}\mathbf{J} = \frac{\mathbf{1}}{\mu+\nu}(\mu\mathbf{S_1} + \nu\mathbf{S_2} + \mathbf{2}\lambda\mathbf{S_1} \times \mathbf{S_2})$$

that contains the (antisymmetric)tensor interaction between $\mathbf{S_1}$ and $\mathbf{S_2}$. For Hydrogen atom $\mathbf{S_1} = \mathbf{L}$ and $\mathbf{S_2} = \mathbf{K}$ (Lung-Lenz vector).

For $S_1 = S_2 = 1/2$, when

$$\mu\nu = \lambda^2$$

we prove that after the similar transformation

$$\mathbf{Y} = A\mathbf{J}'A^{-1}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \nu & i\lambda & 0 \\ 0 & i\lambda & \nu & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the Yangian reduces to SO(4):$(\rho = \nu + i\lambda = \sqrt{\nu^2 + \lambda^2} e^{i\theta})$

$$Y_1 = \begin{bmatrix} M_1 & 0 \\ 0 & L_1 \end{bmatrix}, \ M_1 = \frac{1}{2}\begin{bmatrix} 0 & \rho \\ \rho^{-1} & 0 \end{bmatrix}, \ L_1 = \frac{1}{2}\begin{bmatrix} 0 & \rho^{-1} \\ \rho & 0 \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} M_2 & 0 \\ 0 & L_2 \end{bmatrix}, \ M_2 = \frac{1}{2}\begin{bmatrix} 0 & -i\rho \\ i\rho^{-1} & 0 \end{bmatrix}, \ L_2 = \frac{1}{2}\begin{bmatrix} 0 & -i\rho^{-1} \\ i\rho & 0 \end{bmatrix}$$

$$Y_3 = \begin{bmatrix} \frac{1}{2}\sigma_3 & 0 \\ 0 & \frac{1}{2}\sigma_3 \end{bmatrix}, \ M_3 = \frac{1}{2}\sigma_3$$

and

$$\mathbf{Y}^2 = \frac{1}{2}(\frac{1}{2} + 1) = \frac{3}{4}$$

Namely, under $\mu\nu = \lambda^2$, the $\mathbf{Y}$ reduces to $SO(4)$. By $M_\pm = M_1 \pm iM_2$, $M_+ = \rho\sigma_+$,

$M_- = \rho^{-1}\sigma_-$. The scaled $M_\pm$ and M_3 still satisfy the $SU(2)$ relation:

$$[M_3, M_\pm] = \pm M_\pm$$
$$[M_+, M_-] = 2M_3$$

and the similar relation's for $\mathbf{L}$.

It should be emphasized that here the new "spin" $\mathbf{M}$ (and $\mathbf{L}$) is the consequence of two spin($\frac{1}{2}$) interaction. As usual in Lie algebra

$$\underline{2} \otimes \underline{2} = \underline{3}(spin\ triplet) \oplus \underline{1}(singlet)$$

However, here we meet different decomposition:

$$\underline{2} \otimes \underline{2} = \underline{2}(\mathbf{M}) \oplus \underline{2}(\mathbf{L})$$

The idea can be generalized to $SU(3)$ fundamental rep:

$$J_\lambda = uI_1^\lambda + vI_2^\lambda + \lambda f_{\lambda\mu\nu} \sum_{i<j} F_{1i}^\mu F_{2j}^\nu$$
$$[F_\mu, F_\nu] = if_{\mu\nu\lambda}F_\lambda \qquad (\lambda, \mu, \nu = 1, 2, \cdots, 8)$$

Under the condition

$$uv = \lambda^2 \qquad v + i\lambda = \rho$$

and

$$Y_\mu = AJ_\mu A^{-1}/(u+v)$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & i\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 & 0 & 0 & i\lambda & 0 & 0 \\ 0 & i\lambda & 0 & \nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu & 0 & i\lambda & 0 \\ 0 & 0 & i\lambda & 0 & 0 & 0 & \nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\lambda & 0 & \nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Yangian reduces to

$$Y(I_-) = \begin{bmatrix} \rho^{-1}I_- & 0 & 0 \\ 0 & \rho I_- & 0 \\ 0 & 0 & I_- \end{bmatrix}, \; Y(I_+) = \begin{bmatrix} \rho I_+ & 0 & 0 \\ 0 & \rho^{-1}I_- & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$Y(I_8) = \frac{\sqrt{3}}{3} \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \; Y(I_3) = \frac{1}{2} \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$Y(U_+) = \begin{bmatrix} U_+ & 0 & 0 \\ 0 & \rho U_+ & 0 \\ 0 & 0 & \rho^{-1}U_+ \end{bmatrix}, \; Y(U_-) = \begin{bmatrix} U_- & 0 & 0 \\ 0 & \rho^{-1}U_- & 0 \\ 0 & 0 & \rho U_- \end{bmatrix}$$

$$Y(V_+) = \begin{bmatrix} \rho^{-1}V_- & 0 & 0 \\ 0 & V_- & 0 \\ 0 & 0 & \rho V_- \end{bmatrix}, \; Y(V_-) = \begin{bmatrix} \rho V_- & 0 & 0 \\ 0 & V_- & 0 \\ 0 & 0 & \rho^{-1}V_- \end{bmatrix}$$

The usual decomposition of $\underline{3} \otimes \underline{3} = \underline{6} \oplus \underline{1}$ for $SU(3)$, however, here we have

$$\underline{3} \otimes \underline{3} = \underline{3} \oplus \underline{3} \oplus \underline{3}$$

and

$$\sum_{\lambda=1}^{8} Y_\lambda^2 = \frac{1}{u+v} \sum_{\lambda=1}^{\infty} J_\lambda^2 = \frac{1}{3}$$

It is easy to check that the rescaling factor ρ does not change the commutation relations for $SU(3)$ formed by $I_\pm$, $U_\pm$, $V_\pm$, I_3 and I_8. In general, we guess for the fundamental rep. of $SU(n)$ we shall meet

$$n \otimes n = n \oplus n \oplus n + \cdots + n \quad (n\ times)$$

The Yang-Mills gauge field for reduced $Y(SU(2))$.

For a tensor wave function $(x \equiv \{x_1, x_2, x_3, x_0\})$

$$\Psi(x) = \|\psi_{ij}(x)\| \quad (i, j = 1, 2, 3, 4)$$

An isospin transformation yields

$$\Psi'(x) = U(x)\Psi(x)$$

$$U(x) = 1 - i\theta^a J_a$$

where

$$J^a = uS_a \otimes \mathbf{1} + v\mathbf{1} \otimes S_a + 2\lambda\epsilon_{abc}S^b \otimes S^c$$

or

$$[J_a]^{\alpha\beta}_{\gamma\delta} = u(S^a)_{\alpha\gamma}\delta_{\beta\delta} + v(S^a)_{\beta\delta}\delta_{\alpha\gamma} + i\alpha\varepsilon_{abc}(S^b)_{\alpha\gamma}(S^c)_{\beta\delta}$$

Defining

$$D_\mu = \partial_\mu + gA_\mu$$

i.e.

$$[D_\mu\psi]_{\alpha\beta} = \partial_\mu\psi_{\alpha\beta} + gA^a_\mu[Y_a]^{\alpha\beta}_{\gamma\delta}\psi_{\gamma\delta}(x)$$

$$A_\mu = A^a_\mu J_a$$

The covariant derivative should preserve

$$\delta(D_\mu\psi) = 0$$

i.e.

$$(-i\partial_\mu\theta^a(x) + g\delta A^a_\mu)[Y_a]^{\alpha\beta}_{\gamma\delta} - ig\theta^a(x)A^b_\mu[J_b, J_a]^{\alpha\beta}_{\gamma\delta} = 0$$

When

$$uv = \lambda^2$$

and by rescaling

$$Y_a = (u+v)J_a$$

we have

$$\delta A_\mu^a = \epsilon_{abc}\theta^b(x)A_\mu^c(x) + \frac{i}{g}\partial_\mu\theta^a(x)$$

and

$$F_{\mu\nu} = \frac{1}{g}[D_\mu, D_\nu] = F_{\mu\nu}^a Y_a$$

$$F_{\mu\nu}^a = \partial_\mu A_\gamma^a - \partial_\nu A_\mu^a + ig\epsilon_{abc}A_\mu^b A_\gamma^c$$

Here the tensor isospace has been separated to two irrelevent spaces.i.e. $\Psi = \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix}$ where Ψ_1 and Ψ_2 are 2×2 wavefunction.

(2) Illustrative examples:NMR of Breit-Rabi Hamiltonian and Yangian

$$H = \mathbf{K}\cdot\mathbf{S} + \mu\mathbf{B}\cdot\mathbf{S}$$

where $S=\frac{1}{2}$ and $B=\mathbf{B}(t)$ is magnetic field.

The Hamiltonian can easily be diagonalized for any background angular momentum (or spin) $\mathbf{K}$. The $\mathbf{S}$ stands for spin of electron and for simplicity $\mathbf{K}=\mathbf{S_1}(S_1=1/2)$ is an average background spin contributed by other source,say, control spin. Denoting by

$$H=H_0+H_1(t), \qquad H_0=\alpha\mathbf{S_1}\cdot\mathbf{S_2}, \qquad H_1(t)=\mu\mathbf{B}(t)\cdot\mathbf{S_2}$$

Let us work in the interaction picture:

$$\begin{aligned} H_I &= \mu\mathbf{B}(t)\cdot(e^{i\alpha\mathbf{S_1}\cdot\mathbf{S_2}}\mathbf{S_2}e^{-i\alpha\mathbf{S_1}\cdot\mathbf{S_2}}) \\ &= \mu\mathbf{B}(t)\cdot\mathbf{J} \end{aligned}$$

$$\mathbf{J}=\mu_1\mathbf{S_1}+\mu_2\mathbf{S_2}+2\lambda(\mathbf{S_1}\times\mathbf{S_2})$$

$\mu_1=\frac{1}{2}(1-cos\alpha), \qquad \mu_2=\frac{1}{2}(1+cos\alpha), \qquad \lambda=\frac{1}{2}sin\alpha$

Obviously, here we have $\mu_1\mu_2=\lambda^2$. It is not surprising that the $Y(SU(2))$ reduces to $SO(4)$ here because the transformation is fully Lie-algebraic operation.

For generalization we regard μ_1 and μ_2 as independent parameters,i.e.drop the relation $\mu_1\mu_2 = \lambda^2$.Looking at

$$\mathbf{J} = \mu_1\mathbf{S_1} + \mu_2\mathbf{S_2} - \frac{1}{2}(\mu_1 + \mu_2)(\mathbf{S_1} + \mathbf{S_2}) + \gamma(\mathbf{S_1} + \mathbf{S_2}) + 2\lambda\mathbf{S_1} \times \mathbf{S_2}$$

When $\gamma = \frac{1}{2}$,$\mu_2 - \mu_1 = cos\alpha$ and $\lambda = \frac{1}{2}sin\alpha$ it reduces to the form in the interacting picture.Putting

$$\mathbf{S_1} + \mathbf{S_2} = S$$

$$2\lambda = -\frac{h}{2}(h \quad \text{not} \quad \text{Plank} \quad \text{constant})$$

In accordance with the convention we have

$$\mathbf{J} = \gamma\mathbf{S} + \sum_{i=1}^{2}\mu_i\mathbf{S_i} + \frac{h}{2}\mathbf{S_1} \times \mathbf{S_2} - \frac{1}{2}(\mu_1 + \mu_2)\mathbf{S} = \gamma\mathbf{S} + \mathbf{Y}$$

Since $\mathbf{J} \to \xi\mathbf{S} + \mathbf{J}$ still satisfies Yangian raltions,it is natural to appear the term $\gamma\mathbf{S}$.The interacting Hamiltonian then reads

$$H_I(t) = -\gamma\mathbf{B}(t)\cdot\mathbf{S} - \mathbf{B}(t)\cdot\mathbf{Y}$$

When $\mu_i = 0$, $h = 0$ it is the usual NMR for spin 1/2. To solve the equation, we use

$$i\frac{\partial\Psi(t)}{\partial t} = H_I(t)\Psi(t)$$

$$|\Psi(t)\rangle = \sum_{\alpha=\pm,3;0} a_\alpha(t)|\chi_\alpha\rangle$$

where $\{\chi_\pm, \chi_3\}$ is spin triplet and χ_0 singlet.

Setting

$B_\pm(t) = B_1(t) \pm iB_2(t)$ and $B_3 = \text{const}$

$B_\pm(t) = B_1 e^{\mp i\omega_0 t}$

and rescaling by

$a_\pm(t) = e^{\pm i\omega_0 t} b_{\pm(}t)$

then we get

$$i\frac{db_\pm(t)}{dt} = -\gamma\{\frac{1}{\sqrt{2}}B_1 a_3(t) \mp (\omega_0\gamma^{-1} - B_3)b_\pm(t)\} \pm \frac{1}{2\sqrt{2}}\mu_- B_1 a_0(t)$$

$$i\frac{da_3(t)}{dt} = -\frac{\gamma B_1}{\sqrt{2}}\{b_+(t) + b_-(t)\} - \frac{1}{2}\mu_- B_3 a_0(t)$$

$$i\frac{da_0(t)}{dt} = -\frac{1}{2}\mu_+\{\frac{1}{\sqrt{2}}B_1[b_-(t) - b_+(t)]\} + B_3a_3(t)$$

$$\mu_\pm = (\mu_1 - \mu_2 \pm i\frac{h}{2})$$

i.e.

$$|\Phi(t)\rangle = \begin{bmatrix} b_1(t) \\ a_3(t) \\ b_-(t) \\ a_0(t) \end{bmatrix}, \mathcal{H}_I = \begin{bmatrix} \omega_0 - \gamma B_1 & -\gamma B_1\frac{1}{\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}}\mu_- B_1 \\ -\gamma B_1\frac{1}{\sqrt{2}} & 0 & -\gamma B_1\frac{1}{\sqrt{2}} & -\frac{1}{2}\mu_- B_3 \\ 0 & -\gamma B_1\frac{1}{\sqrt{2}} & -(\omega_0 - \gamma B_1) & -\frac{1}{2\sqrt{2}}\mu_- B_1 \\ \frac{1}{2\sqrt{2}}\mu_+ B_1 & -\frac{1}{2}\mu_+ B_3 & -\frac{1}{2\sqrt{2}}\mu_+ B_1 & 0 \end{bmatrix}$$

$i\frac{d|\Phi(t)\rangle}{dt} = H_I|\Phi(t)\rangle$

Noting that $\mathcal{H}_\mathcal{I}$ is independent of time we get

$$|\Phi(t)\rangle = e^{-iEt}|\Phi(t)\rangle,$$

Then

$$\det|H_I - E| = 0$$

leads to

$$E^4 - [(\omega_1 - \gamma B_3)^2 + \gamma^2 B_1^2 + \frac{1}{4}\mu_+\mu_-(B_1^2 + B_3^2)]E^2 +$$

$$\frac{1}{4}\mu_+\mu_-[B_3^2(\omega_0-\gamma B_3)^2-2\gamma B_3 B_1^2(\omega_0-\gamma B_3)+\gamma^2 B_1^4]=0$$

There is transition between the spin singlet and triplet in the NMR process, i.e. the Yangian transferes the quantum information through the evolution. The simplest case is $B_1 = 0$ then eigenvalues are

$$E=\pm(\omega_0-\gamma B_3), E=\pm\omega=\pm\frac{B_3}{2}\sqrt{(\mu_1-\mu_2)^2+\frac{h^2}{4}}$$

It turns out that there is vabration between s=0 and s=1.

$$< s^2 >= 0 \ \texttt{at} \ t=\frac{\pi}{2\omega} \ \texttt{(total spin=0)}$$
$$< s^2 >= 2 \ \texttt{at} \ t=\frac{\pi}{\omega} \ \texttt{(total spin=1)}$$

Under adiabatic approximation it can be proved that it appears Berry's phase, even there is witness of spin singlet which takes part in the transition process.

(3) Transition between S-wave and P-wave superconductivity

$$S: \quad \texttt{spin singlet,} \quad L = 0$$

$$P: \quad \texttt{spin triplet,} \quad L = 1$$

Balian-Werthamer (1963):

$$\triangle(\mathbf{k}) = -\frac{1}{2}\sum_{\mathbf{k}'} V(\mathbf{k},\mathbf{k}')\frac{\triangle(\mathbf{k}')}{E(\mathbf{k}')}\tanh\frac{\beta}{2}E(\mathbf{k}')$$
$$E(\mathbf{k}) = (\epsilon^2(k) + |\triangle(\mathbf{k})|^2)^{\frac{1}{2}}$$

B-W:

$$\triangle(\mathbf{k}) = \triangle(k)(\frac{4\pi}{3})^{\frac{1}{2}} \begin{bmatrix} \sqrt{2}Y_{1,1}(\hat{\mathbf{k}}) & Y_{1,0}(\hat{\mathbf{K}}) \\ Y_{1,0}(\hat{\mathbf{k}}) & \sqrt{2}Y_{1,-1}(\hat{\mathbf{k}}) \end{bmatrix}^* = (-\sqrt{6})\triangle(k)(\frac{4\pi}{3})^{\frac{1}{2}}\Phi_{0,0}(\hat{\mathbf{k}})$$

$$\Phi_{0,0}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{3}}\{Y_{1,-1}(\hat{\mathbf{k}})\chi_{11} - Y_{1,0}(\hat{\mathbf{k}})\chi_{10} + Y_{1,1}(\hat{\mathbf{k}})\chi_{1-1}\} = \frac{1}{\sqrt{8}}\begin{bmatrix} \hat{\mathbf{k}}_- & -\hat{\mathbf{k}}_z \\ -\hat{\mathbf{k}}_z & -\hat{\mathbf{k}}_+ \end{bmatrix}$$

where χ_{11},χ_{10} and χ_{1-1} stand for spin triplet.

$$\Phi_{0,0} \equiv \Phi_{J=0,m=0}$$

The wave function of SC is

$$\phi_{0,0} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & Y_{0,0} \\ -Y_{0,0} & 0 \end{bmatrix}$$

Introducing

$$\begin{aligned} I_\mu &= \sum_{i=1}^{2} S_\mu(i); \qquad (\mu = 1,2,3) \\ J_\mu &= \sum_{i=1}^{2} \lambda_i S_\mu(i) - \frac{ihv}{4}\epsilon_{\mu\lambda\nu}(S^\lambda(1)S^\nu(2) - S^\lambda(2)S^\nu(1)) \end{aligned}$$

and noting that $J_\mu \to J_\mu + fI_\mu$ does not change the Yangian relations, we choose for simplicity $f = -\frac{1}{2}(\lambda_1 + \lambda_2)$. We obtain

$$\begin{aligned} G\phi_{0,0} &= \hat{\mathbf{k}} \cdot (\mathbf{J} + f\mathbf{I})\phi_{0,0} = \frac{\sqrt{3}}{2}(\lambda_2 - \lambda_1 + \frac{hv}{2})\Phi_{0,0}, \\ G\Phi_{0,0} &= \hat{\mathbf{k}} \cdot (\mathbf{J} + f\mathbf{I})\Phi_{0,0} = \frac{1}{2\sqrt{3}}(\lambda_2 - \lambda_1 - \frac{hv}{2})\phi_{0,0}. \end{aligned}$$

The transition direction depends on the parameters in $Y(SU(2))$. For instance,

$$\begin{aligned} SC \rightarrow PC: \quad G\phi_{0,0} &= \frac{\sqrt{3}}{2}\Phi_{0,0} \quad if \;\; \lambda_1 - \lambda_2 = -\frac{hv}{2} \\ G\Phi_{0,0} &= 0 \end{aligned}$$

and

$$\begin{aligned} PC \rightarrow SC: \quad G\phi_{0,0} &= 0 \\ G\Phi_{0,0} &= -\frac{hv}{2\sqrt{3}}\phi_{0,0} \quad if \;\; \lambda_1 - \lambda_2 = \frac{hv}{2} \end{aligned}$$

We call the type of the transition “directional transition”. The controlled parameters are in the Yangian operation.

We have got used to apply electromagnetic field A_μ to make transitions between l and $l \pm 1$. Now there is Yangian formed by two spins that plays the role changing angular momentum states.

(4) $Y(SU(3))$**-directional transitions**

$$
\begin{aligned}
F_\mu &= \frac{1}{2}\lambda_\mu, \ [F_\lambda, F_\mu] = i f_{\lambda\mu\nu} F_\nu \\
I_\mu &= \sum_i F_i^\nu \\
J_\mu &= \sum_i \mu_i F_i^\mu - i h f_{\mu\nu\lambda} \sum_{i \neq j} w_{ij} F_i^\nu F_j^\lambda, (w_{ij} = -w_{ji})
\end{aligned}
$$

$$[F_i^\lambda, F_j^\mu] = i f_{\lambda\mu\nu} \delta_{ij} F_i^\nu,$$

where F_μ are fundamental rep. of $SU(3)$ and $(i, j, k = 1, 2, ..., 8)$.

$$\triangle_{ijk} = w_{ij} w_{jk} + w_{jk} w_{ki} + w_{ki} w_{ij} = -1$$

(`no summation over repeated indices,` $i \neq j \neq k$)

The reason that such a condition works only for 3-dimensional representation of $SU(3)$ is similar to Haldane's (long-ranged) realization of $Y(SU(2))$. In $SU(2)$ long-ranged form the property of Pauli matrices leads to $(\sigma^{\pm})^2 = 0$. Instead, for $SU(3)$ the conditions of J_μ satisfying $Y(SU(3))$ read

$$\sum_{i\neq j}(1-w_{ij}^2)(I_j^+V_i^+U_i^+ - U_i^-V_i^-I_j^- + I_i^+V_j^+U_i^+ - U_i^-V_j^-I_i^- + I_j^+V_j^+U_i^+ - U_i^-V_j^-I_j^-) = 0$$

and

$$\sum_i (I_i^+V_i^+U_i^+ - U_i^-V_i^-I_i^-) = 0$$

that are satisfied for Gell-Mann matrices.

The simplest realization of $Y(SU(3))$ is then

$$W_{ij} = \begin{cases} 1 & i > j \\ 0 & i = j \\ -1 & i < j \end{cases} \qquad (W_{ij} = -W_{ji})$$

Recalling($I_8 = \frac{\sqrt{3}}{2}Y$)

$$I^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; U^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; V^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$I^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; Y = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

We find

$$\begin{aligned}
J_\mu &= \{\bar{I}_\pm, \bar{U}_\pm, \bar{V}_\pm, \bar{I}_3, \bar{I}_8\} \\
\bar{I}_\pm &= \sum_i \mu_i I_i^\pm \mp 2h \sum_{i\neq j} W_{ij}(I_i^\pm I_j^3 - \frac{1}{2}U_i^\mp V_j^\mp) \\
\bar{U}_\pm &= \sum_i \mu_i U_i^\pm \pm h \sum_{i\neq j} W_{ij}[U_i^\pm (I_j^3 - \frac{3}{2}Y_j) + I_i^\mp V_j^\mp] \\
\bar{V}_\pm &= \sum_i \mu_i V_i^\pm \pm h \sum_{i\neq j} W_{ij}[V_i^\pm (I_j^3 + \frac{3}{2}Y_j) + U_i^\mp I_j^\mp] \\
\bar{I}_3 &= \sum_i \mu_i I_i^3 + h \sum_{i\neq j} W_{ij}[I_i^+ I_j^- - \frac{1}{2}(U_i^+ U_j^- + V_i^+ V_j^-)] \\
\bar{I}_8 &= \sum_i \mu_i Y_i + h \sum_{i\neq j} W_{ij}(U_i^+ U_j^- - V_j^+ V_j^-)
\end{aligned}$$

where μ_i and h(not Planck constant) are arbitrary parameters

When $i = 1, 2$ $Y(SU(2))$ makes transition between spin singlet and triplet. Now

$Y(SU(3))$ transits $SU(3)$ singlet and Octet. For instance for

$$\begin{aligned} |\ \pi^-\rangle &= |d\bar{u}\rangle, \quad |\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) \\ |K^-\rangle &= |d\bar{u}\rangle, \quad |K^0\rangle = |d\bar{s}\rangle \\ |\ \eta^0\rangle &= \frac{1}{\sqrt{(6)}}(-|u\bar{u}\rangle - |d\bar{d}\rangle + 2|s\bar{s}\rangle) \\ |\ \eta^{0'}\rangle &= \frac{1}{\sqrt{(3)}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) \end{aligned}$$

$$\overline{I}_-|\pi^+ >= \frac{1}{\sqrt{6}}(\mu_1 - \mu_2)|\eta^0 > +\frac{1}{\sqrt{2}}(\mu_1 + \mu_2)|\pi^0 > -\frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'} >$$

$$\overline{U}_+|\overline{K}^0 >= \frac{1}{\sqrt{6}}(\mu_1 + 2\mu_2)|\eta^0 > +\frac{1}{\sqrt{2}}\mu_1|\pi^0 > -\frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'} >$$

$$\overline{U}_-|K^0 >= \frac{1}{\sqrt{6}}(2\mu_1 + \mu_2)|\eta^0 > +\frac{1}{\sqrt{2}}\mu_2|\pi^0 > +\frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'} >$$

$$\overline{V}_+|K^+ >= \frac{1}{\sqrt{6}}(2\mu_1 + \mu_2)|\eta^0 > -\frac{1}{\sqrt{2}}\mu_2|\pi^0 > +\frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'} >$$

$$\overline{V}_-|K^- >= -\frac{1}{\sqrt{6}}(\mu_1 + 2\mu_2)|\eta^0 > +\frac{1}{\sqrt{2}}\mu_1|\pi^0 > +\frac{1}{\sqrt{3}}(\mu_1 - \mu_2 + 3h)|\eta^{0'} >$$

$$\overline{I}_3|\pi^0 >= -\frac{1}{2\sqrt{3}}(\mu_1 - \mu_2)|\eta^0 > +\frac{1}{\sqrt{6}}(\mu_1 - \mu_2 + 3h)|\eta^{0'} >$$

$$\overline{I}_8|\eta^0 >= -\frac{1}{3}(\mu_1 - \mu_2)|\eta^0 > -\frac{\sqrt{2}}{3}(\mu_1 - \mu_2 + 3h)|\eta^{0'} >$$

Special interest is the following. When

$$\mu_1 - \mu_2 = -3h, \ f = -\frac{1}{2}(\mu_1 - \mu_2)$$

we obtain

$$(\overline{I}_\pm + fI_\pm)|\eta^{0'} >= \pm 2\sqrt{3}h|\pi^\pm >, \ (\overline{U}_+ + fU_+)|\eta^{0'} >= -2\sqrt{3}h|K^0 >$$

$$(\overline{U}_- + fU_-)|\eta^{0'} >= 2\sqrt{3}h|\overline{K}^0 >, \ (\overline{V}_\pm + fV_\pm)|\eta^{0'} >= -2\sqrt{3}h|K^\mp >$$

$$(\overline{I}_3 + fI_3)|\eta^{0'} >= -\sqrt{6}h|\pi^0 >, \ (\overline{I}_8 + fI_8)|\eta^{0'} >= 2\sqrt{2}h|\eta^0 >$$

and

$$(\overline{I}_\pm + fI_\pm)|\pi^\mp >= \pm\sqrt{\frac{3}{2}}h|\eta^0 >,\ (\overline{U}_+ + fU_+)|K^0 >= -\frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0 > -|\eta^0 >)$$

$$(\overline{U}_- + fU_-)|K^0 >= \frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0 > -|\eta^0 >),$$

$$(\overline{V}_\pm + fV_\pm)|K^\pm >= -\frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0 > +|\eta^0 >)$$

$$(\overline{I}_3 + fI_3)|\pi^0 >= \sqrt{\frac{3}{2}}h|\eta^0 >,\ (\overline{I}_8 + fI_8)|\eta^0 >= \sqrt{3}h|\eta^0 >$$

If

$$\mu_1 - \mu_2 = 3h,\ f = -\frac{1}{2}(\mu_1 + \mu_2)$$

$$(\overline{A}^{(2)} + fA^{(1)})|\eta^{0'} >= 0,\ A = I_\alpha,\ (\alpha = \pm, 3, 8),\ U_\pm,\ V_\pm$$

and

$$(\overline{I}_\pm + fI_\pm)|\pi^\mp > \ = \ \mp\sqrt{\frac{3}{2}}h|\eta^0 > \pm 2\sqrt{3}h|\eta^{0'} >,$$

$$
\begin{aligned}
(\overline{U}_+ + fU_+)|\overline{K}^0> &= \frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0> - |\eta^0>) - 2\sqrt{3}h|\eta^{0'}>,\\
(\overline{U}_- + fU_-)|K^0> &= -\frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0> - |\eta^0>) + 2\sqrt{3}h|\eta^{0'}>,\\
(\overline{V}_\pm + fV_\pm)|K^\pm> &= \frac{\sqrt{3}}{2\sqrt{2}}h(\sqrt{3}|\pi^0> + |\eta^0>) + 2\sqrt{3}h|\eta^{0'}>,\\
(\bar{I}_3 + fI_3)|\pi^0> &= -\frac{\sqrt{3}}{2}h|\eta^0> + \sqrt{6}h|\eta^{0'}>,\\
(\bar{I}_8 + fI_8)|\eta^0> &= h|\eta^0> - 2\sqrt{2}h|\eta^{0'}>
\end{aligned}
$$

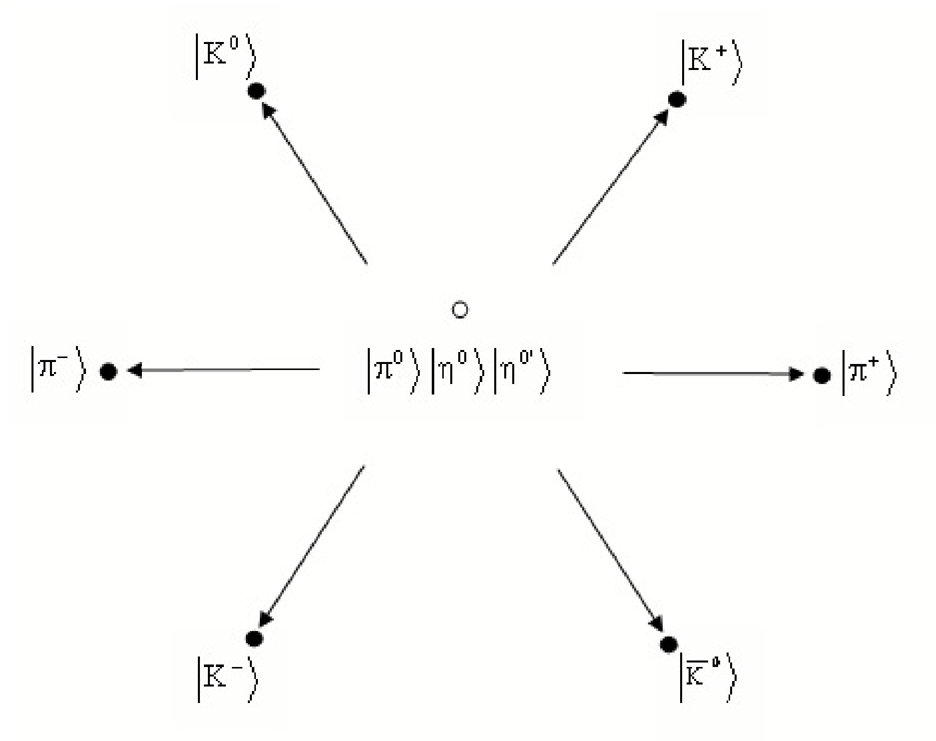

Figure 1: representation of $SU(3)$

(5) $\mathbf{J}^2$ as a new quantum number

Because $[\mathbf{I}^2, \mathbf{J}^2] = 0$, $[\mathbf{I}^2, I_z] = 0, [\mathbf{J}^2, I_z] = 0$, but $[\mathbf{J}^2, J_z] \neq 0$, we can take $\{\mathbf{I}^2, I_z, \mathbf{J}^2\}$ as a conserved set.

Example. $\mathbf{S}_1 \bigotimes \mathbf{S}_2 \bigotimes \mathbf{S}_3 \quad (S_1 = S_2 = S_3 = \frac{1}{2})$

We shall show that instead of 6-j coefficients and Young diagrams, $\mathbf{J}^2$ can be viewed as a "collective" quantum number that describes the "history" besides $S(\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3)$ and S_z

$$(\tfrac{1}{2} \bigotimes \tfrac{1}{2}) \bigotimes \tfrac{1}{2} = (1 \bigoplus 0) \bigotimes \tfrac{1}{2} = \tfrac{3}{2} \bigoplus \tfrac{1}{2} \bigoplus \tfrac{1}{2}'$$

Noting that $|\frac{1}{2}\rangle$ and $|\frac{1}{2}'\rangle$ are degenerate regarding the total spin $\frac{1}{2}$. The usual Lie algebraic base can be easily written as

$$\begin{aligned}
&\phi_{\frac{3}{2},\frac{3}{2}} = |\uparrow\uparrow\uparrow\rangle \\
&\phi_{\frac{3}{2},\frac{1}{2}} = \tfrac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) \\
&\phi_{\frac{3}{2},-\frac{1}{2}} = \tfrac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \\
&\phi_{\frac{3}{2},-\frac{3}{2}} = |\downarrow\downarrow\downarrow\rangle
\end{aligned}$$

and the two degeneracy states to $\mathbf{S}^2$ and S_z:

$\phi'_{\frac{1}{2},\frac{1}{2}} = \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle)$
$\phi'_{\frac{1}{2},-\frac{1}{2}} = \frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle - 2|\downarrow\downarrow\uparrow\rangle)$
$\phi_{\frac{1}{2},\frac{1}{2}} = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - \uparrow\downarrow\uparrow\rangle)$
$\phi_{\frac{1}{2},-\frac{1}{2}} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle)$

To distinguish ϕ' from ϕ we introduce $\mathbf{J}$:

$$\mathbf{J} = \sum_{i=1}^{3} u_i \mathbf{S}_i + ih \sum_{i<j}^{3} (\mathbf{S}_i \times \mathbf{S}_j)$$

and calculate $\mathbf{J}^2$. It turns out that

$$\begin{aligned}
\mathbf{J}^2\phi_{\frac{3}{2},m} &= [\tfrac{3}{4}(u_1^2 + u_2^2 + u_3^2) + \tfrac{1}{2}(u_1u_2 + u_2u_3 + u_1u_3) - h^2]\Phi_{\frac{3}{2},m} \\
\mathbf{J}^2\phi'_{\frac{1}{2},m} &= [\tfrac{3}{4}(u_1^2 + u_2^2 + u_3^2) + \tfrac{1}{2}u_1u_2 - u_2u_3 - u_1u_3 - \tfrac{7}{4}h^2]\Phi'_{\frac{1}{2},m} \\
&\quad -\tfrac{\sqrt{3}}{2}(u_1 - u_2 + h)(u_3 + h)\Phi_{\frac{1}{2},m} \\
\mathbf{J}^2\phi_{\frac{1}{2},m} &= -\tfrac{\sqrt{3}}{2}(u_1 - u_2 - h)(u_3 - h)\Phi'_{\frac{1}{2},m} + [\tfrac{3}{4}(u_1 - u_2)^2 + \tfrac{3}{4}u_3^2 - \tfrac{3}{4}h^2]\Phi_{\frac{1}{2},m}
\end{aligned}$$

In order to make the matrix of $\mathbf{J}^2$ symmetric, one should put

$$u_2 = u_1 + u_3$$

The eigenvalues of $\mathbf{J}^2$ are given by

$$\lambda_{\frac{3}{2}} = 2u_1^2 + 2u_3^2 + 3u_1u_3 - h^2$$
$$\lambda_{\frac{1}{2}}^{\pm} = u_1^2 + u_3^2 - \tfrac{5}{4}h^2 \pm \tfrac{1}{2}[(2u_1^2 - u_3^2 - h^2)^2 + 3(u_3^2 - h^2)^2]^{\frac{1}{2}}$$

The eigenstates of $\mathbf{J}^2$ are the rotation of $\phi'_{\frac{1}{2},m}$ and $\Phi_{\frac{1}{2},m}$:

$$\begin{pmatrix} \alpha^{+}_{\frac{1}{2},m} \\ \alpha^{-}_{\frac{1}{2},m} \end{pmatrix} = \begin{pmatrix} \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} \phi'_{\frac{1}{2},m} \\ \phi_{\frac{1}{2},m} \end{pmatrix}, \quad \mathbf{J}^2\alpha^{\pm}_{\frac{1}{2}} = \lambda^{\pm}_{\frac{1}{2}}\alpha^{\pm}_{\frac{1}{2},m}$$
$$\sin\varphi = \sqrt{3}(u_3^2 - h^2)/\omega$$
$$\omega^2 = (2u_1^2 - u_3^2 - h^2)^2 + 3(u_3^2 - h^2)^2$$

It is worth noting that the conclusion is independent of the order, say, $(\frac{1}{2} \otimes \frac{1}{2}) \otimes \frac{1}{2}$, $\frac{1}{2} \otimes (\frac{1}{2} \otimes \frac{1}{2})$ and the other way. The difference is only in the value of φ.

The above example can be generalized to $\mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{l}$ where $S_1 = S_2 = \frac{1}{2}$.

$$(\frac{1}{2} \bigotimes \frac{1}{2}) \bigotimes l = (1 \bigoplus 0) \bigotimes l = l+1 \qquad l \qquad l-1$$
$$l$$

There are no degeneracy for $l \pm 1$, but two l states can be distinguished in terms of $\mathbf{J}^2$.

$$
\begin{aligned}
\mathbf{J}^2\Phi_{l+1,m} &= \{\frac{3}{4}(u_1^2+u_2^2)+l(l+1)u_3^2+\frac{1}{2}u_1u_2+l(u_2u_3+u_1u_3) \\
&\quad -h^2[l(l+1)+\frac{1}{4}]\}\Phi_{l+1,m} \\
\mathbf{J}^2\Phi_{l-1,m} &= \{\frac{3}{4}(u_1^2+u_2^2)+l(l+1)u_3^2+\frac{1}{2}u_1u_2-(l+1)u_1u_3-(l+1)u_2u_3 \\
&\quad -h^2[l(l+1)+\frac{1}{4}]\}\Phi_{l-1,m} \\
\mathbf{J}^2\Phi^1_{l,m} &= \{\frac{3}{4}(u_1^2+u_2^2)+l(l+1)u_3^2+\frac{1}{2}u_1u_2-u_2u_3-u_1u_3-2h^2[l(l+1)\frac{1}{8}]\Phi^1_{l,m} \\
&\quad -\sqrt{l(l+1)}(u_1-u_2+h)(u_3+h)\Phi^2_{l,m} \\
\mathbf{J}^2\Phi^2_{l,m} &= -\sqrt{l(l+1)}(u_1-u_2-h)(u_3-h)\Phi^1_{l,m}+[\frac{3}{4}(u_1-u_2)^2+l(l+1)u_3^2-\frac{3}{4}]\Phi^2_{l,m}
\end{aligned}
$$

Again in order to guarantee the symmetric form of the matrix we put

$$u_2 = u_1 + u_3$$

then the eigenvalues and eigenstates of $\mathbf{J}^2$ are given by

$$\lambda_l^{\pm} = u_1^2 + [l(l+1) + \frac{1}{4}]u_3^2 - h^2[l(l+1) + \frac{1}{2}] \pm \frac{1}{2}\sqrt{P}$$

$$\omega^2 = P = [2u_1^2 - u_3^2 - h^2(2l(l+1) - \frac{1}{2})]^2 + 4l(l+1)(u_3^2 - h^2)^2$$

$$\sin\varphi = \frac{2\sqrt{l(l+1)}}{\omega}(u_3^2 - h^2)$$

$$\begin{pmatrix} \alpha_{l,m}^{+} \\ \alpha_{l,m}^{-} \end{pmatrix} = \begin{pmatrix} \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} \Phi_{l,m}^{1} \\ \Phi_{l,m}^{2} \end{pmatrix}$$

Example: Spin structure of rare gas

$$H = -a\mathbf{l}\cdot\mathbf{S}_1 - b\mathbf{S}_1\cdot\mathbf{S}_2 \qquad (\lambda = \frac{b}{a})$$

It describes the interaction of spin $\mathbf{S}_1$ of an electron exited from l-shell and the left hole $\mathbf{S}_2$.

$$H\Phi_{l+1,m} = -\frac{1}{2}(al + \frac{1}{2}b)\Phi_{l+1,m}$$

$$
\begin{aligned}
H\Phi_{l-1,m} &= \frac{1}{2}[(l+1)a - \frac{1}{2}b]\Phi_{l-1,m} \\
H\begin{bmatrix} \Phi^{\pm}_{l,m} \\ \Phi^{2}_{l,m} \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} (a-\frac{1}{2}b) & a\sqrt{l(l+1)} \\ a\sqrt{l(l+1)} & \frac{3}{2}b \end{bmatrix}\begin{bmatrix} \Phi^{1}_{l,m} \\ \Phi^{2}_{l,m} \end{bmatrix}
\end{aligned}
$$

The eigenstates of H

$$
\begin{pmatrix} \alpha^{+}_{l,m} \\ \alpha^{-}_{l,m} \end{pmatrix} = \begin{pmatrix} \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}\begin{pmatrix} \Phi^{1}_{l,m} \\ \Phi^{2}_{l,m} \end{pmatrix}
$$

where

$$
\sin\varphi = \frac{\sqrt{l(l+1)}}{\omega}, \omega^2 = (\frac{1}{2} - \lambda)^2 + l(l+1), \lambda = \frac{b}{a}.
$$

The eighenvalues are

$$
\begin{aligned}
\lambda_{l+1} &= -\frac{1}{2}(la + \frac{b}{2}), \quad \lambda_{l-1} = \frac{1}{2}[(l+1)a - \frac{b}{2}] \\
\lambda^{\pm}_{l} &= \frac{1}{4}(a+b) \pm \frac{1}{2}[l(l+1)a^2 + (\frac{a}{2} - b)^2]^{\frac{1}{2}}
\end{aligned}
$$

The rotation comes from the fact

$$
[H, \mathbf{J}^2] = 0
$$

that is satisfied for the matrix of $\mathbf{J}^2$ being symmetric, i.e.

$$\begin{aligned} \gamma &= \frac{\{2u_1^2 - 2h^2[l(l+1)+\frac{1}{4}]\}}{(u_3^2 - h^2)} \\ &= 2(1-\lambda) \end{aligned}$$

Therefore, the parameter γ in $Y(SU(2))$ determines the rotation angle φ. It is reasonable to think that the appearence of "rotation" of degenerate states is closely related to the "quantum number" of $\mathbf{J}^2$. Transition between $\alpha^+_{l,m}$ and $\alpha^-_{l,m}$ $(l=1)$ can be made by J_3. Because there are two independent parameters u_1 and u_3 in $\mathbf{J}$, one can choose a suitable relation between u_3 and $\lambda = \frac{b}{a}$ such that

$$J_3\alpha_1^+ \sim \alpha^-$$

i.e. the transition between two degenerate states in Lie-algebra is made trough J_3 operator. This is because of

$$[\mathbf{J}^2, J_3] \neq 0$$

(6) Happer degeneracy

In the experiment for $^{87}R_b$ molecular there appears new degeneracy (Happer etal. 2002) at the special $\pm B_0$ (magnetic field), i.e. the Zeeman effect disappears at $\pm B_0$. The model Hamiltonian reads

$$H = \mathbf{K} \cdot \mathbf{S} + x(k + \frac{1}{2})S_z$$

where $\mathbf{K}$ is angular momentum and $\mathbf{K}^2 = K(K+1)$. The spin $s = 1$ and x is scaled magnetic field. It turns out that when

$$x = \pm 1, \qquad E = -\frac{1}{2}.$$

The conserved set is $\{\mathbf{K}^2, G_z = K_z + S_z\}$. For $\mathbf{G} = \mathbf{K} + \mathbf{S}$ we have $G = k \pm 1, k$. The eighenstates are specified in terms of three families: T, B and D. Only D-set possesses the degeneracy.

Happer gives, for emple,the eigenstates for $x = \pm 1$:

$$\begin{array}{ll} x = +1 & H\alpha_{DM} = (-\frac{1}{2})\alpha_{DM} \\ x = -1 & H\beta_{DM} = (-\frac{1}{2})\beta_{Dm} \end{array}$$

and shows that

$$\alpha_{Dm} = [2(K+\frac{1}{2})(K+m+\frac{1}{2})]^{-\frac{1}{2}}\{-[\frac{(K-m+1)(K+m+1)}{2}]^{\frac{1}{2}}\alpha_1 + [(K+m)(K+m+1)]^{\frac{1}{2}}\alpha_2 + [\frac{(K-m)(K+m)}{2}]^{\frac{1}{2}}\alpha_3\}$$

$$\beta_{Dm} = [2(K+\frac{1}{2})(K-m+\frac{1}{2})]^{-\frac{1}{2}}\{[\frac{(K-m)(K+m)}{2}]^{\frac{1}{2}}\alpha_1 + [(K-m)(K-m+1)]^{\frac{1}{2}}\alpha_2 - [\frac{(K-m+1)(K+m+1)}{2}]^{\frac{1}{2}}\alpha_3\}$$

where $\alpha_1 = e_1 \otimes e_{m-1}$, $\alpha_2 = e_0 \otimes e_m$ and $\alpha_3 = e_{-1} \otimes e_{m+1}$.

Question: what is the transition operator between α_{DM} and β_{DM} ?

The answer is Yangian.

Introducing

$$J_{\pm} = aS_{+} + bK_{-} \pm (s_{\pm}K_z - s_z K_{\pm})$$

we find

by choosing $a = -\frac{k+1}{2}, b = 0$ $\quad \beta_{Dm} \xrightarrow{J_+} \lambda_1(m)\alpha_{Dm+1}$

and $\quad \alpha_{Dm} \xrightarrow{J_-} \lambda_2(m)\beta_{Dm-1}$

$$\text{by choosing} \qquad a = \frac{k}{2}, b = 0 \qquad \beta_{Dm} \xrightarrow{J_-} \lambda_1'(m)\alpha_{Dm-1}$$
$$\alpha_{Dm} \xrightarrow{J_+} \lambda_2'(m)\beta_{Dm+1}$$

The Yangian introduced here is only for $S = 1$, because for $S = 1$ there are two independent coefficients in the combination of α_1, α_2 and α_3 and there are two free parameters in $\mathbf{J}$. Hence the number of equations are equal to those of free parameters (a and b), so we have solution. The numerical computaion shows that only $s = 1$ gives rise to the new degeneracy that prefers the Yangian operation.

(7) New degeneracy of extended Breit-Rabi Hamiltonian

As was shown in the Happer's model ($H = \mathbf{K} \cdot \mathbf{S} + x(k + \frac{1}{2})S_3$) there appeared new degeneracy for $S = 1$. It has been pointed out that the Zeeman effect cannot appear for spin=$\frac{1}{2}$. Actually, in this case it yields for $S = \frac{1}{2}$

$$E = -\frac{1}{4} - \omega_m S_3$$

where

$$\omega_m^2 = [(1 + x^2)(k + \frac{1}{2}) + 2xm](k + \frac{1}{2}).$$

Therefore the Happer's type of degeneracy can only occur at $\omega_m = 0$ that means

$$x_0 = -\frac{m}{K+1/2}) \pm i\sqrt{1-\frac{m^2}{k^2}} \ \ (k = K + \frac{1}{2})$$

i.e. the magnetic field should be complex.

However, the situation will be completely different, if a third spin is involved. For simplicity we assume $S_1 = S_2 = S_3 = \frac{1}{2}$ in the Hamiltonian:

$$H = -(a\mathbf{S}_2 + b\mathbf{S}_3)\cdot\mathbf{S}_1 + x\sqrt{ab}S_1^z, \lambda = b/a$$

then besides two non-degenerate states, there appears the degenerate family:

$$H\alpha^{\pm}_{D,\pm\frac{1}{2}} = -(\frac{a+b}{4})\alpha^{\pm}_{D,\pm\frac{1}{2}}, \ \text{ for } \ x = \pm 1,$$

where

$$\alpha^{\pm}_{D,+\frac{1}{2}} = -\sqrt{2}\lambda|\uparrow\uparrow\downarrow> \pm\sqrt{\lambda}|\uparrow\downarrow\uparrow +(1\pm\sqrt{\lambda})|\downarrow\uparrow\uparrow>;$$
$$\alpha^{\pm}_{D,-\frac{1}{2}} = -\sqrt{2}\lambda|\downarrow\downarrow\uparrow> \mp\sqrt{\lambda}|\downarrow\uparrow\downarrow +(1\mp\sqrt{\lambda})|\uparrow\downarrow\downarrow> .$$

The expaction value of S_1^z are

$$<\alpha^{+}_{D,\pm\frac{1}{2}}|S_1^z|\alpha^{+}_{D,\pm\frac{1}{2}}>\sim\sqrt{\lambda}\ (x=1)$$

$$< \alpha^-_{D,\pm\frac{1}{2}} | S^z_1 | \alpha^-_{D,\pm\frac{1}{2}} > \sim -\sqrt{\lambda}\ (x = -1)$$

namely, at the special magnetic field ($x = \pm 1$) the observed $< S^z_1 >$ still opposite to each other for $x = \pm 1$, but without Zeeman split.

The reason of the appearance of the new degeneracy is obvious. The two spins $\mathbf{S}_2$ and $\mathbf{S}_3$ here play the role of $S = 1$ in comparison with Happer model.

(8) Super $YM(n = 4)$-Lipatov model and $Y(SO(6))$.

Beisert et al(2002), Dolan-Nappi-Witten, (DNW) $\cdots$ proposed to take the quantum correction of the delitation operator δD ($D \in SO(4,2)$ as Hamiltonian for supper $YM(N = 4)$:

$$H = \sum_\alpha H_{\alpha\alpha+1}$$

$$H_{\alpha\alpha+1} = 2 \sum_j h(j) P^j_{\alpha\alpha+1}, \quad h(j) = \sum_{k=1}^{j} \frac{1}{k}, h(0) = 1.$$

where P^j is projector for the weight j of $SU(2)$ and α stands for "lattice" index. DNW showed that

$$[H, Y(SO(6))] = 0$$

It turns out that the Hamiltonian H is nothing but Lipatov model (1994) which was related to the Yang-Baxter form by Lipatov (1995), Faddeev and Korchemsky (1995).

Based on Tarasov, Takhtajan and Faddeev(1983) the $\breve{R}$-matrix reads

$$\breve{R}(u) = \frac{\Gamma(u-s)\Gamma(u+2s+1)}{\Gamma(u-\hat{J})\Gamma(u+\hat{J}+1)}$$

where u is spectrum parameter and s the spin (arbitrary). The trigonometric Yang-Baxterization (Jimbo) gives

$$\breve{R}(u) = \sum_{j=0} \rho_j(x) P_j(q) \qquad (x = e^{iu})$$

where $P_j(q)$ is the q-deformed prodector with weight j. Taking the rational limit (Cheng, Ge, Xue) we have

$$\rho_j \Rightarrow \frac{\Gamma(u)\Gamma(u+1)}{\Gamma(u-j)\Gamma(u+j+1)}, \quad P_j(q) \Rightarrow P_j$$

The Hamiltonian for the lattices α and $\alpha+1$

$$H_{\alpha\alpha+1} = I_1 \times I_2 \times \cdots \times I_{\alpha-1} \times \frac{d}{du}\breve{R}(u)|_{u=0}[\breve{R}(0)]^{-1} \times I_{\alpha+2} \times \cdots$$

is then

$$H = \sum_{\alpha} H_{\alpha\alpha+1}$$

where

$$\begin{aligned} H_{\alpha\alpha+1} &= \{-\psi(-\hat{J}_{\alpha\alpha+1}) - \psi(\hat{J}_{\alpha\alpha+1}+1) + \psi(1+2s) + \psi(1-2s) - \frac{1}{2s}\}|_{s=0} \\ &= \sum_{j}\{-\psi(-j) - \psi(j+1) + 2\psi(1) - \lim_{x\to 0}\frac{1}{x}\}P^{j}_{\alpha\alpha+1} \end{aligned}$$

It describes the QCD correction to the parton model. The diagonalization of Lipatov model has been achieved by Lipatov and de Vega (2003). Noting that the j indicates the block in the reducible block-diagonal form.

Using

$$\begin{aligned} &\psi(x+1) = \psi(x) + \frac{1}{x} \\ &\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1}\frac{1}{x+k} \\ &\psi(1) = -c \end{aligned}$$

and hence

$$\psi(j+1) = \psi(1) + \sum_{k=1}^{j} \frac{1}{k} = \psi(1) + h(j)$$

$$\psi(-j) = \psi(1) + h(j) - \lim_{x\to 0} \frac{1}{x}$$

We obtain

$$H_{\alpha,\alpha+1} = (-2)\sum_{j} h(j) P^{j}_{\alpha\alpha+1}$$

Separating the finite part from the infinity and normalizing to be unit H is nothing but the δD derived in super $YM(N=4)$ with approximation. Therefore, DNW's result shows that the Lipatov's model possesses $Y(SO(6))$ symmetry.

To obtain $Y(SO(6))$ in terms of RTT relation we start from the rational solution of $\breve{R}$-matrix whose general form for $O(N)$ was firstly by Zamolodchikov and Zamolodchikov (1972) and extended through rational limit of trigonometric Yang-Baxteization (Cheng, Ge, Xue, 1991):

$$\breve{R} = u[u - \frac{1}{2}(N-2)a]P + \alpha u A_N + [-u\alpha + \frac{\alpha^2}{2}(N-2)]I$$

where u is stpectrum parameter and α a free parameter allowed by YBE.

Here we adopt the convention of Jimbo:

$$P^{ab}_{cd} = \delta^a_d \delta^b_c$$

$$(A_N)^{ab}_{cd} = \delta^{a,-b}\delta_{c,-d}$$

$$a, b, c, c = [-(\frac{N-1}{2}), -(\frac{N-1}{2}) + 1, \cdots, (\frac{N-1}{2})]$$

$N = 2n + 1$ for B_n and $N = 2n$ for C_n, D_n.

The R-matrix is given by

$$R = \check{R}P = u(u - 2\alpha)I + u(2u - \alpha)P + 2u\alpha A_N$$

that is coinside with Zamolodchikov's S-matrix (up to an over all factor considering the CDD poles) with $\alpha = 1$ and $u = \frac{\theta}{i\lambda}$.

Actually, Z's s-matrix is universal, i.e. model independent.

$$S(\theta) = R(u) \quad = \quad Q^{\pm}(u)u(u-2)[I + \frac{\sigma_3}{\sigma_2}P + \frac{\sigma_1}{\sigma_2}A_N]$$

$$
\begin{aligned}
&= Q^{\pm}(u)u(u-2)[I - \frac{1}{u}P + \frac{2}{u-2}A_N] \\
Q^{\pm}(u) &= \frac{\Gamma(\pm\frac{\lambda}{2\pi} - i\frac{\theta}{2\pi})\Gamma(\frac{1}{2} - i\frac{\theta}{2\pi})}{\Gamma(\frac{1}{2} \pm \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi})\Gamma(-i\frac{\theta}{2\pi})}
\end{aligned}
$$

where $\lambda = \frac{2\pi}{N-2}$, $\theta = i\lambda u$. Although the spectrum parameter u is one dimensional, but u can be taken to be the cut-off in QFT, for example

$$u \sim \ln \Lambda^2$$

where Λ^2 is Lorentz invariant, i.e. scalar. This is the reason why asymptotic behavior of QFT model may be related to YB system.

For given $\breve{R}(u)$ one can easily obtain Hamiltonian by

$$H = [\frac{\partial \breve{R}(u)}{\partial u}\breve{R}(u)]|_{u=0}$$

for $O(N)$.

However, the essential connection between Lipatov model and $SO(6)$-RTT formulation is still missing.

Conclusion Remark

There are still two open questions:

(1) How can the Yangian representations help to solve physical models.

(2) Direct evidences of Yangian in the real physics.