Standing waves with critical frequency for nonlinear Schrödinger equations

! ! ! ! ! ! ! Happy Paul's 70th Birthday ! ! ! ! ! ! !

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We consider the following equation

$$\varepsilon^2 \Delta u - V(x)u + f(u) = 0, \quad u > 0 \qquad \text{on} \qquad \mathbf{R}^n \tag{1}$$

with $\lim_{|x|\to\infty} u(x) = 0$ and sufficiently small $\varepsilon > 0$.

$$V \in C(\mathbf{R}^n), \quad f \in C(\mathbf{R}), \quad V \geq 0$$

Nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\Delta\psi - V(x)\psi + f(\psi) = 0,\ \ (t,x) \in \mathbf{R} \times \mathbf{R}^n$$

Standing wave

$$\psi(x,t) = \exp(-\frac{iEt}{\hbar})u(x), u(x) \in \mathbb{R}$$

Corresponding equation for u when $f(e^{i\theta}t) = e^{i\theta}f(t)$

$$\frac{\hbar^2}{2}\Delta u - (V(x) - E)u + f(u) = 0 \quad \text{in} \quad \mathbf{R}^n$$

Semiclassical states

$$\varepsilon^2 \Delta u - (V(x) - E)u + f(u) = 0 \quad \text{in} \quad \mathbf{R}^n$$

$$\varepsilon^2 \equiv \hbar^2/2 \longrightarrow 0$$

Normalization $u(x) \longrightarrow u(\varepsilon x)$

$$\Downarrow$$

$$\Delta u - (V(\varepsilon x) - E)u + f(u) = 0, u > 0 \text{ on } \mathbf{R}^n \text{ with } \lim_{|x| \to \infty} u(x) = 0$$

For each $d > 0$,

$$V(\varepsilon x) - E \to V(x_0) - E$$

uniformly on $B(\frac{x_0}{\varepsilon}, d)$ as $\varepsilon \to 0$.

If u_ε is a solution satisfying $\frac{1}{C} \leq \|u_\varepsilon\|_{C^{1,\alpha}(B(\frac{x_0}{\varepsilon},d))} \leq C$ for each $d > 0$,

$u_\varepsilon(\cdot + x_0/\varepsilon) \to u$ in C^1_{loc} and

$$\Delta u - (V(x_0) - E)u + f(u) = 0, u > 0$$

Almost optimal existence result by Berestycki-Lions.

Some criticality of V + some conditions on f

(stronger than or equal to Berestycki-Lions conditions on f)

$$\Downarrow$$

Existence of solution concentrating around critical points of V.

Liapunov-Schmidt reduction method

Pioneering work and development by

Floer-Weinstein, Yonggeun Oh, Ambrosetti, Vieri Benci, B., Yanyan Li, Zhaoli Liu, Juncheng Wei, Oshita, ...

Variational approach

Pioneering work and development by

Paul H. Rabinowitz, Patricio Felmer, del Pino, Gui, B. Zhi-Qiang Wang, Kazunaga Tanaka, Jeanjean, Oshita, ...

Typical nonlinearity $f(u) = u^p, \ 1 < p < \frac{n+2}{n-2}$

No solutions for small $\varepsilon > 0$ if $\inf_{x \in \mathbf{R}^n} V(x) - E < 0$
No solutions for limiting problem if $V(x_0) - E = 0$
[Gidas-Spruck]
A unique(up to a translation) solution U of limiting problem for $V(x_0) - E > 0$

If $V(x_0) - E = 0$, no solution u_ε for small $\varepsilon > 0$ such that $\frac{1}{C} \leq \|u_\varepsilon\|_{C^{1,\alpha}(B(\frac{x_0}{\varepsilon},d))} \leq C$

(We change $V(x) - E \to V$ for simplicity)

If $V(x) \geq V(x_0) = 0$, the functional

$$\Gamma_\varepsilon(u) = \frac{1}{2} \int_{\mathbf{R}^n} |\nabla u|^2 + V(\varepsilon x) u^2 dx - \frac{1}{p+1} \int_{\mathbf{R}^n} u^{p+1} dx$$

has a (local) mountain pass structure.

Indeed, for each isolated zero set $A \subset \mathbf{R}^n$ and $\delta > 0$, there exists a local mountain pass solution u_ε such that $\lim_{\varepsilon\to\infty} \|u_\varepsilon\|_{L^\infty} = 0$, $\liminf_{\varepsilon\to\infty} \varepsilon^{-\frac{2}{p+1}} \|u_\varepsilon\|_{L^\infty} > 0$ and $u_\varepsilon(x) \leq C\exp(-\frac{c}{\varepsilon}\mathrm{dist}(\varepsilon x, A^\delta))$, where $A^\delta = \{x \mid \mathrm{dist}(x, A) \leq \delta\}$.

? Asymptotic profile of u_ε as $\varepsilon \to 0$

$$\Delta u_\varepsilon - V(\varepsilon x)u_\varepsilon + (u_\varepsilon)^p = 0$$

Direct limit

$$\Delta U + U^p = 0, \quad U > 0 \text{ in } \mathbf{R}^n$$

No solutions

$$\Delta u_\varepsilon - V(\varepsilon x)u_\varepsilon + (u_\varepsilon)^p = 0$$

$$\Downarrow$$

$$(\text{renormalization, } w_\varepsilon(x) \equiv (\tfrac{a(\varepsilon)}{\varepsilon})^{2/(p-1)} u_\varepsilon(\tfrac{a(\varepsilon)}{\varepsilon} x))$$

$$\Downarrow$$

$$\Delta w_\varepsilon - V(a(\varepsilon)x)(\frac{a(\varepsilon)}{\varepsilon})^2 w_\varepsilon + (w_\varepsilon)^p = 0$$

Take an appropriate $a(\varepsilon)$ so that $V(a(\varepsilon)x)(\frac{a(\varepsilon)}{\varepsilon})^2 \longrightarrow V_0$ as $\varepsilon \longrightarrow 0$ and there exists a solution of

$$\Delta U - V_0 U + U^p = 0$$

Expamples

Flat cases

$$V = 0 \text{ on } \overline{\Omega} \Rightarrow a(\varepsilon) = 1$$

$$\Delta U + U^p = 0 \text{ in } \Omega, U = 0 \text{ on } \partial\Omega$$

Infinite cases

$$V(x) \approx \exp(-|x|^{-\alpha}) \text{ near } 0 \Rightarrow a(\varepsilon) = (\log \varepsilon^{-2})^{-1/\alpha}$$

$$\Delta U + U^p = 0 \text{ in } B(0,1), \quad U = 0 \text{ on } \partial B(0,1)$$

Finite cases

$$V(x) \approx |x|^\alpha \text{ near } 0 \Rightarrow a(\varepsilon) = \varepsilon^{2/(\alpha+2)}$$

$$\Delta U - |x|^\alpha U + U^p = 0 \text{ in } \mathbf{R}^n$$

Very slow cases

$$V(x) \approx (-\log|x|)^{-1} \text{ near } 0 \Rightarrow a(\varepsilon) = g^{-1}(\varepsilon^2), g(x) = -x^2/\log x$$

$$\Delta U - U + U^p = 0 \text{ in } \mathbf{R}^n$$

A rich variety of solutions following from
a rich variety of decaying behavior of V near zeros

Algebraically small solutions whose interaction is exponentially small

$$\Downarrow$$

It should be possible to glue together the localized solutions

[B. and Oshita, 2004] Yes for general critical points if $f(t) = t^p \in C^4(\mathbb{R})$ and there exist *limiting problems* whose solutions are nondegenerate

(Lyapunov-Schmidt reduction method)

[Ding and Tanaka, 2003], [Cao and Noussair, 2004]

flat cased - variational method

[Cao and Peng,2006] exponentially (same order) decaying case - variational method

[Cao, Noussair and Yan, 2008]

different scales without the restriction $f(t) = t^p \in C^4(\mathbb{R})$

- Lyapunov-Schmidt reduction method

[Sato, 2007] different scales(no requiring the existence of limiting profile) for local minimum points of V and $f(t) = t^p$
- minimization on infinite dimensional torus

Aim

Develop a variational method and a hybrid method combining the variational method and reduction method to glue together localized solutions with different energy scales in a possibly general setting

(V1) $V \in C(\mathbb{R}^n)$; $\liminf_{|x|\to\infty} V(x) > 0 = \min_{x\in\mathbb{R}^n} V(x)$;

(V2-1) there exist disjoint bounded open sets Ω_i with smooth boundary $\partial\Omega_i$, $i = 1, \cdots, l$, satisfying $0 = \inf_{x\in\Omega_i} V(x) < \min_{x\in\partial\Omega_i} V(x)$;

(f1) $f \in C^1(\mathbb{R})$, $f(t) = 0$ for $t \leq 0$ and there exist some $\mu > 1$ and $C > 0$ satisfying $|f(t)| \leq Ct^{\mu}$ for $t \in (0, 1)$;

(f2-1) there exists some $p \in (1, \frac{n+2}{n-2})$ for $n \geq 3$ and $p \in (1, \infty)$ for $n = 1, 2$ such that $\liminf_{t\to 0+} \frac{1}{t^{p+1}} \int_0^t f(s)\, ds > 0$;

(f3) there exists $\mu_2 > 1$ such that $(\mu_2 + 1) \int_0^t f(s)ds \leq f(t)t$ for $t > 0$.

[B. and Wang, 2003] Suppose that (V1),(V2-1),(f1), (f2-1), (f3) hold. Then, for sufficiently small $\varepsilon > 0$, there exists a positive solution $u_\varepsilon^i, i = 1, \cdots, l$ such that for any sufficiently small $d > 0$, there exist $C, c > 0$ satisfying

$$u_\varepsilon^i(x) \leq C \exp(-\frac{c}{\varepsilon}\text{dist}\,(x, (A_i)^d))$$

(Here $B^d \equiv \{x \in \mathbf{R}^n |\ \text{dist}(x, B) \leq d\}$) satisfying

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\Omega_i)} = 0, \qquad \lim_{\varepsilon \to 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_i)} > 0.$$

[A related work of Sirakov]

[B. and Oshita] Suppose that (V1),(V2-1),(f1), (f2-1), (f3) hold. Then, for sufficiently small $\varepsilon > 0$, there exists a positive solution u_ε such that

(i) for any sufficiently small $d > 0$, there exist $C, c > 0$ satisfying

$$u_\varepsilon(x) \le C \exp(-\frac{c}{\varepsilon}\text{dist}\,(x, (A_1 \cup \cdots \cup A_l)^d));$$

(Here $A^d \equiv \{x \in \mathbf{R}^n |\ \text{dist}(x, A) \le d\}$)

(ii) for $i = 1, \cdots, l$, it holds that

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\Omega_i)} = 0, \qquad \lim_{\varepsilon \to 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_i)} > 0.$$

(V2-2) there exist disjoint bounded open sets Ω_i with smooth boundary $\partial\Omega_i$, $i = 1, \cdots, k$, satisfying

$$m_i = \inf_{x\in\Omega_i} V(x) < \min_{x\in\partial\Omega_i} V(x)$$

and

$$0 = m_1 = \cdots = m_l < m_{l+1}, \cdots, m_k;$$

(f2-2) there exists some $p \in (1, \frac{n+2}{n-2})$ for $n \geq 3$ and $p \in (1, \infty)$ for $n = 1, 2$ such that $\limsup_{t\to\infty} \frac{|f(t)|+|f'(t)t|}{t^p} < \infty$.

[B. and Oshita] Suppose that (V1) and (V2-2) hold, and that (f1), (f2-1), (f2-2), (f3-1) hold. Then, for sufficiently small $\varepsilon > 0$, there exists a positive solution u_ε such that

(i) for any sufficiently small $d > 0$, there exist $C, c > 0$ satisfying

$$u_\varepsilon(x) \leq C \exp(-c\mathrm{dist}\,(x, (A_1 \cup \cdots \cup A_k)^d)/\varepsilon);$$

(ii) for $i = 1, \cdots, l$, it holds that

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\Omega_i)} = 0, \qquad \lim_{\varepsilon \to 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_i)} > 0;$$

and for $i = l+1, \cdots, k$, it holds that for a least energy solution U of $\Delta U - m_i U + f(U) = 0$and some $x_\varepsilon^i \in \Omega_i$ with $\lim_{\varepsilon \to 0} x_\varepsilon^i = x_i$, a transformed solution $u_\varepsilon(\varepsilon x + x_\varepsilon^i)$ converges (up to a subsequence) uniformly to $U(x)$ on each bounded set in $\mathbb{R}^n$.

(V3) there are $x_1, \cdots, x_m \in \mathbb{R}^n$ and disjoint bounded open sets $\Omega_{k+1}, \cdots, \Omega_{k+m}$ with smooth boundary $\partial\Omega_{k+j}$ such that $x_j \in \Omega_{k+j}, V \in C^2(\Omega_{k+j})$, $\nabla V(x) \neq 0$ for $x \in \Omega_{k+j} \backslash \{x_j\}$, $\inf_{x \in \Omega_{k+j}} V(x) > 0$ and x_j is a non-degenerate critical point of V for $j \in \{1, \cdots, m\}$;

(f4) for any $a \in \{V(x_1), \cdots, V(x_m)\}$, the problem

$$\Delta u - au + f(u) = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n, \quad u \in H^{1,2}(\mathbb{R}^n)$$

has a radially symmetric solution U_a which is non-degenerate in $H_{\mathrm{r}}^{1,2}(\mathbb{R}^n) \equiv \{w \in H^{1,2}(\mathbb{R}^n); w(x) = w(|x|)\}$, and $f \in C_{\mathrm{loc}}^{1,\gamma}(\mathbb{R})$ for some $\gamma \in (0,1)$.

[B. and Oshita] We assume that (V1), (V2-2) and (V3) hold. Suppose that (f1), (f2-1),(f2-2), (f3) and (f4) hold. Then for sufficiently small $\varepsilon > 0$, there exists a positive solution u_ε such that

(i) for any sufficiently small $d > 0$, there exist $C, c > 0$ satisfying

$$u_\varepsilon(x) \le C \exp(-\frac{c}{\varepsilon} \mathrm{dist}\,(x, (A_1 \cup \cdots \cup A_k \cup \{x_1, \cdots x_m\})^d));$$

(ii) the same behavior around zero A_i, $i = 1, \cdots, k$;

(iii) for each $i = 1, \cdots, m$, there exist $y_\varepsilon^i \in \mathbb{R}^n$ such that $\lim_{\varepsilon \to 0} y_\varepsilon^i = x_i$ and that $u_\varepsilon(\varepsilon x + y_\varepsilon^i)$converges uniformly (up to a subsequence) to $U_{V(x_i)}$ on each bounded set in $\mathbb{R}^n$. Here $U_{V(x_i)}$ is a function given in (f4).

Setting and preliminaries

$\mathcal{Z} := \{x \in \mathbf{R}^n \mid V(x) = 0\}$, $A_0 := \mathcal{Z} \setminus \cup_{i=1}^k A_i$ and $\Omega_0 \supset A_0$ a bounded open set with a smooth boundary such that for $\delta > 0$ and $\Omega_i^\delta = \{x \in \mathbf{R}^n \,|\, \mathrm{dist}(x, \Omega_i) \leq \delta\}$, $\Omega_i^{2\delta} \cap \Omega_j^{2\delta} = \emptyset$ for each $0 \leq i \neq j \leq k+m$, and that $\partial\Omega_i^{\delta'}$ is smooth for each $0 \leq i \leq k+m$ and $\delta' \in [0, 2\delta]$.

For $\lambda << 1$, we define $g(x,t)$ so that $|g(x,t) - g(x,t')| \leq \lambda|t - t'|$ for $x \notin \cup_{i=0}^{k+m}\Omega_{i,\varepsilon}$, $g(x,t) = f(t)$ for small $t > 0$ and some more properties.

$G(x,t) \equiv \int_0^t g(x,s)\,ds$, $\|u\|_\varepsilon^2 \equiv \int_{\mathbb{R}^n} |\nabla u|^2 + V_\varepsilon(x)u^2\,dx$.

For $u \in H_\varepsilon \equiv \overline{(C_0^\infty, \|\cdot\|_\varepsilon)}$,

$$\Gamma_\varepsilon(u) \equiv \frac{1}{2}\|u\|_\varepsilon^2 - \int_{\mathbb{R}^n} G(x,u)\,dx, \quad \Gamma_\varepsilon \in C^2(H_\varepsilon)$$

For $u_i \in H^1(\Omega_{i,\varepsilon})$ (here $\Omega_{i,\varepsilon} = \frac{1}{\varepsilon}\Omega_i$), let

$$X_i^\varepsilon(u_i) = \{u \in H_0^1(\Omega_{i,\varepsilon}^\delta) \mid u = u_i \text{ on } \Omega_{i,\varepsilon}\}.$$

Proposition 0.1. *For each $u_i \in H^1(\Omega_{i,\varepsilon})$, $i \in \{0, \cdots, k+m\}$, there exists a unique minimizer $P_i(u_i)$ of Γ_ε on $X_i^\varepsilon(u_i)$, which satisfies the following:*

(i) *$w = P_i(u_i) \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ solves*

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda(w) = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon} \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon} \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta, \end{cases}$$

(ii) *$P_i : H^1(\Omega_{i,\varepsilon}) \to H_0^1(\Omega_{i,\varepsilon}^\delta)$ is of class C^1,*

(iii) *there exists a positive constant C, independent of small $\varepsilon > 0$ such that*

$$\|P_i(u_i)\|_\varepsilon \leq C\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}} \text{ for all } u_i \in H^1(\Omega_{i,\varepsilon}), i \in \{0, \cdots, k+m\}.$$

For $\vec{u} = (u_0, \cdots, u_{k+m}), u_i \in H^1(\Omega_{i,\varepsilon})$, let

$$X_*^\varepsilon(\vec{u}) = \{u \in H_\varepsilon \,|\, u = u_i \text{ on } \Omega_{i,\varepsilon}, i = 0, \cdots, k+m\}.$$

Proposition 0.2. *For each $\vec{u} = (u_0, \cdots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k+m,\varepsilon})$, there exists a unique minimizer $\varphi(\vec{u})$ of Γ_ε on $X_*^\varepsilon(\vec{u})$, which satisfies the following:*

(i) *$w = \varphi(\vec{u})$ solves*

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda(w) = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \cdots \cup \Omega_{k,\varepsilon})^c \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon}\, (i = 0, \cdots, k), \end{cases}$$

(ii) *$\varphi : H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k,\varepsilon}) \longrightarrow H_\varepsilon$ is of class C^1,*

(iii) *there exists a positive constant C, independent of small $\varepsilon > 0$ such that*

$$\|\varphi(\vec{u})\|_\varepsilon \leq C \|\vec{u}\|_\varepsilon.$$

Let $\tilde{\varphi}(\vec{u}) = \varphi(\vec{u}) - \sum_{i=0}^{k+m} P_i(u_i)$. Then it follows that $\tilde{\varphi}(\vec{u}) \in X_*^\varepsilon(\vec{0})$. Now we obtain the following estimates for $\tilde{\varphi}(\vec{u})$.

Proposition 0.3. *For any $R > 0$ and $\varepsilon_0 > 0$, there exist constants $C, c > 0$ such that*

$$\|\tilde{\varphi}(\vec{u})\|_\varepsilon \leq Ce^{-c/\varepsilon}$$

for $\varepsilon \in (0, \varepsilon_0)$ and $\|\vec{u}\|_\varepsilon \leq R$.

Let $I_\varepsilon(\vec{u}) = \Gamma_\varepsilon(\varphi(\vec{u}))$. Then from Proposition 0.3, we conclude that for any $R, \varepsilon_0 > 0$, there exist constants $C, c > 0$ such that

$$\left| I_\varepsilon(\vec{u}) - \sum_{i=0}^{k+m} \Gamma_\varepsilon(P_i(u_i)) \right| \leq Ce^{-c/\varepsilon}$$

for $\|\vec{u}\|_\varepsilon \leq R$ and $\varepsilon \in (0, \varepsilon_0)$.

Proposition 0.4. *The following hold.*

(i) *A vector function* $\vec{u} = (u_0, \cdots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k+m,\varepsilon})$ *is a critical point of* I_ε *if and only if* $\varphi(\vec{u})$ *is a critical point of* Γ_ε.

(ii) *The functional* $\vec{u} \mapsto I_\varepsilon(\vec{u})$ *satisfies (PS) condition if* Γ_ε *does.*

(iii) *For any* $R > 0$, $i = 0, \cdots, k+m$ *and* $\varepsilon_0 > 0$, *there exist constants* $C, c > 0$ *such that*

$$\Big|\frac{\partial I_\varepsilon}{\partial u_i}(u_0, \cdots, u_{k+m}) - \frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i}\Big| \leq Ce^{-c/\varepsilon} \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0)$$

and $\|\vec{u}\|_\varepsilon \leq R$.

We define $\Gamma_\varepsilon^j(u) = \Gamma_\varepsilon(P_j(u))$ for $u \in H^1(\Omega_{j,\varepsilon})$ and $j \in \{1, \cdots, k+m\}$.

Proposition 0.5. *For each $j \in \{1, \cdots, k+m\}$, the following hold.*

- **(i)** *A function u_j is a critical point of Γ_ε^j if and only if $P_j(u_j) \in H_0^1(\Omega_{j,\varepsilon}^\delta)$ is a critical point of Γ_ε on $H_0^1(\Omega_{j,\varepsilon}^\delta)$.*
- **(ii)** *The functional Γ_ε^j on $H^1(\Omega_{j,\varepsilon})$ satisfies (PS) condition if Γ_ε on $H_0^1(\Omega_{j,\varepsilon}^\delta)$ does.*

Let $\alpha = 2\frac{\mu+1}{\mu-1}$. By Proposition 0.1 (iii), we can choose a constant $M > 1$, independent of small $\varepsilon > 0$, such that

$$\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}} \leq \|P_i(u_i)\|_\varepsilon \leq M\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}} \text{ for all } u_i \in H^1(\Omega_{i,\varepsilon})$$

and $i \in \{0, \cdots, k+m\}$.

Proposition 0.6. *Let $i \in \{0, \cdots, l\}$. For sufficiently small $\varepsilon > 0$, it holds that $\Gamma_\varepsilon(u) \geq 6M^2\varepsilon^{2\alpha}$ for all $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ with $\|u\|_\varepsilon = 4M\varepsilon^\alpha$.*

Proposition 0.7. *Let $i \in \{0, \cdots, l\}$. For sufficiently small $\varepsilon > 0$, it holds that $|\Gamma_\varepsilon(u)| \leq 5M^2\varepsilon^{2\alpha}$ for all $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ with $\|u\|_\varepsilon \leq 3M\varepsilon^\alpha$, and $\Gamma_\varepsilon(u) \geq 0$ for all $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ with $\|u\|_\varepsilon \leq 5M\varepsilon^\alpha$.*

Note that we take $\alpha = 2\frac{\mu+1}{\mu-1}$.

Proposition 0.8. *For sufficiently small* $\varepsilon > 0$,

$$\|I'_\varepsilon(\vec{u})\| \geq \frac{1}{2}\varepsilon^\alpha$$

if $2\varepsilon^\alpha \leq \|P_i(u_i)\|_\varepsilon \leq 3M\varepsilon^\alpha$ *for some* $i \in \{0, \cdots, k\}$.

Proposition 0.9. *Let* $E > 0$ *be a given constant. Then for sufficiently large* $R_1 > 0$*, there exists a constant* ε_0 *such that if* $I_\varepsilon(\vec{u}) \leq E$, $\varepsilon \in (0, \varepsilon_0)$, $\frac{R_1}{4M} \leq \|P_i(u_i))\|_\varepsilon \leq R_1$ *for some* $i = 1, \cdots, k$, *then* $\|I'_\varepsilon(\vec{u})\| \geq 1$.

Scheme of the Proof for $m_1 = \cdots = m_k = 0$.

Let $G_\varepsilon^i = \{\gamma \in C([0,1], H_0^1(\Omega_{i,\varepsilon}^\delta)) \mid \gamma(0) = 0, \gamma(1) = h_i\}$, $\mathit{\Gamma}_\varepsilon(h_i) < 0$, and

$$c_\varepsilon^i := \inf_{\gamma \in G_\varepsilon^i} \max_{t \in [0,1]} \mathit{\Gamma}_\varepsilon(P_i(\gamma(t))) \geq 6M^2\varepsilon^{2\alpha}, i \in \{1, \cdots, k\}$$

We find a good path $\gamma_i \in C([0,1], H_0^1(\Omega_{i,\varepsilon}^\delta)$ so that

$$\max_{t \in [0,1]} \mathit{\Gamma}(\gamma_i(t)) \leq c_\varepsilon^i + \exp(-\frac{1}{\varepsilon}).$$

Now we define $\vec{\gamma}(s) = (0, \gamma_\varepsilon^1(s_1), \cdots, \gamma_\varepsilon^k(s_k))$ with $s = (s_1, \cdots, s_k) \in [0,1]^k$. We define

$$d_\varepsilon := \max_{s \in [0,1]^k} I_\varepsilon(\vec{\gamma}(s)).$$

Then $d_\varepsilon = c_\varepsilon^1 + \cdots + c_\varepsilon^k + O(e^{-c/\varepsilon})$ for some $c > 0$.

Proposition 0.10. *For sufficiently small* $\varepsilon > 0$*, there exists a critical point* $u_\varepsilon \in H_\varepsilon$ *of* Γ_ε *such that* $u_\varepsilon \in \left(\Gamma_\varepsilon^{d_\varepsilon} \setminus \Gamma_\varepsilon^{d_\varepsilon - \varepsilon^{2\alpha}}\right)$

$$u_\varepsilon \in \left\{ \|P_0(u)\|_\varepsilon \leq 5M\varepsilon^\alpha, \varepsilon^\alpha \leq \|P_i(u)\|_\varepsilon \leq 2R, i = 1, \cdots, k \right\}.$$

PROOF

If not, through by the gradient flow, we can find paths

$$\vec{\gamma_*} = (\gamma_*^0, \cdots, \gamma_*^k) \in C([0,1]^k, \Pi_{i=0}^k H_0^1(\Omega_{i,\varepsilon}))$$

such that

$$I(\vec{\gamma_*}) \leq d_\varepsilon - \varepsilon^{2\alpha}/2.$$

On the other hand,

$$I(\vec{\gamma_*}) \geq \sum_{i=1}^k \Gamma(P_i(\gamma_*^i)) + O(\exp(-\frac{c}{\varepsilon})) \geq c_\varepsilon^1 + \cdots + c_\varepsilon^k + O(\exp(-\frac{c}{\varepsilon})$$

$$= d_\varepsilon + O(\exp(-\frac{c}{\varepsilon}); \quad \text{contradiction}$$

Scheme of the Proof for the last gluing result

Simple case. $x_1 = 0$ and $\Omega_{k+1} = B_\sigma(0)$. $(\mathbf{u}, u_{k+1}) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k+1,\varepsilon})$, $\mathbf{u} = (u_0, \cdots, u_k) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k,\varepsilon})$. We write $\Omega_\varepsilon = \Omega_{k+1,\varepsilon}$, $P = P_{k+1}$. For a given $\mathbf{u}$, we define $Q_\varepsilon(u_{k+1}) = I_\varepsilon(\mathbf{u}, u_{k+1})$, $\psi(u_{k+1}) = \varphi(\mathbf{u}, u_{k+1})$, $K_\varepsilon(u_{k+1}) = \Gamma_\varepsilon(P(u_{k+1}))$.

The first step: we find a solution Ψ_ε of the localized problem $K'_\varepsilon(u) = 0$.

The second step: we find a solution $u_{k+1} = \Phi_\varepsilon(\mathbf{u})$ of $Q'_\varepsilon(u_{k+1}) = 0$ in an exponentially small neighborhood of the solution Ψ_ε to the localized problem.

The third step: to find a critical point of $I_\varepsilon(\mathbf{u}, \Phi_\varepsilon(\mathbf{u}))$, we consider the following functional

$$\overline{I}_\varepsilon(\mathbf{u}) \equiv I_\varepsilon(\mathbf{u}, \Phi_\varepsilon(\mathbf{u})) - K_\varepsilon(\Psi_\varepsilon).$$

The last step : we apply the variational method as before to find a critical point of $\overline{I}_\varepsilon$.

What happen when V vanishes at infinity

$$\varepsilon^2 \Delta u - Vu + u^p = 0, \quad u > 0, \quad \text{in } \mathbf{R}^n$$

[Ambrosetti-Felli-Malchiodi, Ambrosetti-Malchiodi-Ruiz, B.-Wang] If $\liminf_{|x|\to\infty} V(x)|x|^2 > 0$, we can construct standing waves with finite energy.

If V has compact support, for $p \in (1, n/(n-2)]$, there exists no positive solution for

$$\varepsilon^2 \Delta u - Vu + u^p = 0, \quad u > 0, \quad \text{in } \mathbf{R}^n$$

[Chang-Yin, Moroz-Van Schaftingen] No decay restriction on $V \geq 0$ when $p \in (n/(n-2), (n+2)/(n-2)), n \geq 3$

What is an optimal threshold of decay rate of V for existence and nonexistence

[Ambrosetti and Malchiodi, 2007, Concentration phenomena for NLS: recent results and new perspectives, Contem. Math.]

[Bae-B.] Suppose that

$$\lim_{|x|\to\infty} V(x)|x|^2 = 0$$

for $1 < p < n/(n-2)$ if $n \geq 3$ and $p > 1$ if $n = 1, 2$, and that $\lim_{|x|\to\infty} V(x)|x|^2 \ln|x| = 0$ for $p = n/(n-2)$ and $n \geq 3$, then there exist no positive solutions for any small $\varepsilon > 0$.

Moreover there exists a potential V satisfying $\lim_{|x|\to\infty} V(x)|x|^2 \ln|x| > 0$ such that for some small $\varepsilon > 0$, there exists a solution of the problem

$$\varepsilon^2 \Delta u - Vu + u^{n/(n-2)} = 0, \quad u > 0 \quad \text{in } \mathbf{R}^n \setminus B(0,1).$$

A goal on the standing waves through by development of variational methods

Construct solutions under the Beresticki-Lions condition on nonlinearity and the optimal decay condition on V