Nontrivial solutions for a a class of singular problems

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Introduction

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Existence of two solutions

Theorem 1

Problem (1) has two distinct nontrivial solutions for $\lambda > 0$ large.

Perturbed Problem

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Consider the perturbation

$$g_\varepsilon(t) = \begin{cases} \dfrac{t^q}{(t+\varepsilon)^{q+\beta}} \text{ for } t \geq 0 \\ 0 \text{ for } t < 0, \end{cases} \tag{2}$$

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The associated functional $I_\varepsilon \in C^1(H_0^1(\Omega), \mathbb{R})$ is given by

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega G_\varepsilon(u) - \frac{\lambda}{p+1}\int_\Omega (u^+)^{p+1}$$

where $G_\varepsilon(u) = \int_0^t g_\varepsilon(s)ds \geq 0$.

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For every $\lambda > 0$, *there is* $\rho > 0$ *such that,* $I_\varepsilon(u) \geq \frac{1}{4}\rho^2$ *whenever* $\|u\|_{H_0^1} = \rho$ *and* $0 < \varepsilon < 1$.

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Proposition 2

For $\lambda > 0$ *large enough, and* $0 < \varepsilon < 1$*; there is* $a > 0$ *and a critical point* $u_\varepsilon^2 \in H_0^1(\Omega)$ *of mountain pass type such that* $I_\varepsilon(u_\varepsilon^2) > a$*.*

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Claim: the upper bound α is independent of $\varepsilon > 0$.
Observe that the solutions u_ε of (3) are a priori bounded: multiply (3) by u_ε, integrate, discard the term involving g_ε and use the Sobolev imbedding, to obtain

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Our objective is to obtain gradient estimates for solutions of (3). Then, taking $\varepsilon \to 0$, we show that the functions u_1 and u_2 are solutions of (1).

Gradient Estimates

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Let the weight ψ be such that

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Lemma 3

If u_ε is a solution of (3), then there is a constant $M > 0$ independent of ε such that

$$\psi(x)|\nabla u_\varepsilon(x)|^2 \leq M(u_\varepsilon(x)^{1-\beta} + u_\varepsilon(x)) \quad \forall x \in \Omega,$$

where M depends only on Ω, N, β, ψ and $\|u_\varepsilon\|_{L^\infty(\Omega)}$.

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- Remark that a nontrivial solution u_ε of (3) is nonnegative and belongs to $C^2(\overline{\Omega})$. However, we cannot use the maximum principle to ensure that u_ε is positive or identically zero, since $u^{q-1}/(u+\varepsilon)^{q+\beta}$ is singular when $u \sim 0$.

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- Consider the functions

$$w = \frac{|\nabla u|^2}{Z(u)}, \qquad v = w\psi,$$

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  where $Z(u_\varepsilon) = u_\varepsilon^{1-\beta} + u_\varepsilon + \delta$, with $\delta > 0$, in order to have $Z > 0$. In the end of the proof, we let $\delta \to 0$.
- We also use the fact that a nontrivial solution u_ε of (3) belongs to C^3 on a neighborhood of every point where it is positive

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Lemma 4

For any $\Omega' \subset \Omega$ there exists C such that

$$|\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| \leq C|x-y|^{\frac{1-\beta}{1+\beta}} \quad \forall x, y \in \Omega'.$$

The constant C depends only on Ω, N, β, p, $\|u_\varepsilon\|_{L^\infty(\Omega)}$, but not on ε.

Considering u, a weak limit of solutions u_ε of (3),

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$\frac{1}{u^\beta}\chi_{\Omega_+} \in L^1_{loc}(\Omega)$, where $\Omega_+ = \{x \in \Omega : u(x) > 0\}$.

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- The proof is done by choosing appropriate test functions for the perturbed problem.

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Given $\varphi \in C_c^1(\Omega)$, for $m > 0$, we have

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by the Dominated Convergence Theorem.

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By the estimate $|\nabla u_\varepsilon|^2 \le M(u_\varepsilon^{1-\beta} + u_\varepsilon)$ in Ω_0 (provided by Lemma 3), we obtain

$$|J_\varepsilon| \le M \int_{\Omega_0 \cap \{\frac{m}{2} \le u_\varepsilon \le m\}} \frac{(u_\varepsilon^{1-\beta} + u_\varepsilon)}{m} \eta'(u_\varepsilon/m)\varphi \to$$

$$\to M \int_{\Omega_0 \cap \{\frac{m}{2} \le u \le m\}} \frac{(u^{1-\beta} + u)}{m} \eta'(u/m)\varphi \quad \text{as } \varepsilon \to 0,$$

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Combining these facts with (5), we obtain

$$\int_\Omega \nabla u \nabla \varphi = \int_{\{u>0\}} \big(-\frac{1}{u^\beta} + \lambda u^p\big)\varphi$$

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This concludes the proof of Theorem 1.

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Theorem 2

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- We are unable to prove that one of the solutions of Theorem 1 is positive. We believe that one of them is positive and the other one vanishes somewhere in Ω. This would be in agreement with the result for the radial problem proved by Ouyang - Shi - Yao.

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- Theorem 2 is related to a result by Dávila: for λ grater than a precise constant, the maximal solution u_λ is a strict local minimizer I in the convex subset of $H_0^1(\Omega)$ of nonnegative functions in Ω.

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Associated with problem (1) we have the functional $I : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u^+),$$

where $f(u) = -\frac{1}{u^\beta} + \lambda u^p$ and $F(u) = \int_0^u f(s)ds$.

It is known (Dávila-Montenegro) that $\underline{u} = c\varphi_1^{\frac{2}{1+\beta}}$ is a subsolution (if λ is large) for the problem (1), which in our new notation is

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Take a sequence of smooth domains

$$\emptyset \neq \Omega_1 \subset\subset \Omega_2 ... \subset\subset \Omega$$

such that $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$.
Define the truncated function

$$\hat{f}(u) = \begin{cases} f(\underline{u}(x)) \text{ for } & s \leq \underline{u}(x) \\ f(s) \text{ for } & s \geq \underline{u}(x) \end{cases} \tag{8}$$

Consider the truncated problems on each domain Ω_k,

$$\begin{cases} -\Delta u_k = \hat{f}(u_k) & \text{in } \Omega_k \\ u_k = \underline{u}(x) & \text{on } \partial\Omega_k. \end{cases} \tag{9}$$

In order to find a solution to (9) we consider the translated problem for $v_k = u_k - \underline{u}$ with homogeneous boundary conditions

$$\begin{cases} -\Delta v_k = \hat{f}(v_k + \underline{u}) - \Delta \underline{u} & \text{in } \Omega_k \\ v_k = 0 & \text{on } \partial\Omega_k. \end{cases} \tag{10}$$

Define the functional $\tilde{I}_k : H_0^1(\Omega_k) \to \mathbb{R}$ by

$$\tilde{I}_k(v) = \int_{\Omega_k} \frac{1}{2}|\nabla v|^2 - \tilde{F}(v) + \nabla \underline{u} \nabla v,$$

here

$$\tilde{F}(v) = \int_0^v \hat{f}(t^+ + \underline{u})dt.$$

Notice that

$$\tilde{F}(v) = \begin{cases} f(\underline{u}(x))v \text{ for } & v \leq 0 \\ \hat{F}(v + \underline{u}) - \hat{F}(\underline{u}) \text{ for } & v > 0 \end{cases} \tag{11}$$

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- $u_k = v_k + \underline{u}$ is a solution of (9).
- $v_k \geq 0$ on Ω_k (by the maximum principle since $\underline{u}$ is a subsolution).
- Given k_0, $\|v_k\|_{H_0^1(\Omega_{k_0})}$ is bounded for every $k \geq k_0$.

Taking a subsequence, we obtain

- $u_k \rightharpoonup u$ in $H_0^1(\Omega)$,
- $u_k \to u$ in L^σ for $1 \leq \sigma < 2N/(N-2)$,
- $u_k \to u$ a.e in Ω.
- Hence $\underline{u} \leq u$ in Ω.

Let φ be a test function in $C_0^\infty(\Omega)$. There is a $k' > 0$ and a bounded domain Ω' such that *support*$(\varphi) \subset\subset \Omega' \subset\subset \Omega_k$ for every $k \geq k'$. Thus,

$$\int_{\Omega'} \nabla u_k \nabla \varphi = \int_{\Omega'} f(u_k)\varphi \quad \text{for every } k \geq k'.$$

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