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A Polynomial Inequality Generalising an Integer Inequality

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Abstract:

For any $\mathbf{a} := (a_1, a_2, \ldots, a_n) \in (\mathbb{R}^+)^n$, we establish inequalities between the two homogeneous polynomials $\Delta P_{\mathbf{a}}(x,t) := (x + a_1 t)(x + a_2 t)\cdots(x + a_n t) - x^n$ and $S_{\mathbf{a}}(x,y) := a_1 x^{n-1} + a_2 x^{n-2} y + \cdots + a_n y^{n-1}$ in the positive orthant $x, y, t \in \mathbb{R}^+$. Conditions for $\Delta P_{\mathbf{a}}(x,t) \le t S_{\mathbf{a}}(x,y)$ yield a new proof and broad generalization of the number theoretic inequality that for base $b \ge 2$ the sum of all nonempty products of digits of any $m \in \mathbb{Z}^+$ never exceeds m, and equality holds exactly when all auxiliary digits are $b-1$. Links with an inequality of Bernoulli are also noted. When $n \ge 2$ and $\mathbf{a}$ is strictly positive, the surface $\Delta P_{\mathbf{a}}(x,t) = t S_{\mathbf{a}}(x,y)$ lies between the planes $y = x + t\max\{a_i : 1 \le i \le n-1\}$ and $y = x + t\min\{a_i : 1 \le i \le n-1\}$. For fixed $t \in \mathbb{R}^+$, we explicitly determine functions $\alpha, \beta, \gamma, \delta$ of $\mathbf{a}$ such that this surface is $y = x + \alpha t + \beta t^2 x^{-1} + O(x^{-2})$ as $x \to \infty$, and $y = \gamma t + \delta x + O(x^2)$ as $x \to 0+$.