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Integrable quantum gases

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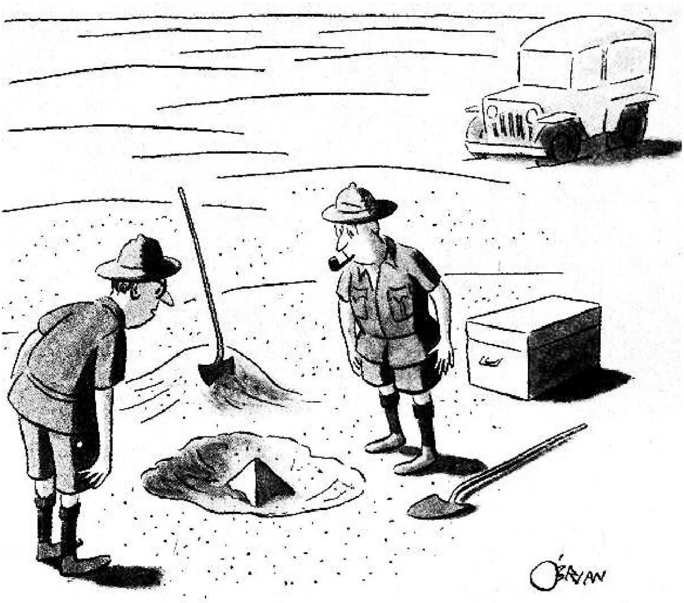
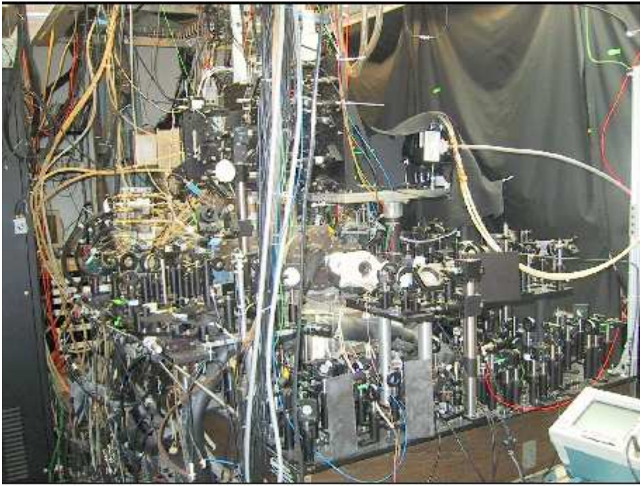


The story so far (how to write a grant proposal)

The study of integrable (exactly solved) models is poised to enter a new golden era. There are compelling reasons for this view:

- The theory is now mature. It has continued to make a huge impact on mathematics and is now expected to be equally important in the experimental study of new states of matter.
- Superior exact results apply where other approaches break down.
- It is now commonplace for experimentalists to manufacture low-dimensional quantum structures – where quantum effects can be most pronounced.
- Key integrable models – like the Lai & Yang interacting mixed boson/fermion model – have gone largely unnoticed for over 30 years.
- New integrable models are beginning to appear for Bose-Einstein condensates and metallic nanograins.

Lasers and magnets as cooling devices



Series of Holy Grails in ultra-cold matter research

- **1a) Bosonic condensates (1995)**
  – Cornell, Weiman & Ketterle

  1b) Realization of an interacting 1D Bose gas through to the strongly interacting Tonks-Girardeau regime (2004)

- **2a) Fermionic condensates (2004)**

  2b) Realization of an interacting 1D Fermi gas (200?)

Talk outline (everything old is new again!)

- **1) Integrable Bose gas**
- **2) Integrable Fermi gas**
- **3) Integrable mixed Bose-Fermi gas**
- **4) Evidence for a quasi-stable attractive Bose gas**

The 1D integrable Bose gas

- Lieb & Liniger (1963)
- McGuire (1964)

N interacting bosons on a line of length L ($\hbar = 2m = 1$)

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\,c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j)$$

kinetic term interaction term

- $c > 0$ repulsive interactions
- $c < 0$ attractive interactions

$\Rightarrow$ variable interaction strength c

The exact solution

Bethe Ansatz wavefunction

$$\psi(x_1,\ldots,x_N) = \sum_p A(p) \exp(\mathrm{i} \sum_{j=1}^{N} k_{p_j} x_j)$$

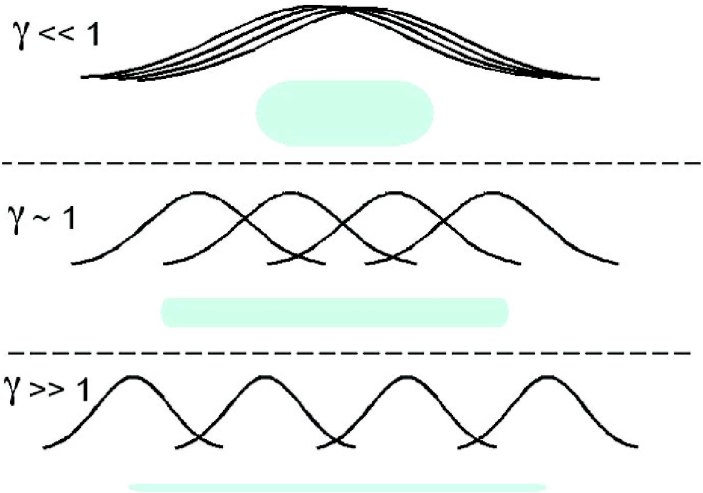
Energy eigenvalues

$$\mathcal{E} = \sum_{j=1}^{N} k_j^2$$

Bethe equations

$$\exp(\mathrm{i} k_j L) = -\prod_{\ell=1}^{N} \frac{k_j - k_\ell + \mathrm{i}\, c}{k_j - k_\ell - \mathrm{i}\, c} \qquad \text{for} \quad j = 1,\ldots,N$$

$\gamma = Lc/N$



The 1D integrable Fermi gas

- Gaudin (1967)
- Yang (1967)

N interacting two-component fermions on a line of length L
$(\hbar = 2m = 1)$
M spin-down fermions (special case $M = N/2$)

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j)$$

- $c > 0$ repulsive interactions
- $c < 0$ attractive interactions

The exact solution

Energy eigenvalues

$$\mathcal{E} = \sum_{j=1}^{N} k_j^2$$

Bethe equations

$$\exp(\mathrm{i} k_j L) = \prod_{\ell=1}^{M} \frac{k_j - \Lambda_\ell + \frac{1}{2}\mathrm{i}\, c}{k_j - \Lambda_\ell - \frac{1}{2}\mathrm{i}\, c}$$

$$\prod_{\ell=1}^{N} \frac{\Lambda_\alpha - k_\ell + \frac{1}{2}\mathrm{i}\, c}{\Lambda_\alpha - k_\ell - \frac{1}{2}\mathrm{i}\, c} = -\prod_{\beta=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}\, c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}\, c}$$

for $j = 1, \ldots, N$ and $\alpha = 1, \ldots, M$.

Weak interaction

For small $|c|$, distinguish between unpaired roots, $k_j^{(u)}$, and paired roots, $k_j^{(p)}$.

Can expand the Bethe equations to $\mathcal{O}(c)$ to obtain

$$k_j^{(\mathrm{u})} = \pi(M - N - 1 + 2j)/L + \delta_j^{(\mathrm{u})},\ j = 1, \ldots, N/2 - M,$$
$$k_j^{(\mathrm{u})} = \pi(3M - N - 1 + 2j)/L + \delta_j^{(\mathrm{u})},\ j = N/2 - M + 1, \ldots, N - 2M,$$

and $k_j^{(\mathrm{p})} = \pi\,[1 - M + 2j_+] + \delta_{j_+}^{(\mathrm{p})} \pm \sqrt{c/L}$,

where $j_+ = j$, if j odd and $j_+ = j - 1$ if j even.

Deviations (BCS-like eqns)

The deviations δ from $k_j^{(\mathrm{u,p})}$ are linear in c, with

$$\delta_j^{(\mathrm{u})} = \frac{c}{L} \sum_{\ell} \frac{1}{k_{j,0}^{(\mathrm{u})} - k_{\ell,0}^{(\mathrm{p})}},$$

$$\delta_j^{(\mathrm{p})} = \frac{c}{L} \left[\sum_{\ell \neq j} \frac{1}{k_{j,0}^{(\mathrm{p})} - k_{\ell,0}^{(\mathrm{p})}} + \frac{1}{2} \sum_{\ell} \frac{1}{k_{j,0}^{(\mathrm{p})} - k_{\ell,0}^{(\mathrm{p})}} \right].$$

Here $k_{j,0}^{(\mathrm{u,p})}$ are the quasimomenta of non-interacting particles as given above. Note that the unpaired momenta do not interact with each other, which is consistent with the Pauli principle.

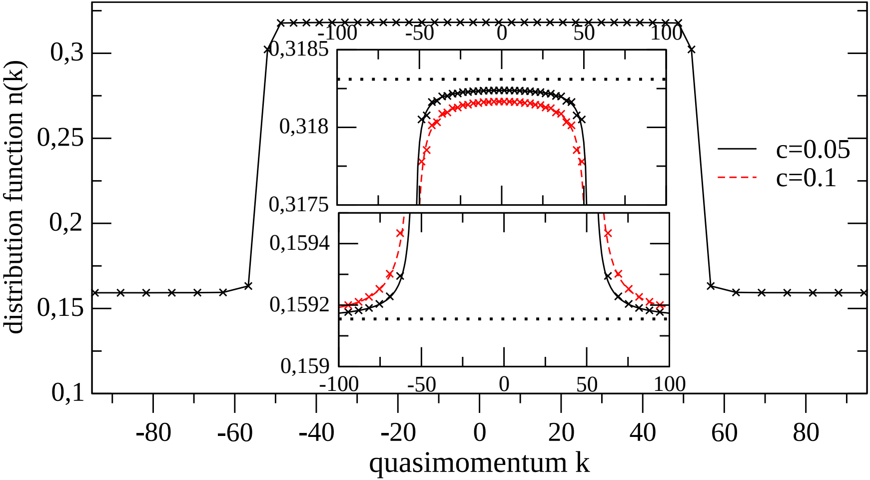
Momentum distribution

From the above, one can derive the momentum distribution fns:

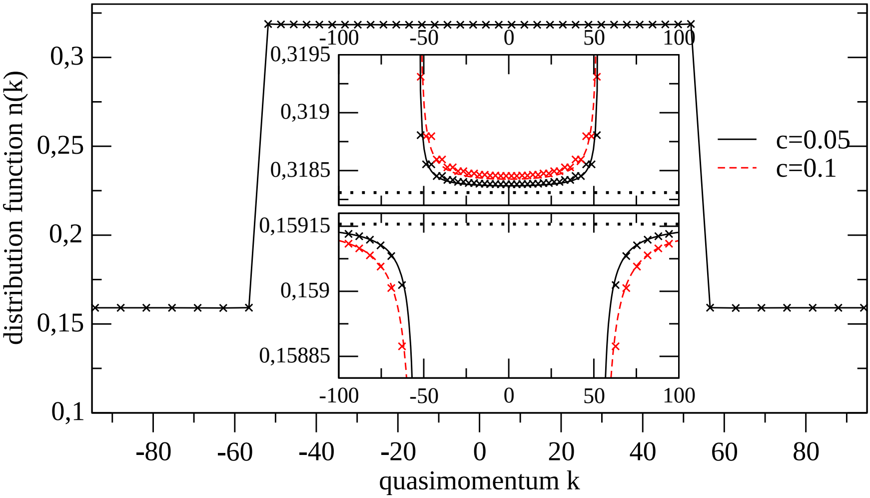
$$n_u(k) = \frac{1}{2\pi} + \frac{c}{2\pi^2}\frac{A}{k^2 - A^2}, \; A < |k| < B,$$
$$n_p(k) = \frac{1}{\pi} - \frac{c}{2\pi^2}\left(\frac{A}{A^2 - k^2} + \frac{B}{B^2 - k^2}\right), \; |k| < A,$$

with $A = \pi M/L$ and $B = \pi N/L$. Especially, for $M = N/2$, $n_{\mathrm{u}} \equiv 0$.

Note the power-law divergence near the cutoffs – an intrinsic property of Luttinger liquids.



Fermion momentum distribution – repulsive regime ($N = 50$ and $M = 16$)



Fermion momentum distribution – attractive regime ($N = 50$ and $M = 16$)

Ground state energy

- Define the polarization $a = 2M/N$.
  $a = 1$ (zero polarized) $a = 0$ (fully polarized).
- The above asymptotic Bethe roots give

$$\frac{E}{N} \approx \frac{\hbar^2 n^2}{2m}\left[\frac{a^2}{2}\gamma + \gamma(1-a)a + \frac{\pi^2}{12}a^3 + \frac{\pi^2}{3}a_0\right]$$

  where $\gamma = Lc/N$ and $a_0 = (1-a)\left((1-\frac{1}{2}a)^2 + \frac{1}{2}a\right)$.
- The first two terms represent the binding and the interaction energy, respectively. Both are linear in γ. This is different from what is obtained in the TL from the Gaudin-Yang integral equations, where the binding energy is $\propto \gamma^2$.
- The other terms are the kinetic energies for paired and unpaired fermions.

Strong coupling – attractive

- For $Lc \ll -1$, tightly bound states with quasi-momentum $k_i^{(\mathrm{p})} \approx \Lambda_i \pm \frac{1}{2}\mathrm{i}c$ separate from unbound states with $k_j^{(\mathrm{u})} = \frac{n_j \pi}{L}\left(1 + \frac{4M}{Lc}\right)^{-1}$ with integers $n_j = \pm M, \pm(M+2), \ldots, \pm(N-M-2)$.
  In the above, $\Lambda_i = \frac{n_i \pi}{L}\left(1 + \frac{M}{Lc} + \frac{2(N-2M)}{Lc}\right)^{-1}$ for $n_i = \pm 1, \ldots, \pm\frac{1}{2}(M-1)$.
- Thus in the strongly attractive limit, a gap appears between the bound paired and the unpaired momenta.
- In this scenario the bound states behave like hard-core bosons while the unpaired fermions interact weakly with the bound states.

Ground state energy

- Using the asymptotic expressions for the Bethe roots given above in the strongly attractive regime, one obtains

$$\frac{E}{N} \approx \frac{\hbar^2 n^2}{2m}\left[-\frac{a}{4}\gamma^2 + \frac{\pi^2}{3}\left(\frac{a^3}{16\, b_1^2} + \frac{a_1}{b_2^2}\right)\right]$$

  where $a_1 = (1-a)(1-\frac{1}{2}a(1-\frac{1}{2}a))$,
  $b_1 = 1 + a/(2\gamma) + 2(1-a)/\gamma$ and $b_2 = 1 + 2a/\gamma$.
- Whereas the first term represents the binding energy, the remaining interaction energies stem from the interaction between pair-pair and pair-unpaired fermions.
- The ground state energy is lowest for $a = 1$, compared to the others for $0 \leq a < 1$.

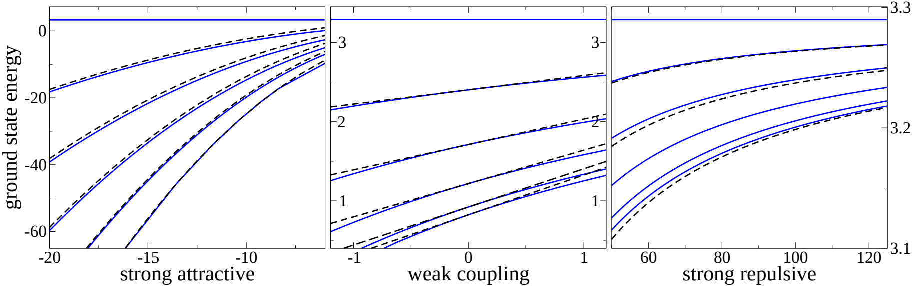
Ground state energy – strong repulsive regime

Straightforward analysis of the Gaudin-Yang integral equations leads to

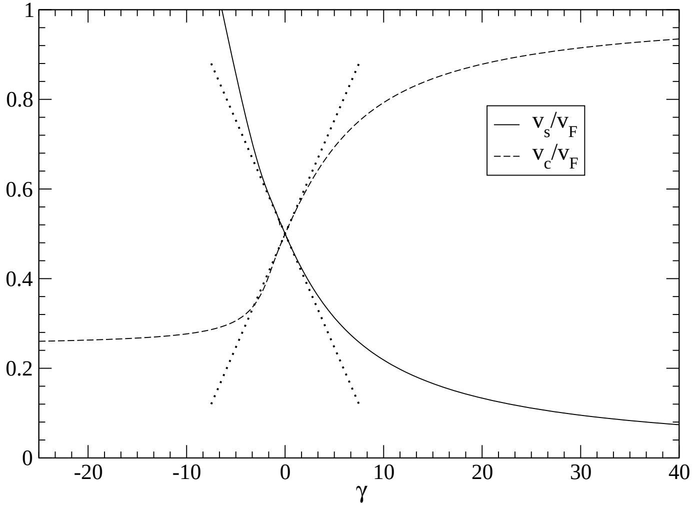
$$\frac{E}{N} \approx \begin{cases} \frac{\hbar^2 n^2}{2m} \frac{\pi^3}{3} \left(1 - \frac{4a}{\gamma}\right) + \mathcal{O}(1/\gamma^2, a^2/\gamma), & a \ll 1 \\ \frac{\hbar^2 n^2}{2m} \frac{\pi^3}{3} \left(1 - \frac{4 \ln 2}{\gamma}\right) + \mathcal{O}(1/\gamma^2), & a = 1 \end{cases}$$

Energy as a fn of interaction strength and polarization

- Use Gaudin-Yang integral eqns to compute E/N as a fn of γ and *a*.
- Dashed lines are the analytic approximations.
- Curves shown are for $a = 0, 0.2, 0.4, 0.6, 0.8, 1$ (top to bottom).



Spin and charge velocities



Mixtures

- Fermi condensates achieved by sympathetic cooling with bosons.

Wise words from the prophet!

- If you want to find a new solvable model – go to the library...

The integrable 1D mixed Bose-Fermi gas

Lai & Yang (1971)

- cited only a handfull of times
- bosons and fermions have equal mass – not so bad

$N_f = N_\downarrow + N_\uparrow$ interacting two-component fermions on a line of length L
$(\hbar = 2m = 1)$
N_b bosons

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j)$$

The exact solution

$$\mathcal{E} = \sum_{j=1}^{N} p_j^2$$

Bethe equations

$$e^{\mathrm{i}\, p_\ell L} = \prod_j \frac{p_\ell - \Lambda_j + \frac{1}{2}\,\mathrm{i}\, c}{p_\ell - \Lambda_j - \frac{1}{2}\,\mathrm{i}\, c}$$

$$\prod_\ell \frac{\Lambda_k - p_\ell - \frac{1}{2}\,\mathrm{i}\, c}{\Lambda_k - p_\ell + \frac{1}{2}\,\mathrm{i}\, c} = -\prod_{j,m} \frac{\Lambda_k - \Lambda_j - \mathrm{i}c}{\Lambda_k - \Lambda_j + \mathrm{i}c} \frac{\Lambda_k - A_m + \frac{1}{2}\,\mathrm{i}\, c}{\Lambda_k - A_m - \frac{1}{2}\,\mathrm{i}\, c}$$

$$1 = \prod_j \frac{A_n - \Lambda_j - \frac{1}{2}\,\mathrm{i}\, c}{A_n - \Lambda_j + \frac{1}{2}\,\mathrm{i}\, c},$$

where $j, k = 1, \ldots, N_b + N_\uparrow$, $\ell = 1, \ldots, N$, $m, n = 1, \ldots, N_b$.

Momentum distribution

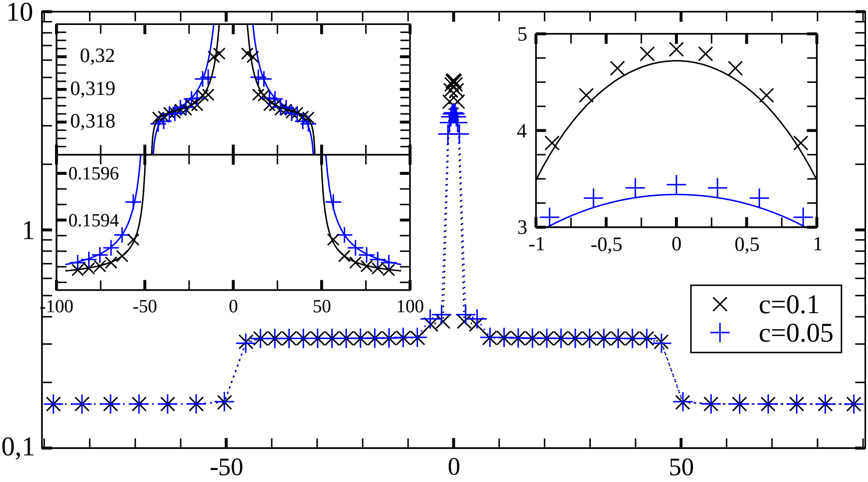
$$\rho_u(k) = \frac{1}{2\pi} + \frac{c}{2\pi^2}\left(\frac{A}{k^2 - A^2} + \frac{B}{k^2}\right), A < |k| < C$$

$$\rho_p(k) = \frac{1}{\pi} - \frac{c}{2\pi^2}\left(\frac{A}{A^2 - k^2} + \frac{C}{C^2 - k^2} - \frac{2B}{k^2}\right),$$

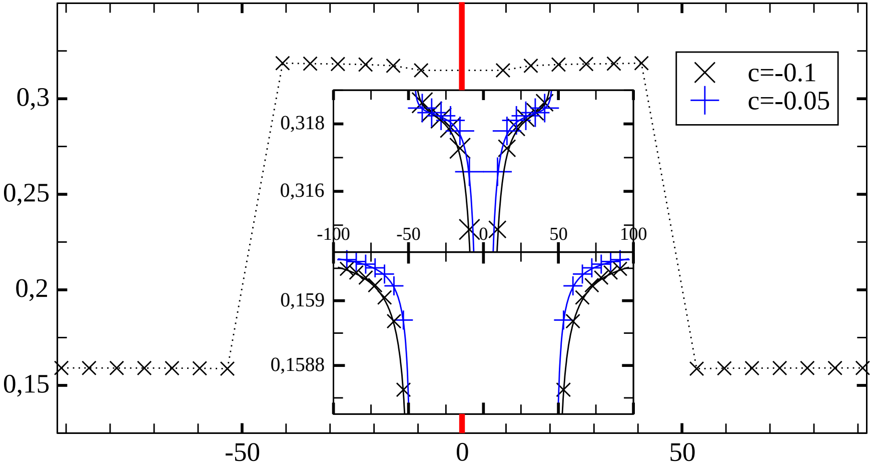
$$2\sqrt{\frac{cB}{\pi}} < |k| < C$$

$$\rho_b(k) = \frac{1}{2\pi c}\left(4cB/\pi - k^2\right)^{1/2}, |k| < 2\sqrt{\frac{cB}{\pi}},$$

where $A = \pi N_\uparrow / L$, $B = \pi N_b / L$, $C = \pi(N_\downarrow - N_\uparrow)/L$

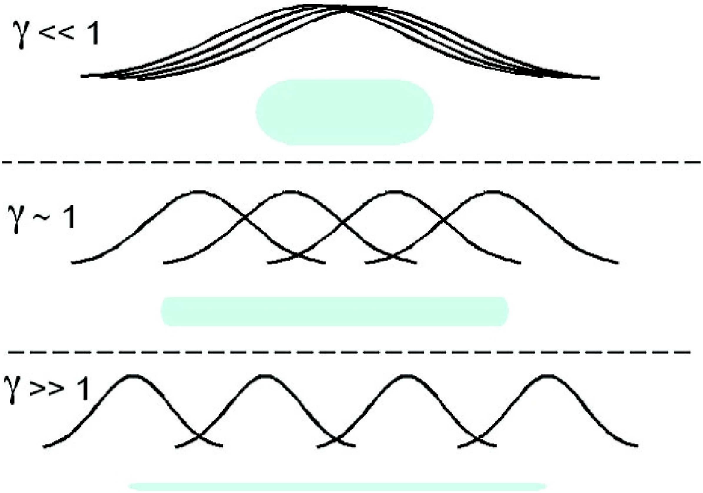


Momentum distribution – repulsive regime ($N_\uparrow = 15$, $N_\downarrow = 31$ and $N_b = 9$)



Momentum distribution – attractive regime ($N_{\uparrow} = 15$, $N_{\downarrow} = 31$ and $N_b = 9$)

The Tonks-Girardeau gas



The *super Tonks-Girardeau gas*

A stable gas-like state in the strongly interacting 1D Bose gas in the attractive regime proposed by Astrakharchik et al. cond-mat/0405225

Using variational Monte Carlo calculations, for a certain range of strong coupling, the energy coincides with the energy

$$\frac{E}{N} = \frac{\pi^2 \hbar^2 n^2}{6m} \left(1 + \frac{2n}{c}\right)^2$$

of a gas of hard rods.

But what about the integrable model?

The weakly attractive regime

Table: $N = 12$ with $c = -0.005$ and $L = 6$

j	P	E_{exact}	E_{BCS}	E_{cloud}	$E_{\text{num.}}$
0	0	-0.11028	-0.11000	-0.10000	-0.11021
1	$\pi/3$	0.96786	0.96809	0.96829	0.96794
2	0	2.04772	2.04791	2.04825	2.04782
3	$2\pi/3$	2.04936	2.04955	2.04991	2.04946
4	$\pi/3$	3.13094	3.13109	3.13153	3.13433
5	π	3.13422	3.13438	3.13487	3.13432
6	0	4.21587	4.21599	4.21649	4.21599
7	$2\pi/3$	4.21751	4.21764	4.21816	4.21763
8	$4\pi/3$	4.22244	4.22260	4.22316	4.22256
9	$2\pi/3$	4.25788	4.25810	4.25816	4.25797

The strongly attractive regime

Take $N = 2M$ bosons with $Lc \ll -1$ and even M.

The Bethe roots for the ground state are

$k_{\pm j} \approx \pm \mathrm{i} \left(\frac{c}{2}(2M - 2j + 1) + \delta_j \right), j = 1, \ldots, M,$

δ_j is small and negligible for $Lc \ll -1$.

The wave function is

$$\psi(x_1, \ldots, x_N) \approx \mathcal{N} \exp \left\{ \frac{c}{2} \sum_{1 \le i < j \le N} |x_j - x_i| \right\}$$

Here $\mathcal{N} = \frac{\sqrt{(n-1)!}}{\sqrt{2\pi}} |c|^{\frac{(n-1)}{2}}$.

The energy eigenvalue is $E_0 = -\frac{1}{12} c^2 N(N^2 - 1)$,
originally obtained by McGuire.

The super attractive state

There is a highly-excited state in which all Bethe roots are real and symmetric about the origin, with $\pm k_{2m-1}$, $m = 1, \ldots, N/2$,
where $k_\ell \approx \frac{\pi \ell}{L} \left(1 + \frac{2}{\gamma}\right)^{-1}$.

The energy of this state follows as

$$\frac{E}{N} \approx \frac{\hbar^2}{2m} \frac{1}{3} (N^2 - 1) \frac{\pi^2}{L^2} \left(1 + \frac{2n}{c}\right)^{-2}$$

which coincides with the result for the gas of hard-rods!

Here the condition $\frac{2N}{L|c|} \ll 1$ is required.

$\Rightarrow$ Inherits hard-core behaviour from the large Fermi-pressure-like kinetic energy from the strongly repulsive interaction.

$\Rightarrow$ Could potentially be reached in experiments.

The analysis of ultracold quantum gases

- *The 1D interacting Bose gas in a hard wall box*
  MTB, X.W. Guan, N. Oelkers and C. Lee, arXiv:cond-mat/0505550
- *Exact results for the 1D interacting Fermi gas with arbitrary polarization*
  MTB, M. Bortz, X.W. Guan and N. Oelkers, arXiv:cond-mat/0506264
- *Exact results for the 1D mixed Bose-Fermi interacting gas*
  MTB, M. Bortz, X.W. Guan and N. Oelkers, arXiv:cond-mat/0506478
- *Evidence for the super Tonks-Girardeau gas*
  MTB, M. Bortz, X.W. Guan and N. Oelkers, arXiv:cond-mat/0508009

$\Rightarrow$ See the talk by Xiwen Guan, Thursday afternoon, for the application of integrable models to the physics of spin chains and ladders.