INVOLUTIVE RATIONAL NORMAL STRUCTURES

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MAY 12, 2009

Moduli spaces: Let X be a complex manifold. When $C \subset X$ is a compact complex submanifold, results of Kodaira show that there is a complex-analytic 'moduli space' M consisting of the 'deformations' of C and whose tangent cone at C is

$$T_C M \simeq H^0(C, \nu) \qquad \text{where} \quad \nu = C^*(TX)/TC.$$

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Moduli of Rational curves: When $C = \mathbb{P}^1$, and $\dim X = n+1$, one has

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so that, when $k_i \geq 0$, one has a (non-canonical) isomorphism

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This idea has been exploited by R. Penrose (and, since, many others) to construct examples of manifolds M endowed with special structures.

Example: (Penrose) When $\nu \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$, the space M has dimension 4 and its tangent spaces carry a structure of the form

$$T_P M \simeq \mathbb{C}^2 \otimes H^0\big(P, \mathcal{O}(1)\big)$$

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A specific example is when C is a line in $\mathbb{P}^3$, so that $M = \mathrm{Gr}(2, \mathbb{C}^4)$.

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Example: When $\nu \simeq \mathcal{O}(n)$, the space M has dimension $n+1$ and its tangent spaces carry a structure of the form

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In particular, each $\mathbb{P}(T_P M)$ contains a rational normal curve

$$Z_P \subset \mathbb{P}(T_P M)$$

that consists of the (projectivized) pure n-th powers in $\mathsf{S}^n(\mathbb{C}^2)$.

Definitions: A rational normal structure (RNS) on a manifold M^{n+1} is a subbundle

$$Z \subset \mathbb{P}(TM)$$

of dimension $n+2$ such that each fiber $Z_x = Z \cap \mathbb{P}(T_x M)$ is a rational normal curve in $\mathbb{P}(T_x M) \simeq \mathbb{P}^n$.

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The coframe bundle of Z is the bundle B consisting of isomorphisms

$$u : T_xM \to \mathsf{S}^n(\mathbb{C}^2)$$

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There is a corresponding notion of real rational normal structure in the smooth category, where $\mathbb{C}$ is replaced by $\mathbb{R}$ everywhere.

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When $n > 2$, an RNS does not correspond to a well-studied structure. However, an RNS $Z \subset \mathbb{P}(TM)$ is equivalent to the choice of a G_n-structure on M, where

$$G_n \subset \mathrm{GL}\big(\mathsf{S}^n(\mathbb{C}^2)\big) \simeq \mathrm{GL}(n{+}1, \mathbb{C})$$

is the group of linear transformations that preserves the cone of pure n-th powers, which is isomorphic to the quotient of $\mathrm{GL}(2, \mathbb{C})$ by its central cyclic subgroup of order n. Thus, an RNS is a choice of a section of the bundle $F/G_n \to M$, whose fibers are the homogeneous spaces $\mathrm{GL}(n{+}1, \mathbb{C})/G_n$.

Some numerology:

The general G_n-structure on $M = \mathbb{C}^{n+1}$ depends (locally, in the sense of germs) on

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This raises the question of characterizing those RNSs that arise from such moduli spaces in terms of differential-geometric invariants of the RNS.

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If one takes X to be $\mathbb{CP}^2$ with one point p blown up and lets C be a conic in $\mathbb{CP}^2$ that passes through p, then M has dimension 4, the induced G_3-structure on M has a (unique) torsion-free connection, and this connection has irreducibly acting holonomy

$$H_3 = G_3 \cap \mathrm{SL}(4, \mathbb{C}) \simeq \mathrm{SL}(2, C)/\mathbb{Z}_3,$$

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This suggested the problem of determining the generality of torsion-free G_3-structures in dimension 4. I showed that, modulo diffeomorphisms, the torsion-free G_3-structures depend (locally, in the sense of germs), on 4 functions of 3 variables.

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When $n = 4$, there are only two distinct torsion-free G_4-structures in dimension 5 and they are both symmetric spaces, while, when $n > 4$, there is (up to diffeomorphism) only one torsion-free G_n-structure in dimension $n+1$, namely, the flat one.

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It turns out that there *is* a twistor-theoretic construction of the general torsion-free G_3-structure, but it involves a different kind of moduli space.

Moduli of rational contact curves: Suppose that X is a holomorphic contact 3-fold, with contact line bundle $L \subset T^*X$ and let $C \subset X$ be an embedded rational curve that is contact (aka *Legendrian*) and suppose that $n \geq 1$ is such that

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Moreover, not only is M smooth and of dimension $2n$ near C, but the subset

$$Y = \{P \in M \mid P \text{ is contact and } P^*(L^*) \simeq \mathcal{O}(n)\}$$

is a smooth subvariety of dimension $n{+}1$, such that, for all $P \in Y$,

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This generalizes the curve-in-a-surface construction as follows: If S is a surface, let $X = \mathbb{P}(TS) = \mathbb{P}(T^*S)$. Then X is endowed with a canonical contact structure $L \subset T^*X$, and every curve $C \subset S$ lifts canonically to a contact curve $C \subset X$. The moduli space Y in this case is equal to the moduli space M associated to $C \subset S$.

Theorem 0: [B—,1987] Every torsion-free G_3-structure is locally constructed as the one arising on the moduli of rational contact curves C in some contact 3-fold X with $C^*(L^*) \simeq \mathcal{O}(3)$.

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Theorem 1: [B—,2008] A RNS $Z \subset \mathbb{P}(TM)$ arises on a moduli space of rational contact curves $C \subset X$ with $C^*(L^*) \simeq \mathcal{O}(n)$ if and only if the dual structure $Z^* \subset \mathbb{P}(T^*M)$ is *involutive*.

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Remark: The involutivity of Z^* is a system of first-order PDE.

Theorem 2: [B—,2008] Modulo diffeomorphism, the local G_n-structures on M^{n+1} whose associated dual variety in $\mathbb{P}(T^*M)$ is involutive depend on $2n-2$ functions of 3 variables.

Involutivity in $\mathbb{P}(T^*M)$: Let M be an $(n{+}1)$-manifold.

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Definition: A submanifold $S \subset \mathbb{P}(T^*M)$ of codimension $k \leq n{+}1$ is said to be *involutive* if

$$S^*\big(\lambda\wedge(\mathrm{d}\lambda)^{n-k+1}\big) = 0.$$

for some (and hence any) nonvanishing contact form λ.

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Definition: A submanifold $S \subset \mathbb{P}(T^*M)$ of codimension $k \leq n{+}1$ is said to be *involutive* if

$$S^*\bigl(\lambda \wedge (\mathrm{d}\lambda)^{n-k+1}\bigr) = 0.$$

for some (and hence any) nonvanishing contact form λ.

Remark: It is easy to show that, if $S^*\lambda \neq 0$, then $S^*\bigl(\lambda\wedge(\mathrm{d}\lambda)^{n-k}\bigr) \neq 0$.

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For $Z^* \subset \mathbb{P}(T^*M)$, which has codimension $n-1$, this is the condition

$$(Z^*)^*\big(\lambda \wedge (\mathrm{d}\lambda)^2\big) = 0.$$

Theorem: If M is a moduli space of rational contact curves $C \subset X$ with $C^*(L^*) \simeq \mathcal{O}(n)$, then the associated RNS $Z \subset \mathbb{P}(TM)$ has its dual variety $Z^* \subset \mathbb{P}(T^*M)$ be involutive.

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Proof idea: Fix $C \in M$ and choose a point $p \in C \subset X$. One can choose C-centered coordinates $c_0, \ldots, c_n$ on M and p-centered coordinates x, y, z on X so that

(1) $\lambda = \mathrm{d}y - z\,\mathrm{d}x$
(2) The curves in M near C are described by the locus $W \subset M \times X$ of the equations

$$y = c_0 + c_1\,x + \cdots + c_n\,x^n + x^{n+1} F(c,x)$$
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and the map $[\lambda] : W \to \mathbb{P}(T^*M)$ thus embeds W as the dual variety Z^*, which is totally unramified on each fiber and hence has its image a rational normal curve in each T_cM. The involutivity follows since $\lambda{\wedge}(\mathrm{d}\lambda)^2 = 0$.

Some representation theory. The standard action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ induces representations of $\mathrm{SL}(2,\mathbb{C})$ on $V_k = \mathsf{S}^k(\mathbb{C}^2)$ for $k \geq 0$, and this gives the entire list of irreducible representations of $\mathrm{SL}(2,\mathbb{C})$.

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Regarding x, y as a basis of $\mathbb{C}^2$, we have V_k as homogeneous polynomials in x and y of degree k. There are $\mathrm{SL}(2,\mathbb{C})$-equivariant pairings

$$\langle , \rangle_p : V_m \times V_n \longrightarrow V_{m+n-2p}$$

for $0 \leq p \leq \min(m,n)$ (called 'transvectants') that are defined by

$$\langle u, v\rangle_p = (-1)^p \langle v, u\rangle_p = \sum_{k=0}^{p} \frac{(-1)^k}{k!\,(p-k)!} \frac{\partial^p u}{\partial x^{p-k}\,\partial y^k} \frac{\partial^p v}{\partial x^k\,\partial y^{p-k}}.$$

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These pairings help make explicit the Clebsch-Gordan formulae

$$\begin{aligned} V_m \otimes V_n &\simeq V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{|m-n|}\,, \\ S^2(V_m) &\simeq V_{2m} \quad\;\; \oplus V_{2m-4} \oplus \cdots\,, \\ \Lambda^2(V_m) &\simeq V_{2m-2} \oplus V_{2m-6} \oplus \cdots\,. \end{aligned}$$

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Proposition: A G_n-structure $B \subset F$ has its associated Z^* be involutive if and only if it is an integral manifold of the V_{3n-2}-valued 4-form

$$\Upsilon = \bigl\langle \mathrm{d}\eta, \langle \eta, \eta\rangle_1 \bigr\rangle_0 .$$

Connections and intrinsic torsion: Let $B \subset F$ be a G_n-structure.

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$$\mathrm{d}\omega = -\rho \wedge \omega - \langle \phi, \omega \rangle_1 + T(\omega \wedge \omega),$$

where $T : B \to \mathrm{Hom}\big(\Lambda^2(V_n), V_n\big)$ is the G_n-equivariant torsion function associated to the connection (ρ, ϕ).

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Since the choice of connection cannot affect the integrability of Z^*, one sees that

$$B^*\Upsilon = \big\langle \mathrm{d}\omega, \langle \omega, \omega \rangle_1 \big\rangle_0 = \big\langle T(\omega \wedge \omega), \langle \omega, \omega \rangle_1 \big\rangle_0 ,$$

so that $B^*\Upsilon = 0$ must represent some number of linear equations on the *intrinsic torsion*, i.e., the part of the torsion unaffected by the choice of connection. (This takes values in $H^{0,2}(\mathfrak{g}_n)$.)

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Proposition: The condition $B^*\Upsilon = 0$ imposes $N = \frac{1}{2}(n{-}1)(n{-}2)(n{+}5)$ linear equations on the intrinsic torsion. The submodule of $H^{0,2}(\mathfrak{g}_n)$ that corresponds to these equations is isomorphic to $V_2 \otimes V_{n-4}$.

Proposition: ($n \geq 3$) If $B \subset F$ is a G_n-structure that satisfies $B^*\Upsilon = 0$, then there is a unique choice of connection (ρ, ϕ) on B for which

$$\mathrm{d}\omega_n = -(\rho - n\,\phi_0) \wedge \omega_n + 2\,\phi_2 \wedge \omega_{n-2}\,.$$

The ideal generated by $\{\omega_n, \omega_{n-2}, \phi_2\}$ is therefore Frobenius and defines a codimension 3 foliation on Z^*.

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Theorem: The ideal $\mathcal{I}$ on F generated by the components of Υ is involutive, with Cartan characters given by

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Remark: This result does not determine the generality modulo diffeomorphism because of the diffeomorphism invariance of the conditions.

The second structure equations: Given a G_n-structure $B \to M$ whose associated dual RNS is involutive, one has the first structure equation

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and this describes all of the relations between the second order invariants. Solving this equation shows that the second-order invariants take values in a module I_2 isomorphic to $V_3 \otimes V_{2n-5}$.

Theorem: The structure equations for a G_n-structure B with involutive dual variety Z^*, namely,

$$\begin{aligned} \mathrm{d}\omega &= -\rho\wedge\omega - \langle\phi,\omega\rangle_1 + T(\omega\wedge\omega),\\ \mathrm{d}\rho &= R(\omega\wedge\omega),\\ \mathrm{d}\phi &= -\tfrac{1}{2}\langle\phi,\phi\rangle_1 + F(\omega\wedge\omega), \end{aligned}$$

where the first order invariants take values in $I_1 \simeq V_2\otimes V_{n-4}$ and the second order invariants take values in $I_2 \simeq V_3\otimes V_{2n-5}$, are involutive, with Cartan characters

$$s_0 = 0,\quad s_1 = 3n{-}9,\quad s_2 = 3n{-}5,\quad s_3 = 2n{-}2,\quad s_4 = 0.$$

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Proof: A calculation.